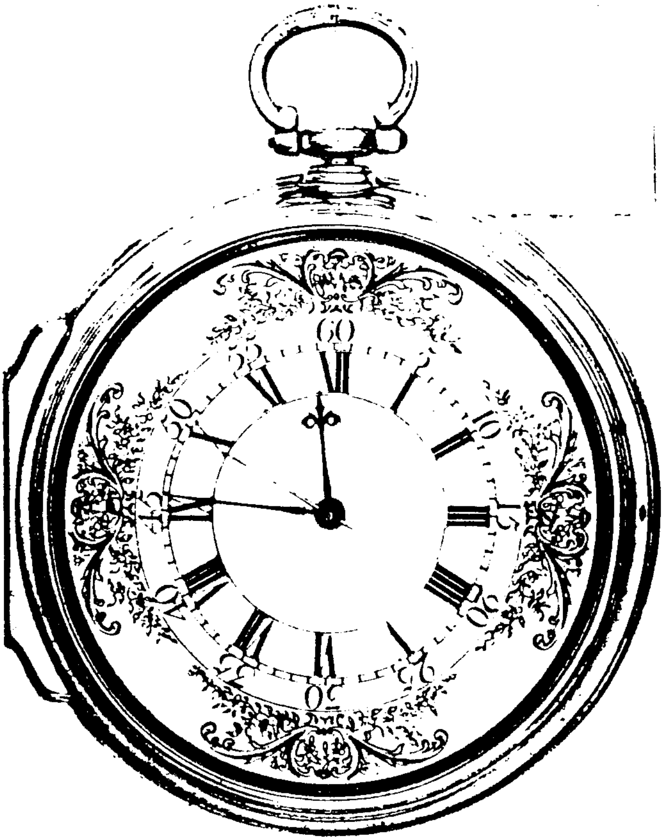
ISSN 0158-0221.

JAMES COOK MATHEMATICAL NOTES

Issue No. 21, Vol. 2

January, 1980.



POLYNOMIALS FROM A RECURSION

G. Szekeres

Let the rational functions $\phi_m(x)$ for $m = 1, 2, \ldots$, be determined from $\phi_0(x) = 1$, $\phi_1(x) = 1$, and

$$\phi_{m-2}(x)\,\phi_m(x) = \left(\phi_{m-1}(x)\right)^2 + x^{2m-3}(1-x)$$

for all $m \geq 2$. Show that $\phi_m(x)$ is a polynomial with integer coefficients. Determine the roots of $\phi_m(x)$.

Your editor, thinking the problem to be hard and trying to be helpful, has worked out the first few of these polynomials.

$$\phi_2(x) = 1 + x - x^2$$
$$\phi_3(x) = 1 + 2x - x^2 - x^3$$
$$\phi_4(x) = 1 + 3x - 3x^3$$
$$\phi_5(x) = 1 + 4x + 2x^2 - 5x^3 - 2x^4 + x^5$$

TWO POINTS IN A CIRCLE

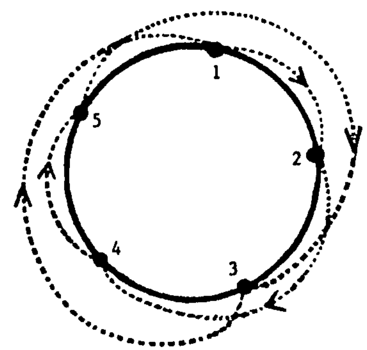
J.B. Parker writes that a mathematician in Wales described to him the following problem, but we do not know the original author.

Given two points, A and B, inside a circle with centre C, find P on the circle so that CP bisects the angle APB.

VISITING ON A CIRCLE

F.J.M. Salzborn (University of Adelaide)

Consider a circle with m points equally spaced, the distance between each pair of adjacent points being 1, travelling round the circle. We are looking for trips $i_1 \to i_2 \to \ldots \to i_m \to i_{m+1} = i_1$ going along the circle clockwise, such that each point is visited exactly once (a point may be passed without visiting). Let $d(i_k, i_{k+1})$ be the distance between two consecutive visiting points. The figure shows the trip $1 \to 2 \to 4 \to 5 \to 3 \to 1$ with the set of distances $\{1, 1, 2, 3, 3\}$. Clearly $\sum d(i_k, i_{k+1})$ is a multiple of m.



