JAMES COOK MATHEMATICAL NOTES

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Capt. James Cook's Coat-of-Arms
Bestowed posthumously by King George III in Sept. 1785

THE DREADED ZETA THREE AGAIN

A. van der Poorten

Show that $10 \int_0^{\log((1+\sqrt{5})/2)} t^2 \coth t \, dt = \sum_1^\infty 1/n^3$.

Also $(288/17) \int_0^{\pi/6} t\big(\log(2 \sin t)\big)^2 dt = \sum_1^\infty 1/n^4$.

ANOTHER BINOMIAL IDENTITY (JCMN 19, p.29)

B.B. Neuman

The problem was to show that if $0 < p < 1$ and k is a positive integer then

$$\lim_{N\to\infty} \sum_{i=0}^{[N/k]} \binom{N}{ki} p^{ik}(1-p)^{N-ik} = 1/k .$$

To simplify notation let us agree that r, s and t, as free or dummy variables, take values $0, 1, \ldots k-1$, that m is a non-negative integer variable, that all limits are as $N \to \infty$ and that all congruences are modulo k. Let u be a primitive kth root of unity.

Put $\quad A(t, N) = \sum\limits_{m \equiv t} \binom{N}{m} p^m (1-p)^{N-m}$, $\quad$ then

$$\sum_t u^{ts} A(t, N) = \sum_t \sum_{m\equiv t} \binom{N}{m} (u^s p)^m (1-p)^{N-m}$$

$$= (1 - p + u^s p)^N$$

which $= 1$ if $s = 0$ and which $\to 0$ if $s \neq 0$.

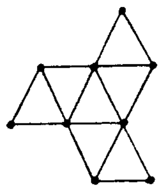
Therefore $\sum\limits_s u^{-rs} \sum\limits_t u^{ts} A(t, N) \to 1$, but also this expression $=$ $\sum\limits_t A(t, N)(1 + u^{t-r} + u^{2t-2r} + \ldots) = k\,A(r, N)$. This shows that $A(r, N) \to 1/k$, a slightly more general result than the one suggested which was the case $r = 0$.

A ROSE BY ANY OTHER NAME

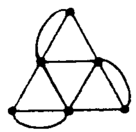
The computer of the North Queensland Electricity Board, formerly called *FRED* (flaming ridiculous electronic device) is to be known as *SOLITUDE* (simultaneous on-line interpretative terminals for user data entry-enquiry).

GRAPHS FOR GROUPS (JCMN 18, p.10)

What are the smallest graphs with automorphism group C_3? *G. Szekeres* writes that *Harary's* book "Graph Theory" (page 170) gives



which has 9 nodes and 15 edges. A smaller graph may be found by those with no prejudice against multiple edges,



which has 6 nodes and 12 edges.

ANOTHER EXPANSION PROBLEM (JCMN 18, p.16)

Alf van der Poorten

Note that $(1 - 6x + x^2)^{-1/2} = \left(1 - 4x(1-x)^{-2}\right)^{-1/2}/(1-x)$

$$= \sum (-4)^n \binom{-1/2}{n} x^n (1-x)^{-2n-1}$$

Because $(-4)^n \binom{-1/2}{n} = \binom{2n}{n}$ it follows that the coefficients b_n of the power series are all integers. A more challenging task would be to show that the recursion

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5)u_{n-1} \quad \text{with } u_o = 1$$

and $u_1 = 5$ gives rise to a sequence of integers.

ANOTHER EXPANSION PROBLEM (JCMN 18, p.16)

C.S. Davis

The question from *A. van der Poorten* was to show that if $(1 - 6x + x^2)^{-1/2} = \sum b_n x^n$ then the b_n are all integers and $b_n = \sum_{k=o}^{n} \binom{n}{k}\binom{n+k}{k}$ and $n b_n + (n-1)b_{n-2} = (6n-3)b_{n-1}$.

