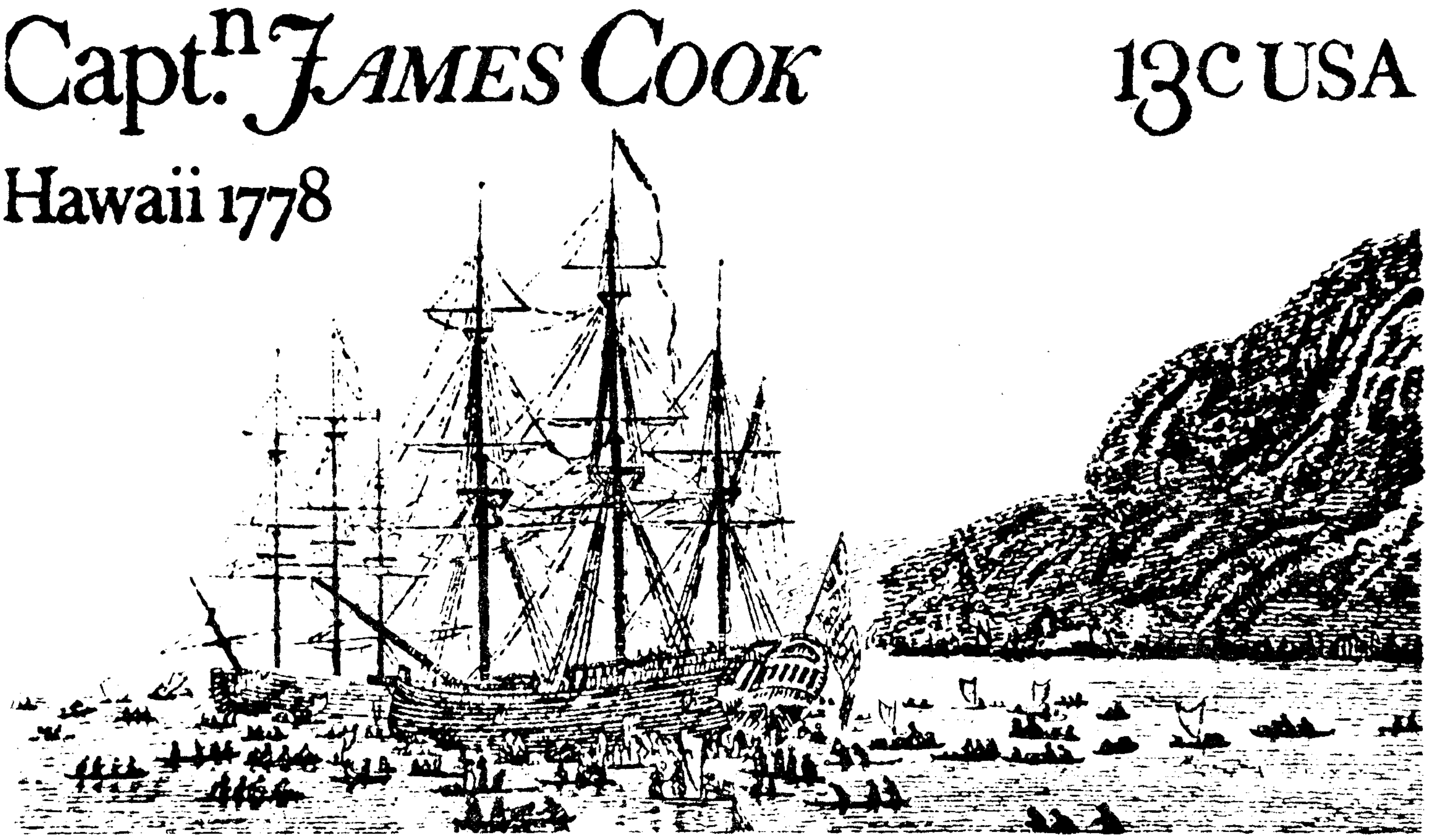
JAMES COOK MATHEMATICAL NOTES

Issue No. 16, July, 1978.