Problem: Given m positive integers $a_1, \ldots a_m$ with their sum a multiple of m, can one always find a trip such that $\{a_1, \ldots a_m\}$ is just the set of distances travelled (not necessarily in the same order) in going round the m points? If so, is there a systematic way of finding such a trip?

VECTOR DIFFERENTIAL EQUATION

H. Kestelman

The following arose from a problem brought to me by a pharmacologist.

A matrix Q has a simple eigenvalue of zero, and all its other eigenvalues have negative real parts. Show that there exist vectors $\underline{v}$ and $\underline{w}$ such that every vector function $\underline{x}(t)$ satisfying $d\underline{x}/dt = Q\underline{x}$ also satisfies $\lim_{t \to \infty} x(t) = \left(\underline{w}^t\, x(0)\right)\underline{v}$. How can $\underline{v}$ and $\underline{w}$ be found from Q?

GEOMETRY

C.P. Moppert

Basil Rennie and I have exchanged many letters over the past few months on the subject of geometry. It all started with a modest problem I sent him for the James Cook **Mathematical Notes.** It ended by him asking me to submit a couple of pages on the subject. Here they are.

Several times in the past I have asked fellow mathematicians: how accurate is Pythagoras? The first response was always perplexity. When I then started explaining, all of my friends said: ah, you think we might be in a curved space and this depends, of course, on the gravitational field, etc. etc..

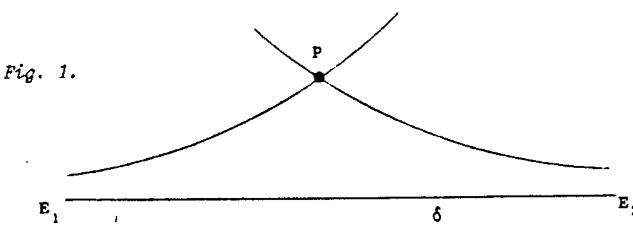
This kind of response is nonsense. Einstein was a priori Kantian in assuming that in a Galilean environment the geometry is Euclidean. Gauss went further: he checked experimentally whether our geometry differs noticeably from the Euclidean one. Gauss's experiment must be repeated periodically. It is possible that tomorrow it will be found that in a gravity-free space the geometry is not Euclidean. If this is not the case and if we assume (it can never be proved experimentally) that the geometry in such a space is Euclidean, then this fact has to be put as the cornerstone of our science and should then, as such, be put into a wider framework.

I take then geometry as an experimental science: lines are light rays. They were not popular when I was a student. They have a

better reputation now as laser beams. There is no doubt that almost every applied science is based on visual observation and thus on geometry (in my sense). A "plane" is formed by a grid of lines. A line has two points at infinity or "ideal points". I use Hilbert's word, "ends", for these. The geometry then has points, ends and lines as objects. Two points determine one line; a point and an end also determine one line. It follows that two lines have at the most one point in common. This excludes elliptic geometry where any two lines have two points in common. Klein said that every geometry is projective. I don't like this approach: the projective plane is an unpleasant thing: it cannot be orientated.

The crucial question: does the parallel axiom hold? Boliay put it as follows: given a line g with the ends E_1 and E_2 and a point P outside g, are then the lines through P and E_1 and through P and E_2 identical or not?

Fig. 1.



An equivalent question: Are there many lines through two ends or is there only one such? In the first case (for both questions) the geometry is Euclidean, in the second case hyperbolic.

<u>If</u> the geometry is euclidean, everything is simpler than otherwise. In particular we have then similarity and an angle can be defined as the constant ratio between arc and radius. As one doesn't know whether Euclidean geometry "exists", one can ask: is Euclidean geometry consistent or does it lead to contradictions? Hilbert answered this question: Euclidean geometry is as consistent as is arithmetic because we can give an arithmetical model. Indeed, this model was given by Descartes, long before Hilbert, in his analytic geometry. We see then that analytic geometry does not make analysis visible but it shows that our Euclidean imagination is consistent. As such, analytic geometry does not get us one inch further than arithmetic.

