JAMES COOK MATHEMATICAL NOTES

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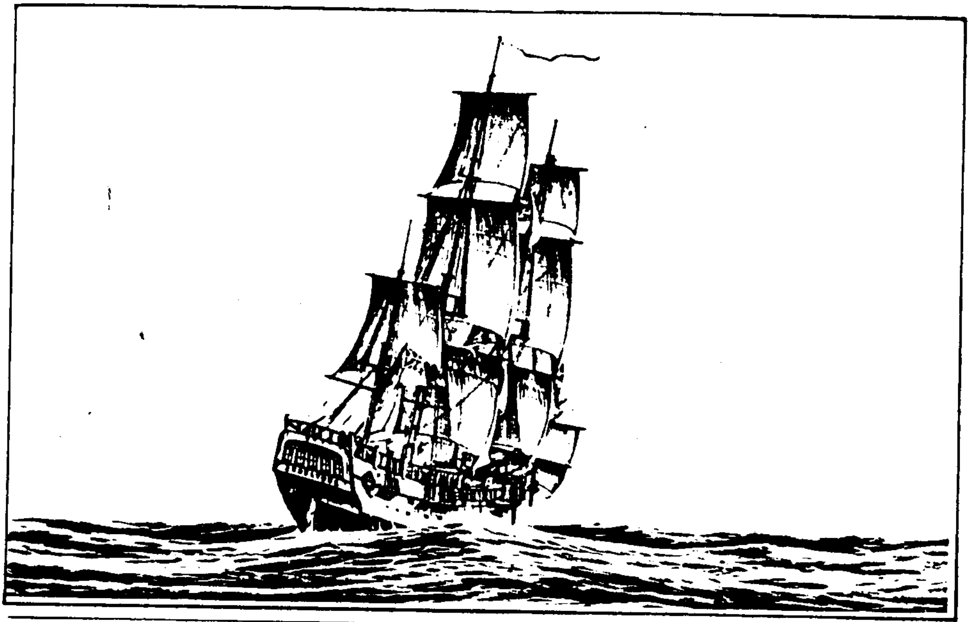


"H.M.S. ENDEAVOUR" - from the painting by Australian marine artist - Richard Linton.

ANOTHER IDENTITY FOR BINOMIAL COEFFICIENTS

$$(-1)^n \binom{2n+1}{n} = 2^{2n+1}\,(2n+1)\binom{1/2}{n+1}$$

The origin of this may be of interest. Some calculations in fluid mechanics using Legendre polynomials indicate that $\int_0^1 P_{2n+1}(\mu)\,d\mu$ ought to be equal to the binomial coefficient $\binom{1/2}{n+1}$, but Rodrigue's formula shows that the integral is $(-1)^n\, 2^{-2n-1}\binom{2n+1}{n}\Big/(2n+1)$. The identity can of course be slogged out, but is there an elegant way?

INTEGRAL ROOTS

For what values of a and b do both the quadratics $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ have both roots integral?

C.J. Smyth.

A NEW PROOF OF A THEOREM OF SCHUR

by *H. Kestelman*

The following result is due to Schur (Amer.J. Math. 67, 1945, p.472)

Theorem Every complex symmetric matrix A can be written $A = WDW^T$ where W is unitary $(WW^* = I)$ and D is real diagonal. Also $AA^* = WD^2W^*$.

Proof. The first step is to show that there is a non-zero column vector $\underline{v}$ and real λ satisfying $A\underline{v} = \lambda\overline{\underline{v}}$. If $A = B + i\,C$ with B and C real and symmetric, this will follow if the symmetric matrix $\begin{pmatrix} B & -C \\ -C & -B \end{pmatrix}$ has a real eigenvalue, which is of course true.

Now assume that the proposition holds for all $k \times k$ matrices, and take any symmetric $(k+1) \times (k+1)$ matrix A. Choose $\underline{v}$ as above and then a unitary matrix U whose first column is $\underline{v}$.

Since AU has $\lambda\underline{v}$ for its first column it follows that

$$U^T A U = \begin{pmatrix} \lambda & \underline{a} \\ 0 & X \end{pmatrix}$$

where $\underline{a}$ is some row vector and X is $k \times k$. Since $U^T A U$ is symmetric it follows that $\underline{a} = 0$ and X is symmetric. By the induction hypothesis $X = Z\Xi Z^T$ where Z is unitary and Ξ is real diagonal. Hence

$$A = \overline{U} \begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \Xi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Z^T \end{pmatrix} U^*$$

Now set $W = \overline{U} \begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix}$ and the formula $A = WDW^T$ follows. Finally $AA^* = WDW^T\overline{W}DW^* = WD^2W^*$. Exercise for the student. Prove that if A is symmetric $n \times n$ with eigenvectors spanning C^n $A = \Omega D \Omega^T$ where D is diagonal and $\Omega\Omega^T = I$.

