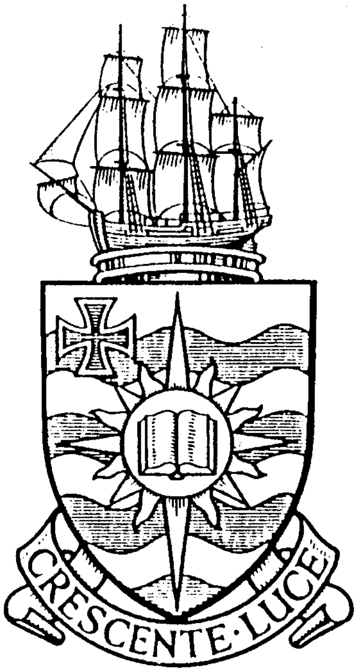
JAMES COOK MATHEMATICAL NOTES

Issue No. 25, Vol. 3

May, 1981.



The Crest of James Cook University of North Queensland incorporating a representation of Captain Cook's ship the *Endeavour* in full sail.

POLYNOMIAL SEQUENCES OF BINOMIAL TYPE

A.P. Guinand

Let $p_n(x)$ (for $n = 0, 1, 2, \ldots$) be a sequence of polynomials with real or complex coefficients. The sequence is said to be "of binomial type" if $p_0(x) = 1$ and

$$p_n(x + y) = \sum_{r=0}^{n} \binom{n}{r} p_r(x)\, p_{n-r}(y) \quad . \tag{1}$$

This condition is sufficient to ensure that each p_n is of degree n or less. The simplest example is the family of powers of x, $p_n(x) = x^n$.

Other examples are the rising factorials $x(x + 1)(x + 2) \ldots (x + n - 1)$ and the Abel polynomials $x(x - n)^{n-1}$. Such polynomial sequences can be generated by products of linear functionals, and this process has many applications, especially in combinatorics. For an exhaustive survey of these methods, see "The Umbral Calculus", by G-C. Rota and S.M. Roman, Advances in Mathematics, 27 (1978), 95-188.

There is, however, a much simpler way of generating such sequences, as follows.

Let $c_0, c_1, c_2, \ldots$ be any sequence. A polynomial sequence of binomial type is generated by $p_0(x) = 1$ and the recurrence

$$p_{n+1}(x) = x\sum_{r=0}^{n} \binom{n}{r} c_{n-r}\, p_r(x) \quad . \tag{2}$$

<u>Proof</u> First note that $p_1(x) = c_0 x$ and so (1) holds for $n = 0$ and 1. The proof is by induction: suppose that (1) holds for $n = 0, 1, 2, \ldots, k$.

Then

$$\frac{y}{x+y}\, p_{k+1}(x+y) = y\sum_{r=0}^{k} \binom{k}{r} c_{k-r}\, p_r(x+y)$$

$$= y\sum_{r=0}^{k} \binom{k}{r} c_{k-r} \sum_{s=0}^{r} \binom{r}{s} p_s(x)\, p_{r-s}(y) \quad . \qquad (3)$$

But $\binom{k}{r}\binom{r}{s} = \binom{k}{s}\binom{k-s}{r-s}$ and $\sum_{r=0}^{k}\sum_{s=0}^{r} = \sum_{s=0}^{k}\sum_{r=s}^{k}$

so that the right hand side of (3) becomes

$$\left\{\sum_{s=0}^{k}\binom{k}{s} p_s(x)\right\}\left\{y\sum_{r=s}^{k}\binom{k-s}{r-s} c_{(k-s)-(r-s)}\, p_{r-s}(y)\right\} \quad .$$

Inclusion of a term $s = k+1$ in the first sum only adds a zero, and by the defining recurrence (2) the second factor is $p_{k+1-s}(y)$, and equation (3) therefore becomes

$$\frac{y}{x+y}\, p_{k+1}(x+y) = \sum_{s=0}^{k+1}\binom{k}{s} p_s(x)\, p_{k+1-s}(y) \quad . \qquad (4)$$

Exchanging x and y and replacing s by k+1-s,

$$\frac{x}{x+y}\, p_{k+1}(x+y) = \sum_{s=0}^{k+1}\binom{k}{k+1-s} p_s(x)\, p_{k+1-s}(y) \quad . \qquad (5)$$