With the incidence relations alone we do not get far in our geometry: we must be able to measure. In Paris there is the standard meter. The length of "two meters" is obtained by moving the standard meter along itself. If a copy of the meter is sent via Suez to Australia and back to Paris via New York it is found on return to have the same length as before. Mathematically speaking: there are motions and they form a group. I find it useful to follow Hilbert, Hjelmslev and Bachmann and start with "reflections" as a basic notion. They are continuous maps on the plane (i.e. the set of ordered pairs (x,y) of real numbers) to itself; to every line corresponds a reflection and the line is the set of points invariant under its reflection. The postulate of the median bisectrix means that to any pair of points corresponds a unique reflection. I use the symbols $\not a, \not b, \ldots$ for lines and $a, b, \ldots$ for the corresponding reflection. We have then $a^2 = b^2 = \ldots = \varepsilon$ (the identity) and it is easy to show that perpendicularity:

$a(\not b) = \not b$ is equivalent to $ab = ba$.

The geometry then has the group of superimposed reflections $abc \ldots$. I call the product of any two reflections an "elementary motion" and have the postulate that the elementary motions $A, B, C, \ldots$ form a group (a subgroup of index 2 of the full group). Any number of reflections can then be represented by at the most three such.

From our incidence relations we are now able to characterise the fixpoints of an elementary motion A depending on the two geometries. Our "points" are ordered pairs of real numbers $P = (x,y)$. By the mapping $(x,y) \to (x',y')$ with $x' = \dfrac{\sqrt{1+x^2+y^2}-1}{x^2+y^2}\,x$, $y' = \dfrac{\sqrt{1+x^2+y^2}-1}{x^2+y^2}\,y$ the "plane" is mapped onto the open disc $x'^2 + y'^2 < 1$ and this disc can be made compact by adding its boundary. *(Fig. 2)* An elementary motion $A = ab$ is a continuous map of this closed disc onto itself and has thus at least one fixpoint. We denote with P the proper points (the inner ones) of the closed disc and with E the ideal ones (the ends). Assume that P is fixpoint for A: $abP = P$ or $bP = aP$. If $P \neq bP$ then $\not b$ is the unique median bisectrix of P and bP and so is $\not a$, thus $a = b$ and $A = \varepsilon$ (the identity). For $A \neq \varepsilon$ we have then $P = aP = bP$ and P is the unique point of intersection of $\not a$ and $\not b$, i.e. A can have at most one proper fixpoint. If an end E is fixpoint for A and $E \neq bE$ then E and bE determine a unique line $\not c$ and $b\not c = \not c$ as b interchanges the ends of $\not c$. Thus $\not c$ is perpendicular to $\not b$ and (for the same reasons) to $\not a$. If $E = bE = aE$ then $\not a$ and $\not b$ have the end E in common. Thus: if $ab = A \neq \varepsilon$ then the lines $\not a$ and $\not b$ have either a proper point or an end in common or they have a common perpendicular. Euclidean and hyperbolic geometry correspond to the two possibilities: two lines with a common end may or may not have a common perpendicular. If some two lines with a common end do have a common perpendicular then the same applies to any two lines with a common end, furthermore in this, the Euclidean case, any two lines with one common end have also their other ends in common. For the elementary motions A we have then:

The fixpoints of A are:

hyperbolic case	Euclidean case
i) one point P	i) one point P
ii) one point E	ii) two points E
iii) two points E	

As I have pointed out earlier, Euclidean geometry "exists" in the sense that there is a model for it, namely analytic geometry. The group of elementary motions is there the group consisting of all rotations and all translations. Within analytic geometry there are many possible models of hyperbolic geometry. The group of elementary motions will be the same for every model. The simplest model for our purpose is due to Poincaré and it can be formulated in the complex plane (of course this word is misleading, all we do is to write $x + iy$ instead of (x,y)). In this model, points P are complex numbers z with $|z| < 1$ and ends E are complex numbers z with $|z| = 1$. "Lines" are arcs of circles intersecting the boundary circle at right angles. *(Fig. 3)*

Easy problems are now coming thick and fast:

i) the hyperbolic incidence relations are satisfied.

ii) The "reflection" on a "line" can be realized as inversion on the circle carrying the arc.

iii) An elementary motion A has the form $w = \frac{az+b}{\bar{b}z+\bar{a}}$ (a and b are arbitrary complex numbers).

iv) The group of elementary motion is isomorphic to the multiplicative group of the matrices $\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$ with $a\bar{a} - b\bar{b} = 1$.

v) Every elementary motion has a unique representation of the form $\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$ with $a\bar{a} - b\bar{b} = 1$ and $\Re(a) \geqslant 0$.

vi) Putting $\Re(a) = \alpha$, the elementary motion $A = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$ with $\alpha > 0$ belongs to type i), ii) or iii) according to whether $\alpha < 1$, $\alpha = 1$ or $\alpha > 1$.