Our cover picture was taken from a stamp, issued by the U.S. Postal Services to mark the 200th anniversary year of Capt. Cook's arrival in Hawaii and his visit to Alaska in 1778. The seascape depicts Capt. Cook's ships, HMS "Resolution" and HMS "Endeavour", anchored in Hawaii. It is based upon a line etching by John Webber, an artist who travelled with Cook, entitled "A View of Karakekooa in Owyhee". We are indebted to Prof. J.D.E. Konhauser for sending some mint copies of the commemorative stamps.

AN EXPANSION PROBLEM

Given the positive integer m, does there exist non-zero k such that all the coefficients in the power series for $(1-kx)^{1/m}$ are integers?

E.O. Tuck

INTELLIGENCE TEST (JCMN 15)

Why is the knob on my washing machine marked *dwnd*? Perhaps the answer had to come from an Australian geometer who lives in the opposite hemisphere. *A.P. Guinand* writes that *dwnd*$^{-1}$ = *pump*.

RIEMANN ARE YOU THERE?

The function $1-z^n-(1-z)^n$ for any positive integer $n \geq 2$ clearly has just two real zeros, 0 and 1. But do all the complex zeros have real part = 1/2? To save you spending too much time on it, the solution is given on the back page, but promise not to look until you have made your guess at the answer.

QUADRATIC FUNCTIONS

Let f be an infinitely differentiable scalar function in n dimensions and let the vector function $\underline{v}$ be its gradient, that is $v_i = \partial f/\partial x_i$. It is quite easy to show that if f is a quadratic function of the variables then $2\underline{h} \cdot \underline{v}(\underline{x}) = f(\underline{x} + \underline{h}) - f(\underline{x} - \underline{h})$ for any vectors $\underline{x}$ and $\underline{h}$, but theinteresting fact is that the converse is also true, if f satisfies the equation then it must be a quadratic. The case n=1 makes a good first year question. The general case can be proved in the analagous way by operating on the equation with $\partial^2/\partial h_i \partial h_j$.

John van der Hoek.

APPLIED MATHEMATICS

Text-books on mechanics often ignore real applications and choose the strangest examples to illustrate their theories. When I was taught about impulsive motion we were bidden to consider such things as four uniform rods smoothly hinged to form a rhombus and sliding on a smooth horizontal table. Now that motor cars are built without starting handles they provide a practical problem in impulsive motion. When your starter has failed and you have mustered some kind friends to start the engine with a push, what gear should you engage when they have got the car moving? This gives a good illustration of Lagrangian methods.

The car can be regarded as a system with two degrees of freedom, take coordinates x, the distance along the road, and θ the angle of rotation of the crankshaft. The kinetic energy is of the form $a\dot{x}^2 + b\dot{\theta}^2$. Engaging a gear and the clutch is imposing a constraint $\dot{x} = k\dot{\theta}$, where the driver has a choice of a few values of k. The imposition of such a constraint invokes impulsive forces and discontinuities in the velocities. The general theory shows that the velocities afterwards are such as to minimize what is called the "kinetic energy of velocity changes". In our example let the velocities $\dot{x}$ and $\dot{\theta}$ be u and zero beforehand, and be $k\omega$ and ω after. Then the actual ω is the value that minimizes $a(k\omega-u)^2 + b\omega^2$, and this value is $kau/(k^2a + b)$. The problem for the driver is to choose k to maximize this function, again a simple exercise in calculus, and the best value for k is $(b/a)^{1/2}$. The average motorist would not want to find the relative speeds of the back axle, the oil pump, and the generator, and the moments of inertia of the road wheels and the fan and so on in order to calculate a and b, and in fact there is no need. With this optimal k the angular velocity of the engine is $\frac{1}{2}u\,(a/b)^{\frac{1}{2}}$ and the road speed is $\frac{1}{2}u$. The best gear ratio is the one that makes the speed drop by half when you engage the gear. The wise driver should be able to determine this best gear before exhausting the strength of the people pushing.

Years ago there used to be some small radial aero engines with inertia starters, can anyone tell us about them?

EASY ALGEBRA

Here's a polynomial query, try it when not feeling weary. Find a polynomial P, non-constant it has got to be. I want two other things to boot, zero must not be a root, and every root a sum must be of some two other roots of P. Now find such P, of least degree.