Because b_n is the Legendre polynomial $P_n(3)$ the recurrence follows from the relation (Whittaker and Watson, page 308)

$$(n+1)P_{n+1}(z) - (2n+1)z\, P_n(z) + n\, P_{n-1}(z) = 0.$$

Also (W. and W. page 311) Murphy's expression for Legendre polynomials as hypergeometric functions is

$$P_n(z) = F(n+1, -n\ ;\ 1\ ;\ (1-z)/2),$$

so that $\quad b_n = F(n+1, -n\ ;\ 1\ ;\ -1)$

$$= \sum \frac{(n+1)(n+2)\ldots(n+k)}{k!} \frac{n(n-1)\ldots(n-k+1)}{k!} = \sum \binom{n+k}{k}\binom{n}{k}.$$

Since Murphy's expression is perhaps a little esoteric, it may be worth giving a direct proof from the formula of Rodrigues:

$$P_n(1-2t) = (1/n!)(d/dt)^n (t - t^2)^n = (d/dt)^n \sum_o^n (-1)^k \binom{n}{k} t^{n+k}/n!$$

$$= \sum (-1)^k \binom{n}{k}\binom{n+k}{k} t^k,$$

and putting $t = -1$ gives the expression for $P_n(3) = b_n$.

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It is interesting to consider what can be found about b_n without appealing to the theory of Legendre polynomials, (at all events, explicitly). Writing $\sum_0^\infty b_n x^n = f(x) = (1-6x+x^2)^{-1/2} = (p(x))^{-1/2}$, we have $p f' + \frac{1}{2} p' f = 0$; equating coefficients of x^{n-1} then gives

$$n b_n + (n-1)b_{n-2} = (6n-3)b_{n-1}$$

Note that $\binom{n}{k}\binom{n+k}{k} = \dfrac{(n+k)!}{k!(n-k)!k!} = \binom{n+k}{2k}\binom{2k}{k} = (-4)^k \binom{n+k}{2k}\binom{-1/2}{k}$.

We do not seem able to get away from the identity $\binom{2k}{k} = (-4)^k \binom{-1/2}{k}$ do we?

Now take a sufficiently small positive x (in fact $0 < x < 3-2\sqrt{2}$ is sufficient). We try to evaluate $S = \sum\sum \binom{n}{k}\binom{n+k}{k} x^n$ with summation over $0 \le k \le n < \infty$. Add on the terms (all zero) for which $-k \le n < k$, and change the notation by putting $m = n+k$, then the sum is for $m \ge 0$, $k \ge 0$. $S = \sum\sum \binom{m}{2k}\binom{-1/2}{k}(-4)^k x^{m-k}$ which is the constant term in the Laurent expansion of $F(t) = \sum x^m(1+1/t)^m(1-4t^2/x)^{-1/2} = (1-x-x/t)^{-1}(1-4t^2/x)^{-1/2}$ in the region of convergence which is between two circles in the t-plane, in fact where $|t - x^2/(1-x^2)| > x/(1-x^2)$ and $|t| < \frac{1}{2}\sqrt{x}$. The constant term is the residue of the function $F(t)/t = (t-tx-x)^{-1}(1-4t^2/x)^{-1/2}$ at the simple pole where $t = x/(1-x)$. Therefore

$$S = \text{Residue} = (1-x)^{-1}(1-4t^2/x)^{-1/2} = (1-6x+x^2)^{-1/2}.$$

This shows that b_n has the given expression as a sum of products of binomial coefficients, and is an integer.

QUOTATION CORNER (2)

Most of the young are not backward, but merely remedial. — A young social worker reported in the Townsville Daily Bulletin on 4th July 1979.

A PARTY GAME (JCMN 18, page 8)

The players in turn add A, B or C to a sequence, and any one completing a repeated segment of length two or more loses a point.

Letters from *V. Klee* of the University of Washington and *P.J. Campbell* of Beloit College both point out that the game can go on for ever with nobody losing a point. It is not clear where is or what was the first proof, but the following references may help those interested.