RANDOM FIGURES (JCMN 15)

Independent random variables x and y each have a Gaussian distribution with mean zero and variance one. Let A(n) be the area of the convex hull of a sample of n points (x, y). What is the distribution of A(n)? The following considerations give a rough answer for large n.

Taking n very large, define R by $R^2\exp(R^2/2) = n$, and put $h = (3/R)\log R$. Then R will be large and h small. Given a sample of n points, the expected number outside a radius R from the origin is $n\int_R^\infty r\exp(-r^2/2)\,dr = n\exp(-R^2/2) = R^2$. The expected number outside the radius R + h is

$$n\exp(-R^2/2 - Rh - h^2/2) = (1/R)\exp(-h^2/2) < 1/R.$$

There will be a large number of points (about R^2) in the narrow annular region at radius between R and R + h from the origin, and probably none outside it. The convex hull is probably roughly circular, with area between πR^2 and $\pi(R+h)^2$. For large n these values are about $2\pi(\log n - \log\log n)$ and $\pi(2\log n + \log\log n)$.

IRREDUCIBLE POLYNOMIALS

Is it possible for one root of an irreducible polynomial (with rational coefficients) to be the mean of two others?

C.J. Smyth

SUMS OF SQUARES

George Szekeres has shown that the information in Sierpinski's Theory of Numbers on sums of four squares leads to every number from 2809 upwards being expressible as the sum of squares of five distinct positive integers. What is the largest number not expressible in this way?

John Mack

NON-NEGATIVE MATRICES

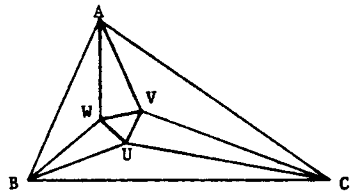
Consider real rectangular matrices $A(m \times n)$ and $B(n \times m)$ in which all elements are non-negative. If for some positive integer k the product $(AB)^k$ has all elements positive, what extra conditions on A and B are necessary and sufficient to ensure that there is some positive integer h for which $(BA)^h$ has all elements positive? (This is for first year students.)

H. Kestelman

AN ELEMENTARY PROOF OF MORLEY'S THEOREM

by E.C.G. Sudarshan (University of Texas at Austin)

Let ABC be any triangle. Let the adjacent trisectors of the angles meet pairwise to form the triangle UVW. Then, UVW is equilateral.



Let $d = 2R$ be the diameter of the circumcircle of the triangle ABC.

Then $\quad BC = d \sin A; \quad CA = d \sin B; \quad AB = d \sin C.$

In triangle WAB

$$W\hat{A}B = \frac{1}{3}A \qquad W\hat{B}A = \frac{1}{3}B \qquad A\hat{W}B = 180^\circ - \frac{1}{3}(A+B)$$

$$AW = \sin W\hat{B}A . \; AB/\sin A\hat{W}B = \frac{d \sin C . \sin B/3}{\sin (A+B)/3} \tag{1}$$

$$AV = \frac{d \sin B \sin C/3}{\sin (A+C)/3}$$

We can simplify using

$$\sin 3\theta = \sin \theta (3 \cos^2\theta - \sin^2\theta)$$
$$= 4 \sin \theta \left(\frac{\sqrt{3}}{2} \cos \theta - \frac{1}{2} \sin \theta\right)\left(\frac{\sqrt{3}}{2} \cos \theta + \frac{1}{2} \sin \theta\right)$$
$$= 4 \sin \theta \sin(60^\circ + \theta) \sin(60^\circ - \theta)$$

and $\sin(A+B)/3 = \sin(60^\circ - C/3)$, so that $\sin C =$
$$4 \sin C/3 \sin(60^\circ + C/3) \sin(A+B)/3.$$

From (1): $\quad AW = 4d \sin B/3 . \sin C/3 . \sin(60^\circ + C/3) = d' \sin(60^\circ + C/3)$
where $d' = d \sin B/3 \sin C/3$.