C.J. Smyth

SOLVING EQUATIONS

Let f, g be two differentiable real functions. Suppose there is known to exist an $x : f(x) = g(x)$, which we want to compute. Newton's method could clearly be tried:

$$x_{n+1} = x_n - \frac{f(x_n) - g(x_n)}{f'(x_n) - g'(x_n)} \quad .$$

If, however, f or g has a differentiable inverse function which can be computed, then using a desk calculator the recurrences

$$(a)\quad x_{n+1} = f^{-1}\{g(x_n)\} \qquad \text{or} \qquad (b)\quad x_{n+1} = g^{-1}(f(x_n))$$

would be easier to use. If $f'(x_1)/g'(x_2)$ is continuous and not equal to ± 1 at (x, x), show that provided the starting value x_0 is sufficiently close to x, exactly one of *(a)*, *(b)* produces a sequence converging to x.

Try it on $\sin^{-1} x = (\sin x)^{-1}$ (that well-known schoolboy identity).

C.J. Smyth

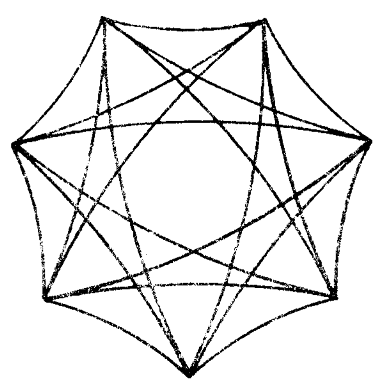
THE FRIENDSHIP THEOREM (JCMN 15)

If every two people have just one common friend is there somebody friend of all?

In algebraic terms the problem is this; on a set S is a binary relation F which is symmetric ($xFy \Rightarrow yFx$) and non-reflexive (never xFx), and for all $x \neq y$ there exists one and only one z such that xFz and yFz; is there p such that for all x, xFp? In graph-theoretical terms suppose that in a simple graph each pair of distinct nodes is joined by just one path of length two, is there one node joined to every other node?

The question can also be put in the language of matrix theory. Let A be a symmetric matrix of zeros and ones with zeros on the principal diagonal and suppose that A^2 has all off-diagonal elements equal to 1. Prove or disprove that one row of A consists entirely of ones except for the zero on the diagonal.

G. Szekeres points out that if the relation of friendship were not symmetric (if the graph were directed) there would be a counter-example in the field of residue classes module 7, with xFy when $x-y \equiv$ any one of the quadratic residues 1, 2 and 4. The reason is that these three quadratic residues form a complete difference set, for any unequal x and y the difference $x-y$ is uniquely expressible as a difference of two members of $\{1, 2, 4\}$ and these two are taken as $x-z$ and $y-z$ respectively, this determines the unique z, friend of both x and y.



This construction is illustrated in the picture above, given any two vertices, there is just one other vertex to which there is, from each of the first two, a path curving gently to the left. The pretty way to draw a directed graph is to curve the edges instead of decorating them with arrows.

The story of how the friendship theorem came to the JCMN is that I went to the July 1969 Combinatorial Conference at Oxford. There H.S. Wilf gave a lecture on finite projective geometries, and the notes that I wrote on the margins of the conference programme mention the friendship theorem with a hint on how to prove it. Sadly I cannot now see the relation between the friendship theorem and finite projective geometries, but perhaps it was clear at the time.

B.C. Rennie.

AFTER DINNER MATHEMATICS (JCMN 15)

Readers will remember how *C. Moppert* recalled for us the beautiful little fact that if a circle rolls inside a fixed circle of double the size then any point on the first circle moves in a straight line. *Dan Pedoe* writes to point out how this result is useful for the geometry of the ellipse. In fact it links the following four theorems.

(1) If two lines ℓ and m meet at right angles and if a straight rod AB moves so that A remains on ℓ and B on m, then the locus of any other point on the line AB is an ellipse.

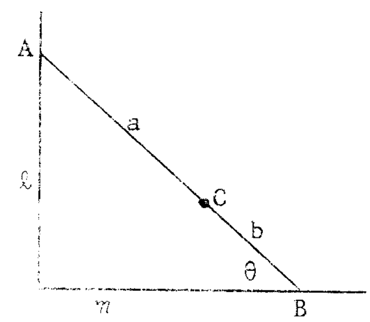
(2) The same except that the two lines need not be perpendicular.