T.C. Brown, Is there a sequence on four symbols in which no two adjacent segments are permutations of one another? Amer. Math. Monthly, 78, 1971, pp. 886-888.
Axel Thue, Uber unendliche Zeichenreihen, Norske Vid. Skr. Mat.-Nat.Kl., Christiania, 1906, No. 7, pp. 1-22.
Martin Gardner, The Incredible Dr. Matrix, Scribner's, 1976 pp. 198-200.

Gardner notes the application to drawn games in chess, which was first noticed by the mathematician and World Chess Champion, Max Euwe.

The finite Thue sequences on two symbols are as follows

01
01, 10
01, 10, 10, 01

etc. Each is obtained from the one before by the rule of putting 01 for 0 and 10 for 1. Each finite Thue sequence contains the one before as its first half, therefore the limit, the infinite Thue sequence, is well defined. The simplest theorem on the subject is that no segment (of length one or more) repeats three times consecutively in the infinite Thue sequence defined above. For a proof, suppose that a triple segment were to appear, then it would be in one of the finite sequences. It can be shown that there would have to be a triple segment in the previous finite Thue sequence. The principle of induction would then lead to a contradiction.

Thue sequences on three symbols are a little more complicated but use the same ideas.
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The transformation taking A to ABC, B to AC, and C to B, leads to the following finite Thue sequences

ABC

ABCACB

ABCACBABCBAC

ABCACBABCBACABCACBACABCB etc.

and as before these give an infinite Thue sequence.

<u>Theorem</u> The infinite Thue sequence on three symbols does not contain any segment (of length one or more) followed by a copy of itself.

Proof. The theorem will follow by induction if we can show that the presence of a repeated segment in one of the finite Thue sequences implies the existence of a repeated segment in the preceding finite sequence. This can be done in eight steps as follows

(i) There cannot be AA or BB or CC or ABA or CBC.

(ii) If a segment x occurs as the image (under the transformation) of a segment y, then it cannot occur as the image of $z \neq y$.

Now we eliminate in turn the six possibilities of a repeated segment, according to the six possible combinations of first and last letter of the segment. (The first and last letter must differ, by (i)).

(iii) There cannot be B x C repeated (where x is a segment, possibly empty). The possibility of x being empty is eliminated by (i). The B of the beginning of the second occurrence of the segment must be the image of a C in the preceding sequence, this C must be followed by A or B and so x must start with A. Therefore xC = A ... C is the image of a segment y, and the preceding sequence contains Cy Cy.

(iv) There cannot be C x A repeated. The AC from the junction of the two segments must be the image of a B in the preceding sequence. The situation may be represented diagrammatically as follows, the preceding sequence above, and the vertical bars showing where one segment is the image of another

?	u	B	u	?
C	x	AC	x	A

The ? symbol in both places must be A or B, but neither can be B because that would give a repeated segment in the preceding sequence. We therefore have to consider if A u B u A is possible. If such a segment were to occur then u would be non-empty by (i), and also u cannot start or end with A or B, therefore u is either C alone or C y C and the segment u B u contains C B C, but this is impossible by (i).

(v) There cannot be A x B repeated. By (i) x cannot be empty and cannot end in A. Then Ax = A ... C must be the image of a segment u and the preceding sequence contains u C u C.

(vi) There cannot be A x C repeated because A x C has to be the image of a segment.

(vii) There cannot be C x B repeated, for by (i) x would have to end in A, and the repeated segment would have to be C y A B C y A B which must be followed by C. This is impossible because y A B C has to be the image of a segment.

(viii) There cannot be B x A repeated because by (i) x would have to start with C and the repeated segment would have to be preceded by A, the argument going as in case (vii) above.

This completes the proof.

P. Campbell writes that the method of Morse and Hedlund (Unending Chess, Symbolic Dynamics and a Problem in Semigroups, Duke Math. Journal 11 (1944) 1-7) is to start with the Thue sequence on two symbols and put A where there is 01, B where there is 00 or 11, and C where there is 10, as follows

0 1 1 0 1 0 0 1 1 0 0 1 0

A B C A C B A B C B A C

Can anyone explain why this should give the same 3-symbol sequence as the other method?