Similarly $\quad AV = 4 d \sin B/3 \sin C/3 . \sin(60^\circ + B/3) = d' \sin(60^\circ + B/3)$

Put $\quad B' = 60^\circ + B/3 \quad C' = 60^\circ + C/3 \quad$ then $\dfrac{B' + C'}{2} = 90^\circ - \dfrac{A}{6}$

$$AV = d' \sin B' \qquad AW = d' \sin C'$$

(continued overleaf)

$$VW^2 = d'^2 \left\{\sin^2 B' + \sin^2 C' - 2 \sin B' \sin C' \cos \frac{A}{3}\right\}$$
$$= d'^2 \left\{ \left(\frac{1 + \cos A/3}{2}\right) (\sin B' - \sin C')^2 + \left(\frac{1 - \cos A/3}{2}\right) (\sin B' + \sin C')^2\right\}$$
$$= 4d'^2 \left\{\cos^2 \frac{A}{6} \cos^2 \left(\frac{B'+C'}{2}\right) \sin^2 \left(\frac{B'-C'}{2}\right) + \sin^2 \frac{A}{6} \sin^2 \left(\frac{B'+C'}{2}\right) \cos^2 \left(\frac{B'-C'}{2}\right)\right\}$$
$$= 4d'^2 \sin^2\frac{A}{6} \cos^2\frac{A}{6} \left(\sin^2 \frac{B'-C'}{2} + \cos^2 \frac{B'-C'}{2} \right)$$
$$= \left(d' \sin \frac{A}{3}\right)^2$$

or $\qquad VW = d \sin \dfrac{A}{3} \sin \dfrac{B}{3} \sin \dfrac{C}{3}$.

Hence $\; UV = VW = WU \;$ and so UVW is equilateral.

BIGGER EIGENVALUES

Notation. Write $A \geq 0$ if the matrix A has every element ≥ 0, similarly > 0 if every element > 0.

Problem. Suppose that A and B are unequal real square matrices and $0 \leq A \leq B$ and $B > 0$. If μ is any eigenvalue of A show that B has a real eigenvalue $> |\mu|$.

It may be observed that Theorem 28 in §41 on page 83 of Beckenbach and Bellman's Inequalities is similar to the proposition suggested above but assumes more and proves less.

H. Kestelman

NOT QUITE ORTHOGONAL

Suppose that the real $n \times n$ matrix A is such that $AA' = D$ and $A'A = E$ are both diagonal, prove or disprove that $D = E$.

PLEA FOR HELP (JCMN 9)

The textbooks tell us that expressions like $\int_0^x e^{-t^2}dt$ and $\int_1^x \frac{e^t}{t}dt$ are "non-elementary" integrals. In this note we look at how this is precisely defined, and sketch how one can show such facts by reasonably direct methods. The original theory of non-elementary integrals was developed by Liouville in the 1830's. His proofs used analytic techniques. This account is based on an algebraic treatment of Rosenlicht (Pacific J. Math. 24 (1968), 153-161).

First of all, we define a 'differential field' F to be a field containing the complex numbers, together with a map $D: F \to F$ satisfying $D(f_1+f_2) = Df_1 + Df_2$, $D(f_1f_2) = f_1Df_2 + f_2Df_1$, for $f_1, f_2 \in F$.

Suppose we want to know whether a function f has an elementary integral. We define F_o to be the smallest differential field containing f, so e.g. if $f = e^{-z^2}$, $F_o = \mathbb{C}(z, e^{-z^2})$, while if $f = e^{z \log z}$, $F_o = \mathbb{C}(z, \log z, e^{z \log z})$. Here we take $D = \frac{d}{dz}$.

We then produce an ascending sequence ("tower") of differential fields $F_o \subset F \subset \cdots \subset F_N$, where for $i = 0, \ldots, N-1$ either

(a) $F_{i+1} = F_i(u)$, where $u \in F_{i+1}$ satisfies a polynomial with coefficients in F_i,

or

(b) $F_{i+1} = F_i(\log u_i)$ for some $u_i \in F_i$,

or

(c) $F_{i+1} = F_i(\exp u_i)$ for some $u_i \in F_i$.

Here $\log u_i$ is an element of F_{i+1} with $D(\log u_i) = Du_i/u_i$, and $\exp u_i$ is an element of F_{i+1} with $D(\exp u_i) = Du_i \exp u_i$.

We then say that our function $f \in F_o$ has an elementary integral if there is a differential field F_N constructed in the above way, and a $y \in F_N$ with $Dy = f$.

The essential result is that if $Dy = f \in F_o$, then

$$f = \sum_{i=1}^{n} c_i \frac{Du_i}{u_i} + Dv, \qquad (1)$$

where the $c_i \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, and the u_i and v belong to F_o.