(3) If ℓ and m are lines meeting at right angles and if the triangle ABC (regarded as a rigid body in the plane of the lines) moves so that A is on ℓ and B on m, then the locus of C is an ellipse.

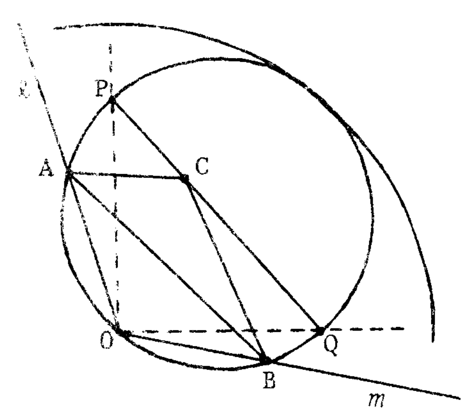
(4) The same as (3) except that the two lines may meet at any angle.

This last result includes the first three as special cases, it was known to *Leonardo da Vinci* (see Geometry and the Liberal Arts by *Dan Pedoe*, Penguin Books, 1976, page 97).

The first result can be seen from this figure, where if the two lines are taken as Cartesian axes x = a cos θ and y = b sin θ. Now to prove the result (4).



Suppose that A and B move on two lines ℓ and m meeting at O. Draw the circle through A, B and O, this is the rolling circle. With centre O draw a circle of twice the size, this is the fixed circle. By attaching A and B to the rolling circle we can make them move along the lines ℓ and m as required. Now draw a line from C through the centre of the small circle meeting it in P and Q. As the circle rolls, P and Q will move along the lines OP and OQ which are perpendicular because PQ is a diameter and which are fixed in direction. By Theorem ([illegible]) C must then trace out an ellipse.



JCMN16.

STILL MORE TRIGONOMETRY (JCMN 15)

This question from *C.J. Smyth* was about showing that $\sum_1^{n-1} \operatorname{cosec}^2 2\pi j/n = (n^2-1)/3$ when n is any odd integer. From (in alphabetical order) *G. Bode, H.O. Davies, A.P. Guinand, V. Laohakosol, B.S. Newman* and *J.B. Parker*, we have seven solutions. In all of them we use S to denote the required sum, and write m for the integer such that $n = 2m + 1$. Since $\operatorname{cosec}^2 2\pi j/n = \operatorname{cosec}^2 2\pi(n-j)/n$ the required sum $= S = \sum_1^{n-1} = 2\sum_1^m$.

<u>First Solution.</u> Let $c = \cos\theta$ and $s = \sin\theta$, then $\sin n\theta$, being the imaginary part of $(c + is)^n$ is a sum of terms with an even power of c and an odd power of s; replacing each c^2 by $1 - s^2$ gives a sum of odd powers of s. In fact $\sin n\theta = ns - n(n^2-1)s^3/6 + \dots$ (a polynomial of degree n). The values of $\sin(2\pi j/n)$ for $j = 1, 2, \dots m$ are non-zero and unequal, and they satisfy the algebraic equation

$$0 = n - n(n^2-1)x^2/6 + \dots \qquad \text{(of degree } n-1\text{)}$$

This equation contains only even powers, and so the m values of $\operatorname{cosec}^2(2\pi j/n)$ all satisfy

$$0 = nx^m - n(n^2-1)x^{m-1}/6 + \dots \qquad \text{(of degree } m\text{)}$$

Since the m values are unequal they must be the roots, and so their sum must be the sum of the roots, that is:

$$\sum_1^m \operatorname{cosec}^2(2\pi j/n) = (n^2-1)/6$$

As noted above this leads to $S = (n^2-1)/3$.

<u>Second solution.</u> If θ is one of the angles $2\pi j/n$ for $j = 1, 2, \dots m$, then $(\cos\theta + i\sin\theta)^n = 1$ and so $(\cot\theta + i)^n = \operatorname{cosec}^n\theta$. Equate imaginary parts.

$$\binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \dots = 0$$

The m values of θ give m distinct values of $x = \cot^2\theta$ all satisfying

$$\binom{n}{1}x^m - \binom{n}{3}x^{m-1} + \dots = 0.$$

These values are the roots and their sum is $(n-1)(n-2)/6$.

$$\sum_1^m \operatorname{cosec}^2 2\pi j/n = m + \sum_1^m \cot^2 2\pi j/n = \tfrac{1}{2}(n-1) + (n-1)(n-2)/6$$

$= (n^2-1)/6$, giving the result as before.