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ANALYTIC INEQUALITY (JCMN 19, page 29)

The problem was, given $f(0,0) = 0$, $f(0,1) = f(1,0) = 2$ and $f(1,1) = 5$, to find the largest M such that (somewhere in the square) $(\partial f/\partial x)^2 + (\partial f/\partial y)^2 \geq M$.

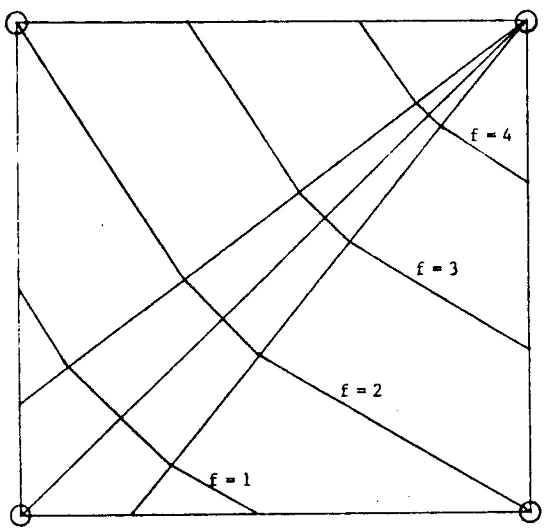
The answer is 25/2. This is clearly a possible value because the mean gradient of the function on the diagonal from (0, 0) to (1, 1) is $5/\sqrt{2}$. To show that $M = 25/2$ is the best (largest) possible value, one way is to observe the piecewise linear function

$$f = 5(x+y)/2 \quad \text{where } (5+\sqrt{14})/11 < (1-y)/(1-x) < 5-\sqrt{14}$$

$$f = 3y + 2 + (x-1)\sqrt{(7/2)} \quad \text{where } (1-y)/(1-x) > 5-\sqrt{14}$$

$$f = 3x + 2 + (y-1)\sqrt{(7/2)} \quad \text{where } (1-x)/(1-y) > 5-\sqrt{14}$$

(The contour lines are sketched below.) This function has the same (scalar) gradient everywhere in the square.



The restriction to differentiable functions in the original problem has no effect on what values of M are permissible.

Another way of showing that this M is the best possible is from the

following proposition.

Let f be a real function on a set F in a metric space, with a Lipschitz condition $|f(x) - f(y)| \leq k\, d(x, y)$, where $d(x, y)$ is the distance between x and y; then, given any set E containing F, there is a function g extending f, defined on E, and satisfying a similar Lipschitz condition with the same constant k. This theorem is a kind of non-linear variation of the Hahn Banach theorem, and it should be of interest to logicians and civil engineers. For the former, here is a problem, does the proposition above imply the axiom of choice?

The interest to civil engineers arises from the fact that the upper surface of the sand or silt on the bed of the sea may be confidently supposed not to have a large gradient, though we can assume nothing about the curvature. Neither practical experience nor the theory of static equilibrium of sand rules out the possibility of a contour line having a sharp corner. Therefore when we try to interpolate contours from a finite set of soundings, we need be concerned only with gradients. There is no obvious best method of interpolating, but it is reasonable that we should in some sense minimize gradients. The theorem above says that we can interpolate depths over the whole region without ascribing to the sand anywhere a gradient greater than that which is made necessary by the data, that is the largest of the ratios (difference of two soundings)/(distance between points). See the contribution "Algorithms Wanted" in JCMN 18, (vol. 2, page 8).

GEOMETRY IS ALGEBRA IS GEOMETRY IS ... (JCMN 19, Vol. 2, p. 27)

The conjecture was put forward in our last issue that (in three dimensions) if the four planes through homologous triads of vertices of three tetrahedra meet in a point, then the four points of intersection of homologous triads of faces are coplanar. This was a pretty idea but sadly not true. Below is given a counter-example, tetrahedra A, B and C, each with vertices numbered 1, 2, 3, 4, and with each face numbered like the opposite vertex.