The proof of this result uses induction on the number N of extensions in the tower. (Trivial if $N = 0$). The induction hypothesis allows us to assume that the result is true for the tower $F_1 \subset \ldots \subset F_N$ (i.e. $N - 1$ extensions), so that we can assume that (1) holds with the u_i and v in $F_1 = F_o(t)$ say. To show that in fact these elements lie in F_o, Rosenlicht first shows by direct methods that they must be polynomials in t over F_o, and then that we can assume that these polynomials do not in fact contain t. The argument is complicated, or at least lengthened, by the need to consider the three possible types of extension F_1/F_o - finite {i.e. type (a)}, log and exp. (The paper is by no means difficult, however, and could be digested in an evening).

The result (1) has narrowly restricted the form of these f with elementary integrals. Using it, results such as the following can be proved:

Theorem: Let $g(z)$, $h(z)$ be rational functions of z. Then $h(z) \exp\{g(z)\}$ has an elementary integral iff there is a rational function $a(z)$ satisfying $h = Da + a\,Dg$.

Applying the theorem to e^{-z^2}, we ask whether there is a rational function $a(z)$ with $1 = Da - 2za$. Expanding a as a polynomial and a sum of partial functions, $a(z) = p(z) + \sum \frac{m_i}{(z-\alpha_i)^{n_i}}$, we see that no such a exists. The same applies to $\frac{e^z}{z}$, which gives rise to the equation $\frac{1}{z} = Da \times a$. Hence the integrals of both e^{-z^2} and $\frac{e^z}{z}$ are non-elementary.

C.J. Smyth

THE FRIENDSHIP THEOREM (JCMN 15 and 16)

If every two people have just one common friend is there somebody friend to all?

Suppose that there are N people satisfying the condition that any two have just one common friend. Then:

Theorem 1 Either one person is a friend of all the others or $N = n^2+n+1$ (for some integer n) and everybody has $n+1$ friends.

Geometrical proof: Let $A = (a_{ij})$ be the matrix of friendships, that is $a_{ij} = a_{ji} = 1$ if i and j are friends, and $= 0$ otherwise. Consider A as an incidence matrix of points and lines, the points being the rows of A and the lines the columns. Then it follows that:

1) There is one and only one line through two distinct point, and

2) There is one and only one point common to two distinct lines.

If in addition there is nobody a friend of all the others then

3) There exist four points, no three of which are on a line.

These are the three axioms for a projective plane (Marshall Hall: Combinatorial Theory, Waltham Mass., Blaisdell, 1967. p.173). Consequently, if the set of points is finite, we must have, for some integer n, exactly $n^2 + n + 1$ lines and points each incident with exactly $n + 1$ points and lines (op. cit. Theorem 12.3.1, p.173).

Graph theory proof. This is much longer, and so we just state the three lemmata which together make up the proof.

Lemma 1. If p and q are two friends and r their common friend and if they have $x + 1$, $y + 1$ and $z + 1$ friends respectively, then $N - 1 = xy + z$.

Lemma 2. Of the three numbers x, y and z of the previous lemma, either they are equal and $N = x^2 + x + 1$, or two of them are 1 and the other $N - 2$.

Lemma 3. There are only two possibilities, (a) all but one of the people have just two friends and the other has $N - 1$, and (b) all the people have the same number $n + 1$ of friends, and $n^2 + n + 1 = N$.

Now that Theorem 1 is established we continue.

Theorem 2. The relation $N = n^2 + n + 1$ cannot hold except in the trivial cases of $N = 1$ and $N = 3$.

Proof $$A = \begin{pmatrix} n+1 & 1 & 1 & \cdots \\ 1 & n+1 & 1 & \cdots \\ \cdots & & & \end{pmatrix}$$

has the simple eigenvalue $N + n = (n + 1)^2$ and the eigenvalue n with multiplicity $N - 1 = n^2 + n$. The eigenvalues of A must be:

$n + 1$ (simple)

$\sqrt{n}$ (multiplicity t)

$-\sqrt{n}$ (multiplicity $N - 1 - t = n^2 + n - t$)

The sum is the sum of the diagonal elements of A which are all zero.

$$n + 1 + 2t\sqrt{n} = n(n + 1)\sqrt{n}$$

Therefore $2t = n(n + 1) - \sqrt{n} - 1/\sqrt{n}$ which is not an integer except when $n = 1$.

This proves the Friendship Theorem in the finite case. For an infinite number of people it is untrue, for let people be lines through the origin in E_3, and let your friends be those that are perpendicular to you.