Problems a bit deeper will appear in the next issue.

Fig. 2.

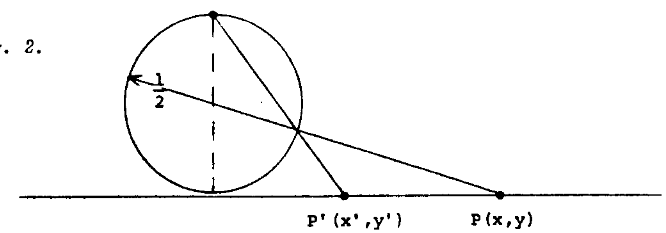
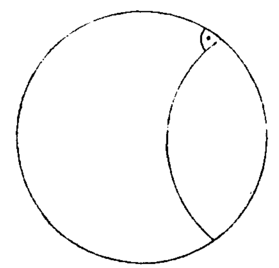


Fig. 3.



COMMUTING MATRICES (JCMN 20, page 60)

Firstly the editor must apologise for a mistake in the original, after "$a_{ij} = 1$ if $i + 1 = j$" there should have been added "$\neq m + 1$.". Use the notation J_m for the $m \times m$ matrix consisting of ones on the super-diagonal and zeros everywhere else, that is with r, s element $= 1$ when $2 \le r + 1 = s \le m$ and with all other elements zero. Let A be the $(m+n) \times (m+n)$ matrix given in block notation by $\begin{pmatrix} J_m & 0 \\ 0 & J_n + c\,I_n \end{pmatrix}$ where $m \ge 2$ and $n \ge 2$.

The problem was to show that if $c \neq 0$ then every B commuting with A must be a polynomial in A, but if $c = 0$ this is not the case. Solutions came in from *H.O. Davies* and *H. Kestelman*, but we shall not print them, for the result is a special case of the more general theorem by *H. Kestelman* printed below, under the heading of "Commuting Matrices II".

COMMUTING MATRICES (II)

H. Kestelman

Let A be a square matrix of complex numbers; it is obvious that $f(A)$ commutes with A if f is a polynomial. In recent issues of JCMN the converse has been mentioned; in what conditions can it be said that every matrix commuting with A is a polynomial in A? There is no difficulty in deciding the question when the $n \times n$ matrix A has n distinct eigenvalues, for then A is similar to a diagonal matrix D and every matrix commuting with D is also diagonal and is a polynomial in D. At the other extreme, with J or J_n denoting the $n \times n$ matrix with ones in the first superdiagonal and zeros elsewhere, every matrix commuting with J is a polynomial in J (see JCMN 18, page 22, and JCMN 19, page 31). The two cases mentioned have this in common: their eigenspaces are all one-dimensional (in the first there are n such eigenspaces and in the second there is just one); it is shown below that this is a necessary and sufficient condition on A to ensure that every matrix commuting with A is a polynomial in A.

1. For any m and n both ≥ 1, $(J_1 = 0)$ suppose that
$$A = \begin{pmatrix} J_m & 0 \\ 0 & J_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0_m & 0 \\ 0 & I_n \end{pmatrix},$$
then if f is a polynomial, $f(A) = \mathrm{diag}\big(f(J_m), f(J_n)\big)$; if this were equal to B it would follow that $f(t)$ would be divisible by t^m (the minimal polynomial of J_m) and consequently that $f(J_n)$ would be singular. But $f(J_n) = I_n$, and this contradiction shows that although $AB = BA$, B is not a polynomial in A.

2. Since the various properties, of one matrix being a polynomial in another, of one commuting with another, and of eigenspaces being one-dimensional, are all unchanged under a similarity transformation, we shall henceforward assume that the $n \times n$ matrix A has the Jordan block form.
$$(1) \quad A = \mathrm{diag}\left(c_1 I_{m_1} + J_{m_1}, c_2 I_{m_2} + J_{m_2}, \ldots, c_j I_{m_j} + J_{m_j}\right)$$
since every A is similar to such a matrix. Here
$$\Pi(t - c_r)^{m_r} = \det(tI - A) \quad \text{and} \quad \sum m_r = n$$
If $\lambda_1, \lambda_2, \ldots, \lambda_q$ are the eigenvalues of A, the nullity of $A - \lambda_s I$, that is the dimension of the eigenspace of A belonging to λ_s, is the number of r for which $c_r = \lambda_s$; this is because the nullity of J is one.