Now $\binom{k}{s} + \binom{k}{k+1-s} = \binom{k+1}{s}$ so adding (4) and (5) we have

$$p_{k+1}(x+y) = \sum_{s=0}^{k+1}\binom{k+1}{s} p_s(x)\, p_{k+1-s}(y) \quad .$$

That is, (1) also holds for $n = k+1$, and by induction for all positive n. This shows that the sequence is of binomial type. It may also be noted that $c_n = p'_{n+1}(0)$ because every polynomial except the first has constant term zero, and by (2)

$$p_{n+1}(x) = x\, c_n + \sum_{r=1}^{n} x\binom{n}{r} c_{n-r}\, p_r(x) \quad .$$

The converse question arises. Can every sequence of binomial type be constructed by the method above? Yes.

If $\{p_n(x)\}$ is a given polynomial sequence of binomial type then let $c_n = p'_{n+1}(0)$ for $n = 0, 1, 2, \ldots$. The sequence $\{c_n\}$ generates (as described above) a polynomial sequence $\{q_n(x)\}$ of binomial type. Since $p_o(x) = 1 = q_0(x)$ it follows by induction that $p_n = q_n$, for both polynomial sequences satisfy (2).

SIMPLE QUESTION FOR UNDERGRADUATES

Is the canvas of a fire-hose more likely to split longitudinally or circumferentially? For the purfst one should add that the water from the hydrant is assumed to be an incompressible non-viscous fluid in steady motion, gravity is negligible, and the canvas is assumed to be infinitesimally thick and to have equal strength in all directions. Those who emphasise the social relevance of their mathematics will add the information that a stout-hearted fireman (or fireperson if necessary) is directing towards the seat of the blaze a jet of water issuing at speed V from the brass nozzle of radius r and area $a = \pi r^2$. If you get from one of your students an answer involving the fourth root of three it will probably be right.

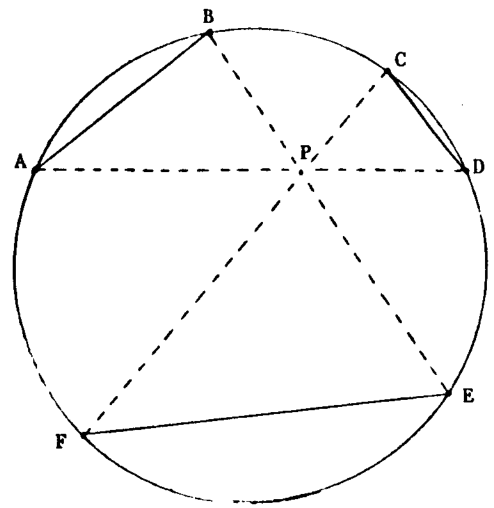
ISOTROPIC VECTORS

H. Kestelman

A vector with complex components is called isotropic if the sum of the squares of the components is zero. If a linear subspace of C^n consists of isotropic vectors show that its dimension cannot exceed $\frac{1}{2}n$.

CYCLIC HEXAGON

E. Szekeres



This problem originates from *E. Straus*. The hexagon ABCDEF has its points (in that order) on a circle of unit radius. The diagonals AD, BE and CF meet in P. Show that AB + CD + EF $\leq$ 4.

NAMES (JCMN 24, Vol. 2, p. 144)

\- First there is the old Latin proverb: *nomen est omen*. The mathematician Kurt Hensel, who was professor in Marburg, was asked whether he would not like going to Goettingen. He answered: *Oh nein, dort sind alle Klein und Schwarz und können einander nicht ausstehen* (there they are short and black and can't stand each other).

C.F. Moppert

\- On 16th December Dr. John Vane spoke on the television news about cardiovascular disease.

CONVERGENCE OF SERIES (JCMN 24, Vol. 2, p. 132)

It was pointed out that inside a certain oval shaped region the series $\sum_0^\infty (-z - z^2)^n = 1 - (z + z^2) + (z^2 + 2z^3 + z^4) - (z^3 + 3z^4 + 3z^5 + z^6) + \ldots$ converges to $1/(1 + z + z^2)$. If we remove the brackets to obtain the series

$$1 - z - z^2 + z^2 + 2z^3 + z^4 - z^3 - 3z^4 - 3z^5 - z^6 + z^4 + \ldots$$

what is the new region of convergence?