M.J.C. Baker

AFTER TEA MATHEMATICS

In Ramsay's Hydrodynamics (Section 9.84, page 247) is quoted the following passage from the epoch-making 1858 paper of Helmholtz in Crelle's Journal, volume 55.

> "We can now see generally how two ring-formed vortex-filaments having the same axis would mutually affect each other, since each, in addition to its proper motion, has that of its elements of fluid as produced by the other. If they have the same direction of rotation they travel in the same direction; the foremost widens and travels more slowly, the pursuer shrinks and travels faster, till finally if their velocities are not too different, it overtakes the first and penetrates it. Then the same game goes on in the opposite order, so that the rings pass through each other alternately.
>
> If they have equal radii and equal and opposite angular velocities, they will approach each other and widen one another; so that finally, when they are very near each other, their velocity of approach becomes smaller and smaller, and their rate of widening faster and faster. If they are perfectly symmetrical, the velocity of fluid elements midway between them parallel to the axis is zero. Here then we might imagine a rigid plane to be inserted, which would not disturb the motion, and so obtain the case of a vortex ring which encounters a fixed plane.
>
> In addition it may be noticed that it is easy in nature to study these motions of circular vortex rings, by drawing rapidly for a short space along the surface of a fluid a half-immersed circular disk, or the nearly semi-circular point of a spoon, and quickly withdrawing it. There remain in the fluid half vortex rings whose axis is in the free surface. The free surface forms a bounding plane of the fluid through the axis, and thus there is no essential change in the motion. These vortex rings travel on, widen when they come to a wall, and are widened or contracted by other vortex rings, exactly as we have deduced from theory."

One of the minor pleasures of the tea table is to make two such vortex half-rings with your teaspoon and to see the second ring overtake the first. Unfortunately I have never had a cup big enough for the first ring to gather itself and start seriously to overtake the upstart. Now, after lecturing on the subject I have had to conclude that I was wrong about it all. The disturbance in a perfect liquid produced in this way by a plane semicircular spoon or paddle (assuming the free surface constrained to be horizontal) is not in fact a vortex ring. The vorticity in the fluid after the spoon is removed is not concentrated in the semicircular rim, it is spread in a sheet over the half-disc.

This motion has the property that it will travel unchanged through the liquid after the disc has been taken away. One might imagine that the velocity potential or the Stoke's stream function would be expressible fairly simply in terms of Legendre polynomials, but such a result has eluded me.

NONLINEAR AERONAUTICS

A.A. Richardson points out that to carry the largest possible case on Trans-Australian Airways you must maximize xyz subject to $0 \le x \le 50$, $0 \le y \le 38$, $0 \le z \le 20$ and $x + y + z \le 100$.

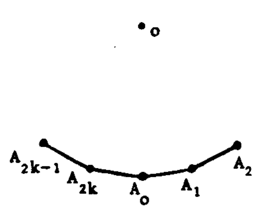
WILL THE REAL GRAM SCHMIDT PLEASE STAND UP?

If $v_1, v_2, \ldots, v_q$ are column vectors in C^n, $q < n$, such that $v_r^T v_s = \delta_{rs}$ when $1 \le r, s \le q$, then $v_{q+1} \ldots, v_n$ exist such that $v_r^T v_s = \delta_{rs}$ when $1 \le r,s \le n$.

H. Kestelman.

COVERING WITH TRIANGLES (JCMN 15)

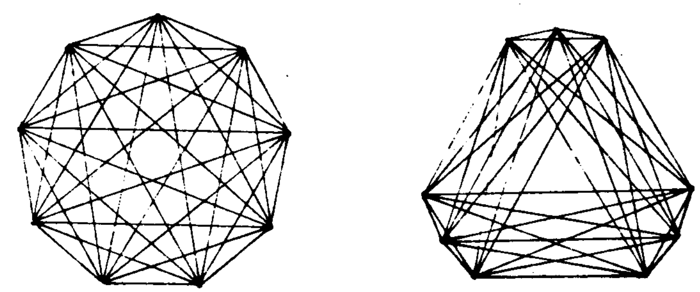
Here is an alternative approach to the calculation of how many triangles cover the centre of a regular $(2k + 1)$ gon. For each such triangle pick out the one side (call it the base) such that the point A_o is either at the clockwise end of the base or is on the part of the circumference of the polygon cut off by the base. The number of sides of the polygon cut off by the base (call it n) may be 1, 2, 3 ... k (triangles with $n > k$ cannot cover the centre). If $n = 1$ there is only one possible base, $A_o A_1$, and only one triangle covering the centre, for the third vertex must be A_{k+1}. If $n = 2$ there are two possible bases, $A_o A_2$ or $A_{2k} A_1$, and for each of them there are two possibilities for the third vertex, this gives $2^2 = 4$ triangles. In fact for each n there are n possible bases and for each base there are n choices for the third vertex to



make the triangle cover the centre. This shows that the number of triangles covering the centre is $1^2 + 2^2 + \ldots + k^2$.