Tetrahedra:		A	B	C	Planes of three vertices
Vertices:	1	(1,0,0,0)	(0,2,2,1)	(2,1,1,1)	(0, 1,-1,0)
	2	(0,1,0,0)	(1,2,1,1)	(2,0,2,1)	(1, 0,-1,0)
	3	(0,0,1,0)	(1,1,2,1)	(1,1,0,1)	(1,-1, 0,0)
	4	(1,1,1,1)	(2,0,1,1)	(1,3,2,1)	(1, 1,-2,0)

					Intersection of faces
Faces	1	(1,0,0,-1)	(2,1,1,-5)	(3,1,-1,-4)	(1, 2, 1,1)
	2	(0,1,0,-1)	(1,1,0,-2)	(1,1,-1,-2)	(1, 1, 0,1)
	3	(0,0,1,-1)	(2,1,2,-6)	(3,1, 1,-8)	(3,-2, 1,1)
	4	(0,0,0, 1)	(1,1,1,-4)	(-1,1, 1, 0)	(0, 1,-1,0)

The four planes that each contain a homologous triad of vertices A_i, B_i and C_i for $i = 1, 2, 3, 4$, all meet in (0, 0, 0, 1), but the four points where homologous triads of faces meet are not coplanar.

MATRIX NUMBER THEORY (JCMN 19, Vol. 2, p.39)

Sholander [1] has proved that the most general proper 3 × 3 orthogonal matrix with rational elements is

$$A = \frac{1}{a^2+b^2+c^2+d^2}\begin{bmatrix} a^2+b^2-c^2-d^2 & 2(ac+bd) & 2(ad-bc) \\ 2(ac-bd) & b^2+c^2-a^2-d^2 & 2(ab+cd) \\ 2(ad+bc) & 2(cd-ab) & b^2+d^2-a^2-c^2 \end{bmatrix}$$

with a, b, c, d integers. With the special choice $a = 3$, $b = 0$, $c = d = 2$.

$$A = \frac{1}{17}\begin{bmatrix} 1 & 12 & 12 \\ 12 & -9 & 8 \\ 12 & 8 & -9 \end{bmatrix},$$

so that

$$M = \begin{bmatrix} 12 & 12 & 1 \\ 8 & -9 & 12 \\ -9 & 8 & 12 \end{bmatrix}$$

satisfies $MM^T = 17^2 I$, contrary to the assertion made in the note. The matrix A, being symmetric, is an example of a square root of the identity matrix.

[1] M. Sholander, Rational orthogonal matrices, Am. Math. Monthly, 63, (1961), 350.

E.S. Barnes, R.B. Potts.

The editor apologises for his carelessness in the last issue. The suggestion there can now be revived as follows.

Conjecture 1. Given any integer vector of integer length, there exists another of the same length orthogonal to the first. In other words if $a^2 + b^2 + c^2 = m^2$ (all integers) there exist integers x, y and z such that $ax + by + cz = 0$ and $x^2 + y^2 + z^2 = m^2$. The three-dimensional result implies that in any other number of dimensions.

Conjecture 2. In three dimensions if two orthogonal integer vectors have the same integer length m, then their vector product has every component divisible by m.

There has been some interest in symmetric orthogonal matrices. (R.B. Potts, Symmetric square roots of the finite identity matrix, Utilitas Mathematica, 9, 1976, 73-86.) The unique proper symmetric orthogonal matrix with top row a/m, b/m, c/m, is

$$\frac{1}{m}\begin{pmatrix} a & b & c \\ b & \frac{b^2}{m+a} - m & \frac{bc}{m+a} \\ c & \frac{bc}{m+a} & \frac{c^2}{m+a} - m \end{pmatrix}$$

The improper one may be found by changing the signs of the whole matrix above and of m. This formula is of no help with Conjecture 1 above, for example $39^2 = 34^2 + 14^2 + 13^2$ but there is no way of choosing a, b and c as a permutation of 34, 14 and 13 to make

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$(m \pm a)$ a factor of b^2 and c^2. Both conjectures are satisfied in this case because (19, -22, -26) is orthogonal to (34, 14, 13) and their vector product is 39(2, -29, 26).