WELL KNOWN FUNCTION (JCMN 24, Vol. 2, p. 136)

G. Szekeres

What is the function f such that the coefficient of x^n in $(f(x))^{n+1}$ is 1? Using n = 0, 1, 2, 3 we may find

$$f(x) = 1 + x/2 + x^2/12 - x^4/720 + \ldots$$

which everybody recognizes as $x/(1 - e^{-x})$, used as a generating function for the Bernoulli numbers.

We must verify that if $x^{n+1}(1 - e^{-x})^{-n-1} = \sum_0^\infty a_n x^n$ then all $a_n = 1$. But this is a simple exercise in the theory of residues.

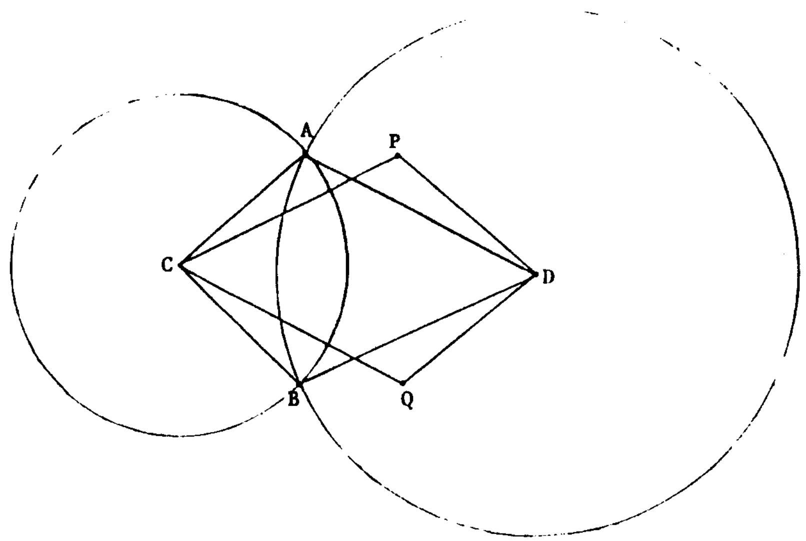
$$a_n = \operatorname*{res}_{z=0} \left(z^{-n-1} \left(\frac{z}{1 - \exp - z} \right)^{n+1} \right) = \operatorname*{res}_{z=0} (1 - \exp - z)^{-n-1}$$

or with the substitution $1 - e^{-z} = t$,

$$a_n = \operatorname*{res}_{t=0} \left(t^{-n-1} \frac{1}{1 - t} \right) .$$

This is just the coefficient of t^n in $\frac{1}{1 - t} = 1 + t + t^2 + \ldots$, hence equal to 1.

MOVING ROUND CIRCLES (JCMN 24, Vol. 2, p. 135)



Question 3 in the 1979 International Mathematical Olympiad was about two points moving in the same sense with equal angular speeds round two circles in the plane, starting simultaneously at A. It asked for a point P always equidistant from the two moving points. The question in JCMN 24 was whether the phrase "in the same sense" was necessary.

C.S. Davis writes that the answer to the query is "No": there is such a fixed point Q if the points move in opposite senses. Indeed, it is the reflection of P in the line CD of centres. In the picture above CBDP and CQDA are parallelograms.

A CIRCLE AND A TRIANGLE (JCMN 24, Vol. 2, p. 141)

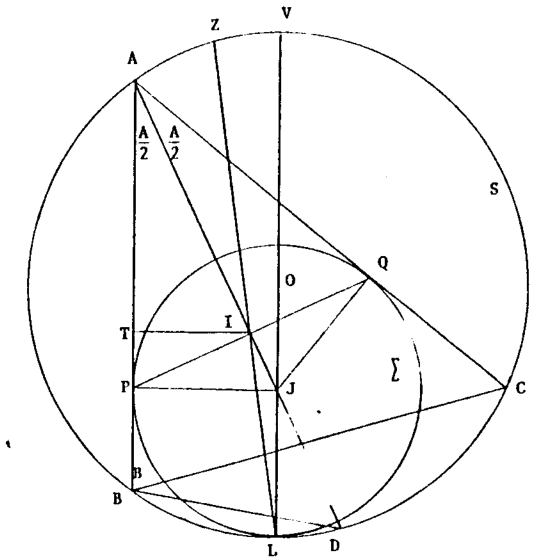


Figure 1.