Third solution. For $1 \le j$ [illegible] the pair of equal terms:

$$\operatorname{cosec}^2 2\pi j/n + \operatorname{cosec}^2 2\pi(n-j)/n = 2\operatorname{cosec}^2 \pi r/n \qquad \text{(where } r = n - 2j\text{)}$$

$$= 2/(1 - \cos^2\pi r/n) = 1/(1 - \cos\pi r/n) + 1/(1 + \cos\pi r/n)$$

$$= 1/(1 - \cos\pi r/n) + 1/(1 - \cos\pi(n-r)/n)$$

Since r takes all the odd values from 1 to $n-2$ and $n-r$ takes the even values from 2 to $n-1$ it follows that the required sum is

$$S = \sum_1^{n-1} 1/(1 - \cos\pi r/n)$$

The Chebychev polynomials of the second kind may be defined by $u_0(x) = 1$, $u_1(x) = 2x$ and $u_{n+1}(x) = 2x\,u_n(x) - u_{n-1}(x)$, or by $u_{n-1}(\cos\theta) = \sin n\theta/\sin\theta$. The $n-1$ zeros of $u_{n-1}(x)$ are $\cos \pi r/n$ for $r = 1, 2, \ldots n-1$.

$S(1 - \cos\pi/n)(1 - \cos 2\pi/n)\ldots(1 - \cos(n-1)\pi/n)$ is the sum of products $n-2$ at a time of the $1 - \cos r\pi/n$ which are the zeros of $u_{n-1}(1-x)$. Therefore $S = -(\text{coefficient of } x)/(\text{constant term}) = u'_{n-1}(1)/u_{n-1}(1) = (n^2-1)/3$.

Fourth Solution. Start with the identity $\operatorname{cosec}^2 z = \sum_{-\infty}^{\infty}(z-k\pi)^{-2}$ (Titchmarsh, page 113). For n odd and j not a multiple of n,

$$\operatorname{cosec}^2(2\pi j/n) = (n^2/\pi^2)\sum_{-\infty}^{\infty}(2j-nk)^{-2}$$

$$(\pi^2/n^2)S = \sum_{-\infty}^{\infty} I_k \quad \text{where}$$

$$I_k = (2-nk)^{-2} + (4-nk)^{-2} + \ldots + (2n-nk-2)^{-2}$$

Each I_k is a sum of inverse squares of [illegible] integers as follows

I_{-1} the odd integers from [illegible] $n-2$ inclusive

I_0 the even integers from [illegible] -2 inclusive

I_1 the odd integers from $-n+2$ to $n-2$ inclusive

I_2 the even integers from [illegible] to [illegible] inclusive

I_3 the odd integers from [illegible] to $3n-2$ etc.

This listing includes (just once) each positive or negative integer except the multiples of n. The sum of the inverse squares is therefore

$$2\sum_1^{\infty} r^{-2} - 2\sum$$ [illegible]

Now we can use the [illegible] which has become familiar by appearing in JCMN 5, 6, 7 and 12.

$$(\pi^2/n^2)S = \sum I_k = 2(1-n^{-2})\pi^2/6$$

$$S = (n^2-1)/3.$$

Fifth solution. Consider the complex function $f(z) = 4z(z^2-1)^{-2}(z^n-1)^{-1}$. The integral round a large circle is zero and so the sum of the residues inside is zero. The residue at the simple poles $\exp(2i\pi j/n)$ is $-(1/n)\operatorname{cosec}^2 2\pi j/n$. The residues at 1 and -1 are $(n^2-4)/(12n)$ and $n/4$ respectively, and the given result follows.