SUMS OF RESIDUE CLASSES

Given $2n - 1$ residue classes mod n, can you choose n of them with sum congruent to zero? This is an old Erdős problem, and is on the tough side.

G. Szekeres

NATURAL PHENOMENON (JCMN 19, page 29)

Why was the strand of barbed wire in the old fence vibrating slowly up and down on that cold calm January morning near Blanefield? The question puzzled me until Ann Evans told me the following answer. There would have been drops of water formed by melting of snow in the sun, and they would fall off the points of the barbed wire. The first drop to fall would start a small vertical oscillation of the wire, then the drops would tend to fall off when the wire was of the lowest point of its oscillation, thereby putting more energy into the oscillation.

Does the idea lead to a good question in elementary mechanics? A bucket hangs from a length of elastic, rain falls steadily into it, and water drips from small holes in the bottom at an average rate equal to that of the rain coming in. Assuming that the drops are small, that the interval between successive drips from one hole is large compared with the period of free vertical oscillation of the bucket, and that friction is negligible, show that the semi-amplitude of vertical oscillation of the bucket is $L(1 - \exp - t/T)$ where L is the length of a simple pendulum with frequency the same as that of vertical oscillations of the bucket, and T is the time taken for the bucket to collect an amount of rain equal to the average mass of the bucket and water.

SOME MATRIX POLYNOMIAL QUESTIONS (JCMN 18, p.22 and 19, p.31)

The question was asked whether if B commutes with the non-degenerate matrix A, then B must be a polynomial in A. Some readers have expressed doubt about the meaning of the word "degenerate", your editor takes it to mean "having two eigenvalues equal" but has not been able to find an authoritative book to support him, the nearest is G. Arfken's "Mathematical Methods for Physicists" where the word is used in this sense to describe self-adjoint operators in Hilbert space.

We might re-cast the problem as follows. What conditions on a square matrix A are sufficient to ensure that every matrix B commuting with A (that is AB = BA) is a polynomial in A? Is it sufficient that the eigenvalues of A be all unequal? Note that $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has the property that every matrix B commuting with A is a polynomial in A, in fact $B = b_{11} I + b_{12} A$, but A is degenerate, having repeated eigenvalue zero.

In the case where A is diagonal with unequal eigenvalues we can answer the question whether every matrix commuting with A is a polynomial in A. For in fact AB = BA will imply that B is diagonal and vice versa, so that the set of all matrices commuting with A is precisely the set of all diagonal matrices, which is the set of all polynomials in A. On the other hand if A is diagonal with a pair of equal eigenvalues then it does not have the property, for we can find a diagonal B not a polynomial in A.

The result can be extended from diagonal A to any A that can be made diagonal by a similarity transformation $A = T^{-1} D T$ (where D is diagonal and T has an inverse). There remains the question of non-diagonalizable matrices. See the contribution "Commuting Matrices" on the next page.

COMMUTING MATRICES

H. Kestelman

Take integers m and n both ≥ 2, define the $(m+n) \times (m+n)$ matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & & & \\ 0 & 0 & 1 & \cdots & & & \\ \cdot & \cdot & \cdot & \cdots & c & 1 & 0 \\ & & & & 0 & c & 1 \\ & & & & 0 & 0 & c \end{pmatrix}$$

In other words

$$a_{ij} = 1 \text{ if } i+1 = j \neq m+1$$
$$= c \text{ if } m < i = j$$
$$= 0 \text{ otherwise.}$$

Show that if $c \neq 0$ then every matrix B commuting with A must be a polynomial in A, but that if $c = 0$ this is not the case.