The circle Σ (centre J) touches internally at L the circumcircle S (centre O) of the triangle ABC, and touches AB at P and AC at Q. The problem was to prove the midpoint I of PQ to be the centre of the inscribed circle of ABC. The proof below comes from *E. Szekeres*, others were sent in by *B.B. Newman* and *S. Collings*.

The bisector of the angle at A passes through I and J, and meets S at D, which is also where the perpendicular bisector of BC meets S. Draw LI and LJ to meet S in Z and V respectively. Drop the perpendicular IT from I to AB.

In the usual notation the radii R and r of the circumcircle and incircle are given by

$$\frac{r}{2 \sin A/2 \sin B/2 \sin C/2} = 2R = \frac{a}{\sin A} = \text{etc.}$$

2. Starting with Σ and Σ_1, the point A is the external centre of similitude, and AI is the axis of symmetry. There are just two circles through A touching both Σ and Σ_1. One is S with centre O on line AO. Since AI bisects the angle HAO (where H is the orthocentre of ABC) it follows that the other circle touching both has its centre on AH. By inversion with respect to A to interchange Σ with Σ_1 it follows that one common tangent to Σ and Σ_1 is parallel to BC (see Figure 2).

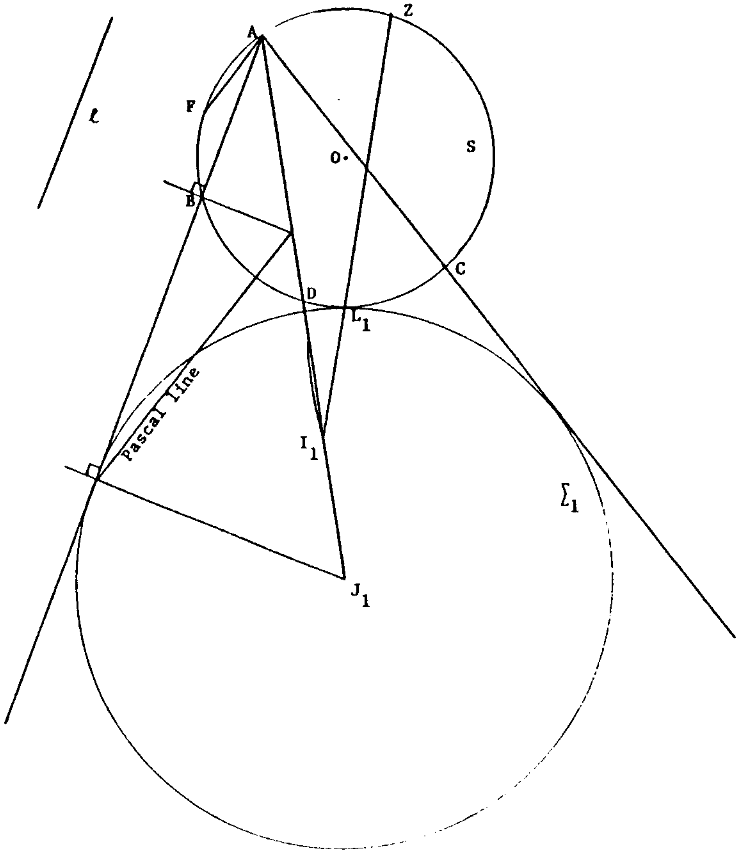


Figure 3.

3. Take the line ℓ parallel AB and at a distance R outside it. The parabola with ℓ as directrix and O as focus must pass through A, B and J_1. Let F be the intersection of S with the perpendicular bisector of AB. Then the line AF makes the same angle 90° - C/2 with OA and with the axis, so that AF, and similarly BF are tangents to the parabola. Pascal's theorem gives the following construction to find the point J_1 (Figure 3).

The hexagon $AAB\infty\infty J_1$ is inscribed in the parabola, and so the intersections of opposite sides are collinear. Here ∞ means the point at infinity, the line $\infty\infty$ is the tangent, which is the line at infinity. The three Pascal points are where AA meets $\infty\infty$, that is the point at infinity on the line AF, the intersection $B\infty$ with AJ_1, that is where the perpendicular from B to AB meets the bisector AI, (these two points enable us to draw the Pascal line as the line parallel to AF through the second point) and where AB meets $J_1\infty$. The Pascal theorem locates the third point where AB meets the Pascal line, and J_1 where the perpendicular to AB from this point meets AI.