Sixth Solution. For odd n the polynomial

$$(z+1)^n - (z-1)^n = 2nz^{n-1} + 2\binom{n}{3} z^{n-3} + \dots + 2$$

has zeros given by $z + 1 = (z-1)\exp 2\pi ir/n$, whence the roots are $z = \pm i \cot \pi r/n$ (for $r = 1, 2, \dots m$). Hence $(z+1)^n - (z-1)^n = 2n\Pi(z^2 + \cot^2 \pi r/n)$. Equating coefficients of z^{n-3} gives:-

$$2n \sum_1^m \cot^2 \pi r/n = n(n-1)(n-2)/3.$$

Therefore $\sum_1^m \operatorname{cosec}^2 \pi r/n = (n^2-1)/6$

which gives the result as before.

Seventh Solution. Let Σ and Π denote sums and products with the dummy suffix r taking values $\pm 1, \pm 2, \dots \pm m$ (but not zero)

$$z^n-1 = (z-1)\Pi(z-\omega_r) \qquad \text{where } \omega_r = \exp 2i\pi r/n$$

$$\ln\{(z^n-1)/(z-1)\} = \sum \ln(z-\omega_r)$$

The logarithm having many values does not matter because we are going to differentiate. Changing the sign of z gives $\ln\{(z^n+1)/(z+1)\} = \sum \ln(-z-\omega_{-r})$. Adding gives: $\ln\{(z^{2n}-1)/(z^2-1)\} = \sum \ln(1-z^2+2i\,z\sin 2\pi r/n)$. Let $\phi(z)$ be the second derivative of either side of this equation.

$$\text{From RHS,} \quad \phi(z) = \sum \frac{-2}{1-z^2 + 2i\,z\sin 2\pi r/n} - \sum \frac{(2z - 2i\sin 2\pi r/n)^2}{(1-z^2 + 2i\,z\sin 2\pi r/n)^2}$$

$$\phi(1) = -2m + \sum \operatorname{cosec}^2 2\pi r/n .$$

The LHS has a removable singularity at $z=1$ and so we must use limits.

$$(d/dz)^2 \ln(z^{2n}-1) = 2n\, z^{2n-2}(1-2n-z^{2n})(z^{2n}-1)^{-2}$$

Putting $z = 1 + u$ and expanding in powers of u, this is:

$$-u^{-2} + (2n-1)(2n-5)/12 + \text{powers of } u .$$

Subtracting the expression when $n=1$ gives:

$$\phi(z) = (n-1)(n-2)/3 + \text{powers of } u$$

$$\phi(1) = (n-1)(n-2)/3 .$$

Comparing this with the other values found above:

$$\sum \operatorname{cosec}^2 2\pi r/n = 2m + \phi(1) = n-1+(n-1)(n-2)/3 = (n^2-1)/3$$

and $$S = \sum_1^{n-1} = 2\sum_1^{m} = (n^2-1)/3.$$

It might be interesting to work out another proof based on a calculation of

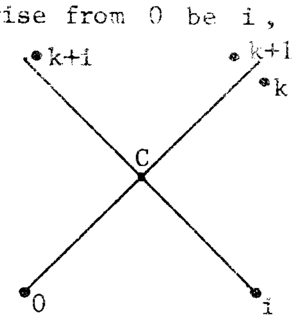
$$\sum_{j=0}^{n-1} \operatorname{cosec}^2(\alpha + 2\pi j/n)$$

where α is any angle and n any integer.

COVERING WITH TRIANGLES (I) (JCMN 15)

The centre of a regular $(2k+1)$gon is in how many triangles made from its vertices? Three readers sent in the answer $k(k+1)(2k+1)/6$, two of them commenting that it was equal to $1^2 + 2^2 + \ldots + k^2$ but that they could not see the geometrical significance of the sum of squares.

From *E. Szekeres* and *H.O. Davies*. Let the centre be C and the vertices numbered 0, 1, ... 2k in order anticlockwise. We shall count the triangles containing C with 0 as a vertex. Let the next vertex anticlockwise from 0 be i, then $i \le k$ and the third vertex (because 0C and iC are lines of symmetry) can be chosen from among i. The number of triangles with 0 as a vertex is therefore $1+2+\ldots+k = \frac{1}{2}k(k+1)$. Adding the numbers for all vertices gives each triangle three times, hence the result.