BASIC CALCULUS (JCMN 19, Vol. 2, p. 32)

Does $n\,f(x/n) \to x$ imply that $f(x)/x \to 1$? The answer in general is NO. Set $g(x) = f(x)/x$. If we know that (for each fixed x) $g(x/n) \to 1$, then does it follow that $g(x) \to 1$ as $x \to 0$? Take a rationally independent sequence $a_1, a_2, \ldots$ in the interval $(1, 2)$ (so that no member of the sequence is a rational multiple of any other). For instance the powers of e or π reduced (mod 1) to the interval $(1, 2)$ will do. Define $g(x) = 1 + kx$ for x any rational multiple of a_k, and $g(x) = 1$ for any other x. Then for $x = r\,a_k$ with r rational, $g(x/n) = 1 + kx/n \to 1$ as $n \to \infty$, but $g(a_k/k) = 1 + a_k > 2$. Since $a_k/k \to 0$ as $k \to \infty$ it follows that $g(x)$ does not tend to any limit as the real variable x tends to zero. I am sure that with some non-elementary effort one could even construct a continuous (though of course non-rectifiable) $f(x) = x\,g(x)$ giving a counter example.

G. Szekeres

A similar answer has come in from *H. Kestelman*.

RECYCLING CONTAINERS

R.B. Potts and J. van der Hoek

A crushed beer can of varying cross-section (but with no holes) is gradually filled with a non-frothy liquid. When is the centre of mass of can and liquid at its lowest point?

INTEGRAL CALCULUS (JCMN 18, page 13)

J.B. Parker

The problem was to show that

$$\frac{1}{2\pi}\int_0^\pi\int_0^\pi \frac{1-\cos mx\cos my}{2-\cos x-\cos y}\,dx\,dy = 1 + \frac{1}{3} + \ldots + \frac{1}{2m-1}\,.$$

We must consider

$$J = \frac{1}{2\pi}\int_0^\pi\int_0^\pi \frac{\cos(m-1)x\cos(m-1)y - \cos mx\cos my}{2-\cos x-\cos y}\,dx\,dy\,.$$

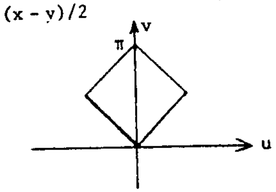
First notice that putting $\pi - x$ for x and $\pi - y$ for y leaves the integral unchanged except for replacing each minus sign by plus in the bottom line of the integrand. Also

$$\frac{1}{2-\cos x-\cos y} + \frac{1}{2+\cos x+\cos y} = \frac{4}{4-(\cos x+\cos y)^2} = \frac{1}{1-\cos^2\frac{x+y}{2}\cos^2\frac{x-y}{2}}$$

With a trigonometric substitution in the top line this leads to

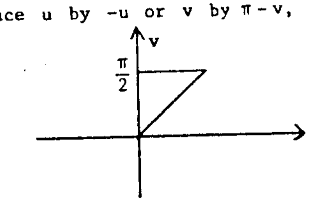
$$8\pi J = \int_0^\pi\int_0^\pi \frac{\cos(m-1)(x+y) + \cos(m-1)(x-y) - \cos m(x+y) - \cos m(x-y)}{1-\cos^2(x+y)/2\,\cos^2(x-y)/2}\,dx\,dy$$

Now change the variables by $2u = x - y$ and $2v = x + y$, the Jacobian is 2, and the region of integration becomes as shown.

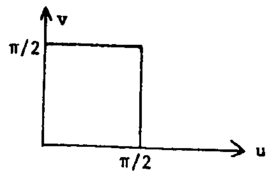


$$2\pi J = \iint \frac{\sin(2m-1)u\,\sin u + \sin(2m-1)v\,\sin v}{1-\cos^2 u\,\cos^2 v}\,du\,dv$$

The integrand is unaltered if we replace u by $-u$ or v by $\pi - v$, and therefore by putting in a factor 4 we may change the region of integration to the triangle where $0 < u < v < \pi/2$.