EASY HAHN BANACH

A real Lipschitz function on a finite subset of the Euclidean plane can be extended to the whole space with the same Lipschitz constant. Can this be proved by elementary methods? That is, not using the axiom of choice or the Boolean prime ideal theorem. The Lipschitz condition for a function f is that $|f(P) - f(Q)| \leq k$ (distance from P to Q). Background to this question may be found in JCMN 20, page 54 and JCMN 24, page 135.

TRIGONOMETRIC FUNCTIONS

Let $f(x) = \sum_1^N a_r \exp(ib_r x)$ (with a_r complex and b_r real). If $f(n) = O(1/n)$ for large positive integers n, does it follow that $\sin \pi x$ is a factor of $f(x)$?

S.U.M.S. COMPETITION 1980 (JCMN 23 & 24, pp.124 & 137)

S.J. Surma

Let A be an arbitrary set. Define the set A^+ of all finite length strings formed from elements of A as the least set such that

(1) $A \subset A^+$,

(2) if $x, y \in A^+$ then $xy \in A^+$.

<u>Remark</u> Let $a, b, c \in A$. It may well happen that strings (ab)c and a(bc) are distinct as are strings ab and ba.

The set P of all primitive strings has been defined in JCMN 23, Vol. 2, p. 124 as well as in JCMN 24, Vol. 2, pp. 137-138 as follows

$P = \{xx: x \in A^+\}$

while the set G of all good strings has been defined there as the least set such that

(1) $P \subset G$,

(2) if $x, xy \in G$ then $y \in G$, for any $x, y \in A^+$.

R.N. Buttsworth claims that the problem whether or not a string is good is decidable. However, his argument seems to assume that, in particular, all the strings below

(+) (AyA)y, (xAx)(xx), ((yx)(yx))(y(xyx)), (x(yy)x)((xy)yx)

are good, which cannot be proved with the definition of P as above. In fact, assuming that $A = \{a, b\}$, as in the original formulation of the problem, the problem whether or not a string is good is undecidable. The proof of the last statement is too lengthy to be quoted herewith but it can be made available upon request.

Fortunately, Buttsworth's argument can be salvaged. In fact, it works with the following definition of the set of all primitive

strings

$$Q = \{(xy)(yx), (x(yz))((xy)z) : x, y, z \in A^+\}$$

with the other definitions unchanged. Namely, with Q instead of P all strings mentioned under (+) above as well as string xx can be shown to be good. The rather lengthy calculations are omitted.

Although Buttsworth's argument (with the modification suggested) looks quite nice, some other nice arguments can also be referred to, for instance, the following one.

With the definitions of A^+, Q and G unchanged let $x, y \in A^+$. Define a relation $x \sim y$ as follows

$x \sim y$ iff $xy \in G$.

One can show that $\sim$ is an equivalence relation on A^+ and that the partition $A^+/\sim$ of A^+ determined by $\sim$ is a Boolean group i.e. an Abelian group with the equivalence class of xx as its identity element and such that each element of the group is of rank 2.

As an easy corollary to the above statement we have that $x \in G$ iff $N_a(x)$ is even for each $a \in A$, where $N_z(x)$ is the number of occurrences of a in string x.

QUOTATION CORNER (10)

'It was Voltaire who wrote "it is magnificent but it isn't war" when discussing the charge of the Light Brigade --- '
From the sports page of the *Weekend Australian*, 3-4 January 1981.
Perhaps the writer was misquoting P. Bosquet's version of what Adam said to Eve in the Garden of Eden, "C'est magnifique mais ce n'est que la poire".

EXP-LORE

C.J. Smyth

We people of Htrae find multiplication quite easy; however, historically addition has posed much greater difficulties. Our calculations were much speeded up when one Nhoj Reipan invented Exp tables. To add two numbers together, one simply looked both up in the Exp tables, multiplied the resulting values together, and then looked up the product in anti-Exp tables.