3. Suppose that A has a multidimensional eigenspace, belonging to eigenvalue 0, say. We can then write $A = \mathrm{diag}(J_r, J_s, Q)$, and if we set $B = \mathrm{diag}(0_r, I_s, I)$ we have $AB = BA$, but, as in paragraph 1 above, B is not a polynomial in A.

4. It remains to prove the converse. Suppose that all the c_r in equation (1) are distinct and that $AB = BA$. We have to show that $B = F(A)$ for some polynomial F. The first step is to show that $B = \mathrm{diag}(M_1, M_2, \ldots, M_j)$ for some suitable matrices,

each M_r being of m_r rows and columns. We can write $B = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ with P a square matrix of m_1 rows and columns. If we can prove that $Q = 0$ and $R = 0$ an obvious inductive argument will display B as required. We can assume that $c_1 = 0$ and that $c_2, \ldots, c_j$ are all non-zero. Then there is an invertible H such that $A = \begin{pmatrix} J_{m_1} & 0 \\ 0 & H \end{pmatrix}$.

The equation $AB = BA$ implies that $J_{m_1} Q = QH$ and by induction $J_{m_1}^{k} Q = QH^{k}$. Take $k = m_1$, to get $QH^{k} = 0$. Since H is invertible, $Q = 0$. Similarly $R = 0$.

5. Now it is established that B must be of the form above, $B = \mathrm{diag}(M_1, M_2, \ldots, M_j)$, and for each $r = 1, 2, \ldots j$, the matrix M_r commutes with J_{m_r}. From the result of JCMN 19, page 31, there exist polynomials $g_1, \ldots g_j$ such that $M_r = g_r(J_{m_r})$.

We have to find a polynomial F such that

$$F\left(c_s I_{m_s} + J_{m_s}\right) = g_s\left(J_{m_s}\right) \tag{2}$$

for all $s = 1, 2, \ldots, j$.

Now we proceed by induction. Suppose that, for some k, $1 \le k < j$, there is a polynomial F_k such that equation (2) holds for $s = 1, 2, \ldots, k$ when F is replaced by F_k. Evidently we can take $F_1(t) = g_1(t - c_1)$. Since the c_r are all different, and so $\prod_{r \le k} (t - c_r)^{m_r}$ and $(t - c_{k+1})^{m_{k+1}}$ are prime to one another in the ring of polynomials, there are polynomials ϕ and θ such that

$$F_k(t) + \phi(t) \prod_{r \le k} (t - c_r)^{m_r} = g_{k+1}(t - c_{k+1}) + \phi(t)(t - c_{k+1})^{m_{k+1}}. \tag{3}$$

Now set F_{k+1} to be equal to either side of this equation (3). From F_{k+1} being equal to the left hand side we have (for $s = 1, 2, \ldots, k$)

$$F_{k+1}\left(c_s I_{m_s} + J_{m_s}\right) = F_k\left(c_s I_{m_s} + J_{m_s}\right) + \phi\left(c_s I_{m_s} + J_{m_s}\right) \prod_{r \le k} \left((c_s - c_r) I_{m_s} + J_{m_s}\right)^{m_r}.$$

In the second term on the right, one of the factors (the one for which $r = s$) is zero, and therefore $F_{k+1}\left(c_s I_{m_s} + J_{m_s}\right) = F_k\left(c_s I_{m_s} + J_{m_s}\right) = g_s\left(J_{m_s}\right)$. From the expression for F_{k+1} as the right hand side of (3) it follows that $F_{k+1}\left(c_{k+1} I_{m_{k+1}} + J_{m_{k+1}}\right)$ differs from $g_{k+1}\left(J_{m_{k+1}}\right)$ by a product of some matrix with $J_{m_{k+1}}^{m_{k+1}}$ which is zero. This has established that F_{k+1} has the desired properties. By repetition we ultimately reach F_j, and this may be taken for the F that we have been seeking.