From *A.P. Guinand*. It is easier to count the triangles not containing the centre. Consider such a triangle, let the right-hand vertex seen from C be numbered 0, there are $k-1$ triangles with second vertex 1, viz. (0, 1, 2), ... (0, 1, k), and similarly $k-2$ with second vertex 2, and so on, making $(k-1)+(k-2)+\ldots+2+1 = \frac{1}{2}k(k-1)$. The number of triangles not containing the centre is therefore $\frac{1}{2}k(k-1)(2k+1)$. Subtracting this from the total $k(4k^2-1)/3$ gives the result above.

JCMN16.

DOES IT CONVERGE? (JCMN 15)

This problem from *V. Laohakosol* was about

$$S_n(x) = (1+x(1+x^2(\ldots(1+x^n)^{\frac12}\ldots)^{\frac12})^{\frac12})^{\frac12}.$$

Our first solution is due jointly to *G. Szekeres* and *B.C. Rennie*. The expression is monotonically increasing with x for each positive integer n, and with n for each $x > 0$.

$$S_n^{\,2}-1 = x(1+x^2(\ldots + x^{n-1}(1+x^n)^{\frac12}\ldots)^{\frac12})^{\frac12} \quad \text{----------} \quad (1)$$

In each bracket on the right of the equation divide both terms by x^2 and increase by one the power of x outside the bracket on the left

$$S_n^{\,2}-1 = x^2(x^{-2}+x(\ldots x^{n-2}(x^{-2}+x^{n-2})^{\frac12}\ldots)^{\frac12})^{\frac12}$$

For all $x \geq 1$, since $x^{-2} \leq 1$ and $x^{n-2} \leq x^{n-1}$ we have:

$$S_n^{\,2}-1 \leq x^2(1+x(\ldots x^{n-2}(1+x^{n-1})^{\frac12}\ldots)^{\frac12})^{\frac12} = x^2 Sn-1$$

$$S_n^{\,2}-1 \leq x^2 S_{n-1} \leq x^2 S_n$$

Therefore S_n is between the two roots of the quadratic $b^2 - x^2 b - 1 = 0$, that is $S_n(x) \leq \frac12 x^2 + \frac12(x^4+4)^{\frac12}$ for $x \geq 1$. It follows that $S_n(x)$ converges to some limit $S(x)$ for all $x \geq 1$, and also it converges for $0 \leq x \leq 1$ because $S_n(x) \leq S_n(1) \leq S(1)$. For $0 \leq x \leq 1$ reasoning like that above can be used, but since $x^{-2} \geq 1$ and $x^{n-2} \geq x^{n-1}$ the inequality is reversed and $S(x) \geq \frac12 x^2 + \frac12(x^4+4)^{\frac12}$.

Now to find inequalities the other way. For $0 < x \leq 1$, since $x^2 \leq x, \ldots x^n \leq x^{n-1}$ it follows from (1) that

$$S_n^{\,2}-1 \leq x(1+x(\ldots+x^{n-2}(1+x^{n-1})^{\frac12}\ldots)^{\frac12})^{\frac12} = x\,S_{n-1}$$

and taking the limit as n tends to infinity

$$S(x) \leq \tfrac12 x + \tfrac12(x^2+4)^{\frac12}.$$

For $x > 1$ a simple inequality comes from:

$$S_n(x) > (x(x^2(\ldots(x^n)^{\frac12}\ldots)^{\frac12})^{\frac12})^{\frac12}$$

and since the infinite sum $1/2 + 2/4 + 3/8 + \ldots = 2$, it follows that $S(x) \geq x^2$. To sum up

$$\tfrac12 x^2+\tfrac12(x^4+4)^{\frac12} \leq S(x) \leq \tfrac12 x+\tfrac12(x^2+4)^{\frac12} \quad \text{for } 0 < x \leq 1$$

and

$$x^2 \leq S(x) \leq \tfrac12 x^2+\tfrac12(x^4+4)^{\frac12} \quad \text{for } x \geq 1.$$

In the first of these inequalities the difference between the upper and lower bounds becomes zero at $x = 0$ and $x = 1$. In the second the difference is $O(1/x^2)$ for large x.