JCMN20.



Next, as the integrand is symmetrical between u and v we may take the integral over the square where $0 < u,\ v < \pi/2$ if we put in a factor of 1/2.
Then, again using the symmetry, we may omit one of the two terms and put in a factor of 2. And finally we come to

$$(\pi/2)\, J = \int_0^{\pi/2}\int_0^{\pi/2} \frac{\sin(2m-1)u\ \sin u}{1 - \cos^2 u\ \cos^2 v}\, du\, dv.$$

But
$$\int_0^{\pi/2} \frac{dv}{1-\cos^2 u\ \cos^2 v}$$

$$= \int_0^{\pi/2} \frac{\sec^2 v\ dv}{1+\tan^2 v - \cos^2 u} = \int_0^{\infty} \frac{dt}{\sin^2 u + t^2} = \frac{\pi}{2\sin u}$$

$$J = \int_0^{\pi/2} \sin(2m-1)u\ du = 1/(2m-1)$$

MISTAKES IN OUR LAST ISSUE

On page 31, line 3, in the list giving the known Mersenne primes, the number at the end of the line should be 44497, not 4497.

On page 35, line 3, the equation $\beta^2 = \beta + 2$ should be amended to $\beta^3 = \beta + 2$.

On page 39, line 6, the rational orthogonal matrix should be described as $n \times n$, not $n \times m$.

The mistaken assertion on page 39 about the integer vector (12, 12, 1) has been corrected in the note (page *56*) by *E.S. Barnes* and *R.B. Potts*, and *G. Szekeres* also gave the vector (-8, 9, -12).

Mistakes are being corrected for the reprinting in Volume 2.

ANOTHER BINOMIAL IDENTITY (JCMN 19, p. 29)

V. Laohakosol gives a solution rather like that of *B.B. Newman* above, (page *47*) and comments that the case $p = \frac{1}{2}$ is closely related to a result (problem 7) given in J. Riordan's book "An Introduction to Combinatorial Analysis", pages 40-41.

ANOTHER EXPANSION PROBLEM (JCMN 18, p.16 and 20, p.49)

A.P. Guinand

This problem of *A. van der Poorten* was about the coefficients in the expansion $(1 - 6x + x^2)^{-\frac{1}{2}} = \sum b_n x^n$. They are the Legendre polynomials $P_n(3)$ for which the recurrence relation is well known. The expression $b_n = \sum_0^n \binom{n}{k}\binom{n+k}{k}$ is obtained from equation (3·135) in Gould's book (reference below). Last week I programmed my HP97 to calculate the coefficients and I attach the result, with written in supplements because the machine gives only 10 figures. This led to some conjectures.

n	$P_n(3)$
0	1.
1	3.
2	13.
3	63.
4	321.
5	1683.
6	8989.
7	48639.
8	265729.
9	1462563.
10	8097453.
11	45046719.
12	251595969.
13	1409933619.
14	7923848253.
15	44642381823.
16	252055236609.
17	1425834724419.

(a) All the coefficients are odd.

(b) They are alternately congruent to 1 and -1 mod 4.

(c) The b_n modulo 2^{k+1} have periodic residues with period 2^k (which reduces to (a) and (b) for $k = 0$ and 1).

(d) For $n \geq 9$, b_n is a multiple of 3.

(e) From $n = 5$ onwards the last digit repeats 3, 9, 9, 9, 3.
Some of these questions can be answered easily from the book of Gould's mentioned below.

Reference: Combinatorial Identities, by H.W. Gould, revised edition, privately published at Morgantown, W.Va., U.S.A., 1972.

QUOTATION CORNER (3)

I was able to put my hand on the tiller and everyone fell in behind.
"Northern Churchman" September 1979, page 12.

Your editor would like to hear from you anything connected with mathematics or with James Cook.

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