Remark. The condition postulated for A, that all the eigenspaces are one-dimensional, has the alternative forms: $\det(tI - A)$ is the minimal polynomial of A, or $zI - A$ has rank n or $n-1$ for every complex number z.

BINOMIAL IDENTITY NUMBER SIX

Supposing that m and n are positive integers, then

$$\sum_{r=n}^{n+m-1} \binom{n+m-1}{r} x^r (1-x)^{n+m-1-r} = \sum_{r=0}^{m-1} \binom{n+r-1}{r} x^n (1-x)^r,$$

and if $|x| < 1$ these also equal:

$$\sum_{r=n}^{\infty} \binom{m+r-1}{r} x^r (1-x)^m.$$

G. Bode

BINOMIAL IDENTITY NUMBER SEVEN

A.P. Guinand writes to say that one of his students, *Michael Lazure* modified a Pascal triangle by taking the reciprocal of each number and changing alternate signs, and then added each row. Alternate rows, of course, add to zero, but the sum of row 2n is

$$\sum_{r=0}^{2n} (-1)^r / \binom{2n}{r} = \frac{2n+1}{n+1} .$$

More generally, put $S_k^{2n} = \sum_{r=0}^{2n} (-1)^r \binom{2n}{r}^k$. There are simple formulae for $k = 1, 2, 3$, but for the formula $S_3^{2n} = (-1)^n (3n)! (n!)^{-3}$ is there any simple proof? See page 4012

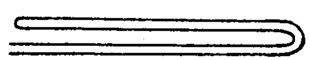
BINOMIAL IDENTITY NUMBER EIGHT

$$\sum_{r=0}^{n} 1/\binom{n}{r} = (n+1) \sum_{r=0}^{n} 2^{-r}/(n-r+1)$$

KEANE SEQUENCES

Alf van der Poorten

Take a sheet of paper and fold it in half, so that the right hand edge comes down on top of the left hand edge. Then do the same, fold the new right hand edge over on top of the left hand edge. The paper is now like this



After n foldings, unfold the paper and you will find $2^n - 1$ creases in it, some are valleys and some are ridges. Assign to valleys the value 1 and to ridges the value 0, and read them from left to right. You will have one of the finite Keane sequences

1
110
1101100
110110011100100

They have the same property as the Thue sequences, that each is an initial segment of the next, and consequently there is an infinite Keane sequences, denote it by $w_1, w_2, \ldots$. An interesting exercise for the student is to find a functional equation connecting $F(z)$ with $F(z^2)$ when $F(z) = \sum w_j z^j$.

PRODUCT OF SINES

There is a well-known trigonmetric identity

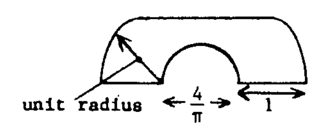
$$\sin n\theta = 2^{n-1} \sin\theta \; \sin(\theta + \pi/n) \ldots \sin\big(\theta + (n-1)\pi/n\big).$$

How many proofs can we uncover?

E.C.G. Sudarshan

GEOMETRIC INEQUALITY (JCMN 19, Vol. 2, page 32)

This question from *M.J.C. and B.J.W. Baker* was about carrying a horizontal table round a right-angled bend in a corridor of unit width. What is the largest area of table for which this is possible? The proposers point out that $\pi/2 + 2/\pi = 2{\cdot}207$ can be attained by this shape of table. It has quarter-circles of unit radius at each end and a semicircular bite of radius $2/\pi$ taken out of one side. So far nobody has produced any improvement on this area.



A related problem has been suggested by *G.P. Henderson* in *Crux Math-*

ematicorum Vol. 5, No. 3, March 1979, page 77. The table is restricted to being rectangular but the two bits of corridor are of unequal widths, a and b. Both these problems may be made a little more difficult by making the angle of the corner in the corridor either bigger or smaller than a right angle.

MATRIX NUMBER THEORY (JCMN Vol. 2, pages 39 and 56)

E.S. Barnes

In the earlier contributions two conjectures were made.

Conjecture 1. Given any integer vector of integer length there exists another of the same length orthogonal to the first.

Conjecture 2. In three dimensions if two orthogonal integer vectors have the same integer length m, then their vector product has every component divisible by m.