H.O. Davies

COVERING WITH TRIANGLES III



Two nonagons are illustrated. The regular one has regions covered by 7, 12, 15, 16, 17, 20, 21, 23, 24, 26, 27, 28, 29, 30 triangles. The other one has regions covered by 7, 12, 15, 16, 17, 20, 21, 23, 24, 25, 26, 27, 28, 29 triangles. This shows two things: first the various thicknesses that occur and in particular the greatest thickness depend on the arrangement of the points; and secondly it shows that the thickness of covering can go down in the middle. For in the second nonagon the central triangle is covered 27 thick, the regions just across its sides 28 thick, and the regions diagonally across its vertices 29 thick.

I find that the quickest way of counting triangles is to consider the change in covering as you cross a line joining two points, say A and B. The only triangles you leave are Δs ABP with P on the hither side of AB, and the only ones you enter are Δs ABQ with Q on the further side. Thus for instance in the nonagon you can change thickness by 7 (at the outside only), or 5 or 3 or 1. This argument shows that for an even family of points the triangle covering is everywhere even.

I guess a) that the above numbers give the only thicknesses that can occur with nonagons (whether the nine points are all extreme points of the convex hull or not) and b) that the regular polygon gives the thickest covering.

M.J.C. Baker.

CONVEX N-GONS WITH EMPTY INTERIORS

Hugh Morris (University of Sydney)

It is an unsolved problem whether, for any given $n \geq 3$, there exists a number $C(n)$ with the property that given $C(n)$ points in the plane, no three collinear, one may find a convex n-gon with its vertices at n of these $C(n)$ points and such that no other of these points is in its interior.

For $n = 3$ the result is trivial, and it is known that $C(4) = 5$. $C(5)$ was known to exist, but it has only recently been shown that, in fact, $C(5) = 10$. The proof given below was found independently in the course of solving Problem 10 of the 1978 Sydney University Mathematical Society's Problems Competition.

We wish to show that given 10 points in the plane (no 3 collinear), we can always find 5 which are the vertices of a convex pentagon, such that none of the other points lie in the interior of the pentagon.

The proof described below is based on a division of the problem into 19 cases, and then finding solutions for each individual case. This makes for a fairly long and tedious proof, so in the following, I will merely describe how the cases arise, and then give some examples of the individual proofs.

If we have a set of N points in the plane, we can form their convex hull in the usual way. If we remove the vertices of that hull, we can form the convex hull of the remaining points. Proceeding in this way through all the points, we get a nested sequence of convex polygons. We can describe any given configuration by a sequence

$$(n_1, n_2, \ldots, n_k)$$

where the outer hull is an n_1-gon

the next hull is an n_2-gon

etc.

The conditions on this sequence are that

$$\sum n_i = 10 \quad \text{and}$$

$$n_i \geq 3 \text{ for all } i \text{ except possibly the last.}$$

It is easy to show that there are exactly 19 possible sequences:

(10) (9, 1) (8, 2) (7, 3) (6, 4) (5, 5) (4, 6) (3, 7)

(6, 3, 1) (5, 4, 1) (4, 5, 1) (3, 6, 1) (5, 3, 2) (4, 4, 2)

(3, 5, 2) (3, 3, 4) (4, 3, 3) (3, 4, 3) (3, 3, 3, 1)

These configurations of ten points correspond to the 19 different cases of the proof.

I shall now give 2 examples of the proofs, proving the existence of an empty convex pentagon (ECP) for the configurations (4, 4, 2) and (3, 3, 3, 1).

Notation: In the following,

a_i denotes a point of the innermost hull

b_i denotes a point of the 2nd hull

c_i etc.

(i) (4, 4, 2).

Draw the line ℓ though a_1 and a_2. If it cuts the quadrilateral $b_1\, b_2\, b_3\, b_4$ in adjacent sides, we are done (see diagram 1). Otherwise, ℓ cuts the quadrilateral in opposite sides. Then divide the part of the plane outside the quadrilateral into four regions, A B C and D as shown in diagram 2.

If there is any one of the points $c_1, \ldots, c_4$ in A or C, we are done (e.g. pentagon P_1 in the diagram). If there are 2 or more points in either B or D, we again have an ECP (e.g. pentagon P_2 in diagram).