The other solution came from *J.B. Parker* who proved convergence and gave a smaller upper bound for the limit as follows, taking $x \geq 1$

$$x^{-2}(S_n^{\,2}-1)^2-1 = x^2(1+x^3(\dots x^{n-1}(1+x^n)^{\frac{1}{2}}\dots)^{\frac{1}{2}})^{\frac{1}{2}}$$

$$= x^3(x^{-2}+x^2(\dots x^{n-2}(x^{-2}+x^{n-2})^{\frac{1}{2}}\dots)^{\frac{1}{2}})^{\frac{1}{2}}$$

$$\leq x^3(1+x^2(\dots x^{n-2}(1+x^{n-1})^{\frac{1}{2}}\dots)^{\frac{1}{2}})^{\frac{1}{2}} = x^2(S_{n-1}^{\,2}-1)$$

An argument very much like the one above proves convergence and gives

$$S(x) \leq (1+\tfrac{1}{2}x^4+\tfrac{1}{2}(x^8+4x^2)^{\frac{1}{2}})^{\frac{1}{2}} .$$

The correspondingly improved lower bound comes from (1)

$$S_n^{\,2}-1 > x(x^2(\dots x^{n-1}(x^n)^{\frac{1}{2}}\dots)^{\frac{1}{2}})^{\frac{1}{2}} = x^{1+2/2+\dots n/2^n}$$

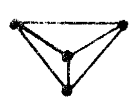
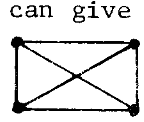
Taking the limit as n tends to infinity, $S(x) \geq (1+x^4)^{\frac{1}{2}}$ for $x > 1$.

The two bounds establish that $S(x) = x^2 + \tfrac{1}{2}x^{-2} + O(x^{-4})$ for large x.

Another question now suggests itself, does this function $S(x)$ continuous on the positive real axis have an analytic continuation into the complex plane?

COVERING WITH TRIANGLES (II)

Given any n points in the plane, the line segments joining them divide the convex hull into polygonal regions. Supposing that no three of the n points are in a line and that no three of the line segments meet in a point, what are the possible values of the number of regions? For example $n=4$ can give either three or four regions:

For each region we can count the number of triangles formed from the vertices that cover the region (considering only the interior and ignoring the boundary points). Let T be the largest of the counts for the different regions. We have seen that for the regular $(2k+1)$gon $T = k(k+1)(2k+1)/6$. Are there other arrangements of $2k+1$ points giving a smaller value of T than this? or giving a larger value?

M.J.C. Baker

THE PANCAKE PROBLEM (JCMN 14)

A stack of n pancakes sits on a plate. They are to be rearranged so that the smallest is at the top, the next smallest second, and so on down to the largest at the bottom. The only permitted move is put a lifter anywhere in the stack and invert the pile above it. What is the minimum number $f(n)$ of moves that is needed to achieve the final arrangement?

G. Szekeres writes that he had suggested this problem for a group of fifth-formers at a Summer Research Project run by the UNSW for talented high school students last December. The group consisted of *David Batts, Sam Needham, Alison Nicholson, Barry Martin, Cecilia Bjorksten* and *Christina Gunner,* supervised by *M.D. Hirschhorn.*

They found by hand the values

n	1	2	3	4	5
f(n)	0	1	3	4	5

and the inequalities $f(n) \geq n$ for $n \geq 4$ and $f(n+1) \leq 2 + f(n)$.

Turning then to a computer they were able to find $f(6) = 7$ and $f(7) = 8$.

There are two arrangements of six pancakes that require seven flips, they are 536142 and 462513 (where the pancakes are numbered in increasing order of size and where the left of the row means the top of the pile) and there are 35 configurations of 7 needing 8 flips.

A natural conjecture is that $f(n+1) \geq 1 + f(n)$. Can you prove it? or even just $f(8) \geq 9$?

SOLUTION FOR "RIEMANN ARE YOU THERE?"

For values of n up to 7 it is true that all the complex roots have real part a half, but if $z = \cdot 7187284 + \cdot 7158679i$ then $(1-z)^8 = 1-z^8 = -\cdot 1211643 + \cdot 0178858i$.

USELESS INFORMATION

$$0 < 10691/462 - \exp \pi < 10^{-8}$$

Your editor would like to hear from you anything connected with mathematics or with James Cook.

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