Both conjectures are true. Their truth follows easily from results of Marshall Hall [1], which are based on work of Hall and Ryser [2].

<u>Theorem 1.</u> Suppose that X is an integral $r \times n$ matrix $(0 < r < n)$ satisfying $XX^T = m^2 I_r$ (m integral). Then there exists a <u>rational</u> $n \times n$ matrix A satisfying $AA^T = m^2 I$, with X as its first r rows.

Hall then considers integral completion and proves:

<u>Theorem 2.</u> In Theorem 1, there exists an integral A satisfying the given conditions, provided that $n - r = 1$ or 2, or that $r = 1$ and $n = 4$.

The case $r = 1$, $n = 3$ of Theorem 2 shows immediately that if $a_1^2 + a_2^2 + a_3^2 = m^2$ (all integers), then there exists an integral A

with $AA^T = m^2 I$ and (a_1, a_2, a_3) as its first row. Thus Conjecture 1 is true; indeed there are two linearly independent integral vectors of length m orthogonal to (a_1, a_2, a_3).

For Conjecture 2, we can use a slightly stronger version of Theorem 2, namely: <u>if</u> $n - r = 1$, <u>then any rational completion A is necessarily integral</u>.

<u>Proof.</u> Since $AA^T = m^2 I$, it follows that $A^T A = m^2 I$. Hence, for $j = 1, \ldots, n$, $\sum_{i=1}^{n} a_{ij}^2 = m^2$ and so $a_{nj}^2 = m^2 - \sum_{i=1}^{n-1} a_{ij}^2$ is integral. But a_{nj} is rational, and so it is integral.

Let now $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be orthogonal integral vectors of length m. Then we have a rational completion of $X = \begin{bmatrix} \underset{\sim}{a}^T \\ \underset{\sim}{b}^T \end{bmatrix}$ by $\frac{1}{m}(\underset{\sim}{a} \times \underset{\sim}{b})^T$. The last result now shows that $\frac{1}{m}(\underset{\sim}{a} \times \underset{\sim}{b})$ is integral.

[1] Marshall Hall, Integral matrices A for which $AA^T = mI$, <u>Number Theory and Algebra</u> (Ed. H. Zassenhaus, Academic Press, 1977), 119-134.

[2] Marshall Hall and H.J. Ryser, Normal completions of incidence matrices, <u>Amer. J. Math.</u> 76 (1954), 581-589.

MURPHY'S EXPRESSION FOR A LEGENDRE POLYNOMIAL

The formula $P_n(z) = F(n+1, -n; 1; (1-x)/2)$ was mentioned by *C.S. Davis* in our last issue (page 49) as being a little esoteric. However if we confine ourselves to positive integer n, as Murphy did in his 1833 book Electricity, then there is a simple proof suggested by *A. van der Poorten's* contribution (page 48).

First note the following binomial identities which will be needed:

$(-4)^r \binom{-1/2}{r} = \binom{2r}{r}$, $(-1)^r \binom{-n}{r} = \binom{n+r-1}{r}$, and $\binom{x}{b}\binom{b}{c} = \binom{x}{c}\binom{x-c}{b-c}$.

The Legendre polynomials are defined by their generating function:

$$\sum_0^\infty P_n(1-2z)t^n = (1-2t+4tz+t^2)^{-\frac{1}{2}} = \left((1-t)^2+4tz\right)^{-\frac{1}{2}}$$

$$= (1-t)^{-1}\left(1+4tz/(1-t)^2\right)^{-\frac{1}{2}}$$

$$= (1-t)^{-1}\sum_0^\infty \binom{-\frac{1}{2}}{k}(4tz)^k(1-t)^{-2k}$$

$$= \sum_0^\infty (-tz)^k \binom{2k}{k} (1-t)^{-2k-1}$$

$$= \sum_0^\infty \sum_0^\infty \binom{2k}{k} \binom{-2k-1}{r} (-t)^{k+r} z^k$$

Now equating coefficients of t^n and noting that

$$(-1)^r \binom{-2k-1}{r} = \binom{2k+r}{r} = \binom{2k+r}{2k},$$

$$P_n(1-2z) = \sum_{k=0}^n \binom{2k}{k}\binom{n+k}{2k} (-z)^k$$

$$= \sum_0^n \binom{n+k}{k}\binom{n}{k} (-z)^k$$

$$= \sum \frac{(n+1)(n+2)\dots(n+k)}{k!}\frac{(-n)(-n+1)\dots(-n+k-1)}{k!} z^k$$

Here the sum may be taken from 0 to n or from 0 to infinity because the terms from $n+1$ onwards are all zero. This expression is the hypergeometric function $F(n+1, -n; 1; z)$.