But we have to distribute the 4 points c_1, c_2, c_3, c_4 amongst A, B, C, D, and so one of these cases must occur.

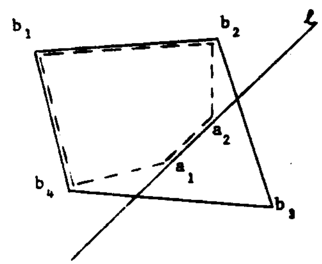


Diagram 1

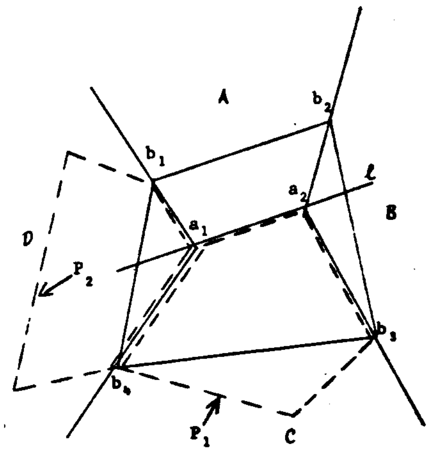


Diagram 2

(ii) (3, 3, 3, 1).

Divide the plane outside the triangle $b_1\ b_2\ b_3$ into three regions R_i ($i = 1, 2, 3$) using the rays ab_i ($i = 1, 2, 3$). If any of these regions contains two of the three points c_i, we have an ECP. (e.g. Pentagon P_3 in diagram 3.)

Otherwise, each of the regions R must contain exactly one of the points c_i. Using these points, divide the plane as shown in diagram 4.

The region A for instance is bounded by the bent line $c_2\ b_1\ c_3$. If any one of A B and C contains two of the points d_i then we have an E.C.P. If one of the regions contained three points, then the triangle $d_1\ d_2\ d_3$ would not contain $c_1\ c_2\ c_3$: therefore each of A, B, C contains one of the points d_i, and the intersections $A \cap B$ etc. do not contain any. But if for instance B were to contain d_3 as shown in diagram 4, then $c_1\ b_3\ a\ b_2\ d_3$ would be an E.C.P. This disposes of all the possibilities for case (ii) of (3, 3, 3, 1).

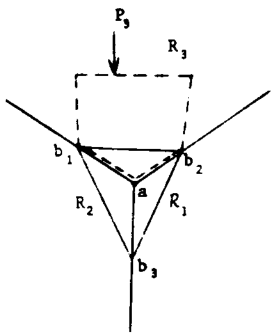


Diagram 3

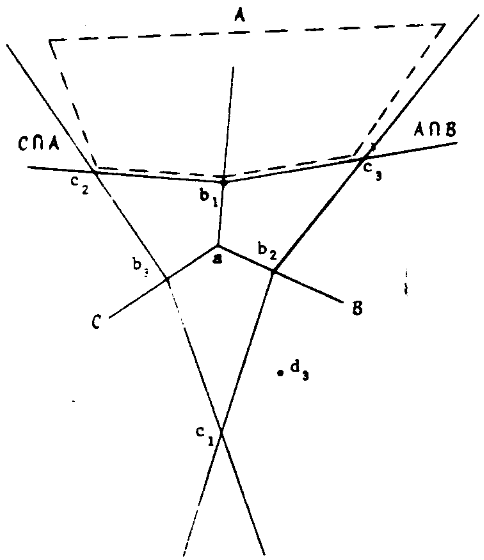


Diagram 4

INACCURACIES IN A RESISTANCE NETWORK

P.A.P. Moran and B.C. Rennie

Suppose that n electrical resistors are to be connected as a network with two external terminals, the resistors have nominal resistances of R_i ohms ($i = 1, 2, \ldots, n$) but owing to inaccuracy of manufacture their actual resistances are $P_i\ R_i$ where the P_i are random variables. The errors are assumed small, so that second order terms in them can be ignored. What is the error in the resistance of the network?

Given a fixed input current, the currents in the network are given by Kelvin's theorem, as follows. Of all the current distributions that satisfy the conservation law, the actual one, the one that satisfies Ohm's law, is the one that minimizes the total heat generated.