Imagine for a moment a planet (let's call it EARTH (Htrae spelt backwards!)) with strange beings who find multiplication much more difficult than addition. Note that for EARTH-lings, the product of two numbers could be found as follows: look up the two numbers in the anti-Exp tables, add them together (an operation which poses no problem to these strange creatures, remember!); then look up the sum in the Exp tables. Easy as falling off an Exp!

PLANE SAILING

C.F. Moppert

A plane L can be shifted about on a plane E, on which it lies flat. L takes three positions L_1, L_2, L_3. Any point P in L takes then three positions P_1, P_2, P_3 (with respect to E). Which are the points P in L such that P_1, P_2, P_3 are collinear?

BLACK AND WHITE CUBES (JCMN 23, Vol.2, p. 125)

C.J. Smyth

We were given adjacent piles of blocks of heights $h_1 \leq h_2 \leq \cdots \leq h_J \geq h_{J+1} \geq \cdots \geq h_n$, and the blocks were to be coloured black or white but so that no two adjacent blocks (horizontally or vertically) were black. The problem was to show that one such colouring with the maximum number of black cubes is one of the two 'chess-board' colourings of the blocks.

The idea of the proof is the following: we show that we can pair (denoted •——• or | in the example of Figure 4) most of the blocks to adjacent ones, such that the few remaining unpaired blocks all have the same parity (i.e. would be the same colour in a chess-board colouring). Then since in any pairing only one of the two blocks can be coloured, the maximum number of blocks which can be coloured black is

$$\frac{1}{2}(\text{\# of paired blocks}) + (\text{\# of unpaired blocks}).$$

But this number is actually attained for the chess-board colouring which colours all the unpaired blocks black, so is best possible. Hence a chess-board colouring attains the maximum number of black squares coloured.

To show that such a pairing exists, we use induction on the number of *rows* (i.e. blocks at a given height). Note that because of the inequality for the h_i's, each row is completely supported (in the obvious sense) by the row below.