HERMITEAN MATRICES (JCMN 19, page 36)

Let A and B be complex square matrices, $m \times m$ and $n \times n$ respectively. A is hermitean ($A^* = A$) and B is skew-hermitean ($B^* = -B$). There is a non-zero $m \times n$ matrix X such that $AX = XB$. Prove that A and B are both singular.

Solution from *E.C.G. Sudarshan*:

Put $X^*X = Y$, and then $YB = X^*XB = X^*AX = (AX)^*X = (XB)^*X = -BX^*X = -BY$. Also note that all the eigenvalues of Y are non-negative, because if $Yu = \lambda u$ then $\lambda\|u\|^2 = u^*\lambda u = u^*Yu = u^*X^*Xu = \|Xu\|^2$.

If B were non-singular then we would also have $B^{-1}Y = -YB^{-1}$ and $\lambda B^{-1}u = B^{-1}\lambda u = B^{-1}Yu = -YB^{-1}u$, so that $-\lambda$ would also have to be an eigenvalue, which would imply that all the eigenvalues of Y are zero, and therefore Y (being Hermitian symmetric) would have to be identically zero. In particular each diagonal element of Y ($= X^*X$) would be zero and so each column of X would be zero. The contradiction implies that B must be singular.

Because $AY = AX^*X = (XA)^*X = (BX)^*X = -X^*BX = -X^*XA = -YA$ it follows from a similar argument that A must be singular.

Solution from *B.B. Newman*:

The eigenvalues of A are real. The eigenvalues of B are purely imaginary or zero. The equation $AX = XB$, $X \neq 0$ implies that A and B have a common eigenvalue (Theorem 5.17, Matrices and Linear Transformation - Cullen.). Hence A and B both have a zero eigenvalue, so that both are singular.

For those that do not have Cullen's book handy, the theorem may be proved as follows. Both A and B have full sets of eigenvectors, and so there exist eigenvectors $\underline{u}^*$ and $\underline{v}$ so that $\underline{u}^*X\underline{v} \neq 0$ and $\underline{u}^*A = \lambda\underline{u}^*$ and $B\underline{v} = \mu\underline{v}$. Therefore $(\lambda - \mu)\underline{u}^*X\underline{v} = \underline{u}^*AX\underline{v} - \underline{u}^*XB\underline{v} = 0$.

Solution from *H. Kestelman*:

Let $k_1, k_2, \ldots k_q$ be the eigenvalues of B, then the minimal polynomial f of B has $f(t)$ equal to a product of powers of $(t - k_r)$, and so $f(A)$ is a product of powers of $(A - k_r I)$. But $AX = XB$ and $A^j X = XB^j$ (for all $j = 1, 2, \ldots$) and $f(A)X = Xf(B) = 0$ and so $f(A)$ is singular. Therefore, for some r, $A - k_r I$ is singular and k_r is an eigenvalue of A, and since A is hermitean, k_r is real; but iB is also Hermitean and so $i k_r$ is real, therefore $k_r = 0$. The matrices A and B both have a zero eigenvalue and so are singular.

Editorial note. We are open minded about whether to call a matrix hermitean or hermitian, but *H. Kestelman* who sent in the problem favours the first, writing that although in a minority he follows Hermann Weyl. And there is the question of whether to use a capital H or small h; we try to give both sides an innings by being consistently inconsistent.

QUOTATION CORNER (4)

Lovely lace and silk shirt open to the waste, hair looking a little blow-waved. —— A sentence from an article about a "rock star" in the Week-End Australian Magazine, July 2-3, 1977, page 7.

An examination candidate in my class mentioned the "Principal of Inductance". Is this the mysterious being otherwise known as "Lord of the Rings"?

Your editor would like to hear from you anything connected with mathematics or with James Cook.

Prof. B.C. Rennie,
Mathematics Department,
James Cook University of North Queensland,
Post Office,
James Cook University, Q. 4811,
Australia.