Suppose that with a unit input current the nominal currents in the various resistors are C_i. The nominal network resistance R is equal to the nominal rate of heat generation, that is

$$R = \sum C_i^2 R_i .$$

The actual resistance is the actual rate of heat generation, which (because it is a minimum) we may calculate from the nominal currents instead of the actual currents, that is

$$PR = \sum C_i^2 P_i R_i = \sum P_i H_i$$

where H_i is the (nominal) rate of generation of heat in resistor i when the network input current is one unit. Since $\sum H_i = R = H$ is the total heat generated with nominal resistance values and with unit current input the formula above may be written $P = \sum P_i (H_i / H)$, or expressed by saying that the percentage error in the network resistance is the weighted sum of the individual percentage errors, weighted in proportion to the heat generated in each resistor.

QUERY POSITIVE DEFINITE

by P.A.P. Moran

Given n disjoint spheres in three dimensions define a real symmetric matrix M as follows; each diagonal element m_{rr} is the reciprocal of the radius of sphere r, and the off-diagonal elements are given by $m_{rs} = m_{sr}$ = the reciprocal of the distance between the centres of spheres r and s. Is this matrix positive definite?

EASY ALGEBRA (JCMN 16)

There are polynomials of degree 6 but no lower of which each zero is the sum of two others, for instance $z^6 - 1$ or generally the polynomial with roots $\pm a$, $\pm b$ and $\pm(a + b)$. A less easy problem is to find a polynomial of lowest degree with the same property of each root being the sum of two others and also with no two roots having the same modulus. One suggestion is the polynomial with the following nine roots:

$$7,\ 5,\ 3,\ 2,\ -1,\ -4,\ -6,\ -8,\ -10.$$

J.B. Parker

ONE PER CENT PERSPIRATION

Find the last hundred digits of 404!

C.J. Smyth

DOUBLY STOCHASTIC MATRICES (JCMN 15)

Are doubly stochastic matrices diagonalizable by a similarity transformatic
Answer: NO.

Let $$M = \frac{1}{3}\begin{pmatrix} 1+2\lambda & 1-\lambda & 1-\lambda \\ 1-\lambda+2\mu & 1+2\lambda-\mu & 1-\lambda-\mu \\ 1-\lambda-2\mu & 1-\lambda+\mu & 1+2\lambda+\mu \end{pmatrix}$$

The row-sums and column-sums are all equal to 1, and for sufficiently small λ and μ all elements are positive and so M is doubly stochastic. The orthogonal matrix

$$P = \begin{pmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix}$$

makes $$P^{-1}MP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & -2\mu\sqrt{3} \\ 0 & 0 & \lambda \end{pmatrix}$$

which clearly cannot be diagonalized. Alternatively (a special case of the

above) just note that $\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & 1/6 & 1/6 \\ 0 & 1/2 & 1/2 \end{pmatrix}$ has a zero as a double eigenvalue but the corresponding eigenspace is only one-dimensional.

E.C.G. Sudarshan.

CABLE LIMIT

Define $g(x, y)$ for $0 < x < 1$ and $y > 0$ by

$$y\, g(x, y) = x^y/y - \log(1 - x) - \int_0^x t^{y-1}/(1 - t)\,dt.$$

What can you say about the behaviour of g as $x \to 1$ and $y \to 0$?

This problem arose in a study of electric cables.

E. Kestelman.

THE DAUGHTER OF TIME

In A.S. Ramsey's Hydrodynamics, (Fourth Edition, 1935) there is quoted on page 90 the following question, ascribed to the University of London, 1911.

If q is the resultant velocity at any point of a fluid which is moving irrotationally in two dimensions, prove that

$$\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 = q\,\nabla^2 q.$$

The suggested proposition is untrue (for instance take $u = 1 + x$, $v = 0$), perhaps the examiner forgot to specify incompressibility.

AN ELECTRICAL CIRCUIT (JCMN 15)

An electrical circuit is made from the n-dimensional Cartesian lattice points by joining each adjacent pair with a one-ohm resistor. What is the resistance of the circuit between two adjacent points A and B? The answer is $1/n$ ohms. To prove this, first consider unit current going in at point A and dispersing to infinity, the voltage drop from A to B is $1/(2n)$ because equal currents must flow in all the $2n$ resistors joined to A. Now consider unit current coming out at B, again there must be a voltage drop of $1/(2n)$ from A to B. Superimposing the two current patterns (as we may because Ohm's law is linear) we see the required result.

For the two dimensional case an amusing proof has been circulating for some years in the folk-lore of circuit theory. Replace one of the resistors by a two-volt battery with one ohm of internal resistance. Then all the components are self-dual and so the circuit is self-dual, so that the battery current and voltage must be equal, therefore both one unit.

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Your editor would like to hear from you anything connected with mathematics or with James Cook.

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