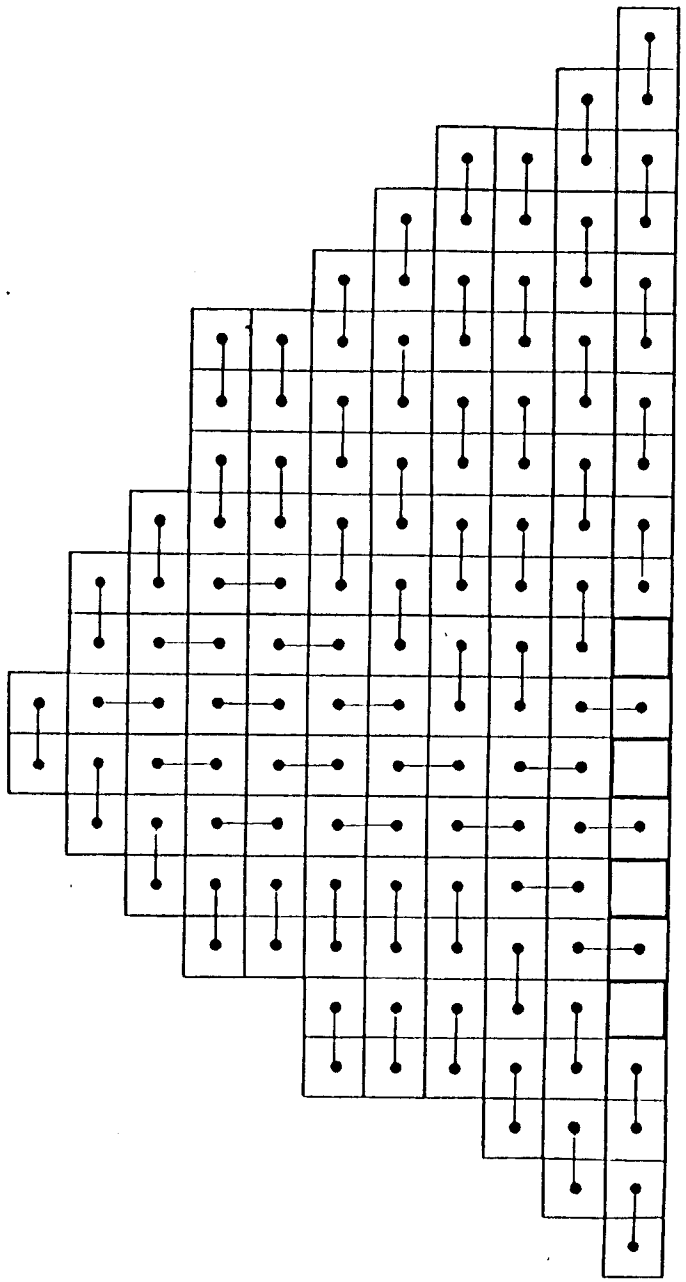


Figure 4

How the proof works for n = 11 rows (pile heights 1 < 2 < 3 < 6 = 6 < 8 < 9 < 10 < 11 = 11 > 10 = 10 > 9 > 8 = 8 = 8 > 6 > 5 > 4 > 2 > 1)

The inductive hypothesis is as follows: that the n rows can be paired in such a way that all the unpaired blocks are on the bottom row, and alternate there with paired blocks.

Proof for n = 1: Start pairing on the left (say). Then at most one block (the right-most one) will be left unpaired.

Assume the result for n. To prove it for n + 1 rows, we first use the induction hypothesis to pair the top n rows, such that the unpaired cubes are in the next-to-bottom row, and alternate there with paired cubes. Then we pair these unpaired ones with the only possible candidates for pairing, namely the cubes directly below them in the bottom row. Next, we pair the rest of the bottom row, working inwards from both sides. The effect of this is easily seen to be that all unpairable blocks in the bottom row alternate with paired blocks. This completes the induction. (It might be a good idea to see how it works in the example).

Generalisations This result was applied to a problem of number theory (JCMN 24, Vol. 2, pp. 130-132). Generalisations to higher dimensions would be useful for this problem, too. In particular, if in R^k we colour (with the same adjacency restriction) lattice points $(n_1, n_2, \ldots, n_k)$, n_i non-negative integers with $\sum_{i=1}^{k} \lambda_i n_i \leq 1$ (all $\lambda_i > 0$ and fixed), then it would be of definite interest to know whether one of the two 'chess-board' colourings (colour a lattice point $(n_1, \ldots, n_k)$ black or white depending on the parity of $n_1 + \ldots + n_k$) always gives the maximum number of black lattice points. It is far from certain that this will be the case, although I have not been able to find a counterexample. The best I can offer is a three-dimensional example which shows at least that the pairing technique, used in the two-dimensional proof, fails for $k = 3$. Consider the set of lattice points (n_1, n_2, n_3) satisfying $n_i \geq 0$, $n_1/3 + n_2/3 + n_3/2 \leq 1$. (See Figure 5). For these lattice points there is no way of pairing them so that the unpaired ones all have the same parity. Try it for yourself and see! However, 'chess-board' colourings do still give the maximum number of black lattice points.

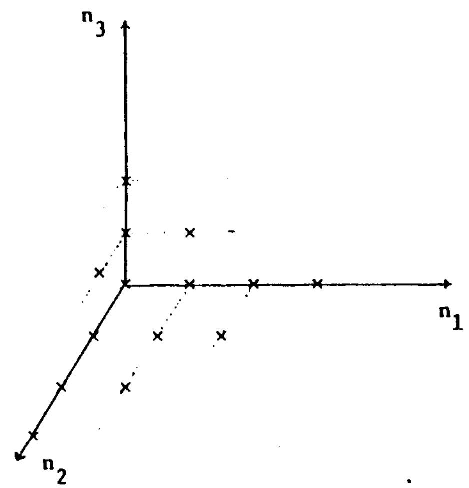


Figure 5

Lattice points (n_1, n_2, n_3) satisfying $n_i \geq 0$ and $n_1/3 + n_2/3 + n_3/2 \leq 1$.

BOUND VOLUME

Issues 18 to 24 inclusive of the James Cook Mathematical Notes are now being reprinted in a single volume. Volume 2 will be available at a price of \$5.00 (including postage). Customers in Australia are asked to send cheques payable to James Cook University. Those overseas are invited to send any kind of currency of roughly equivalent value (for example \$5.50 (U.S.A.) or £2.60 (United Kingdom) etc.)

Basil Rennie is on study leave in Sheffield until October 1981. In the meantime your acting editor would like to hear from you about anything connected with mathematics or James Cook, R.N.

Dr. Chris Smyth,
Mathematics Department,
James Cook University of
North Queensland,
Townsville, 4811,
Australia.