JAMES COOK MATHEMATICAL NOTES

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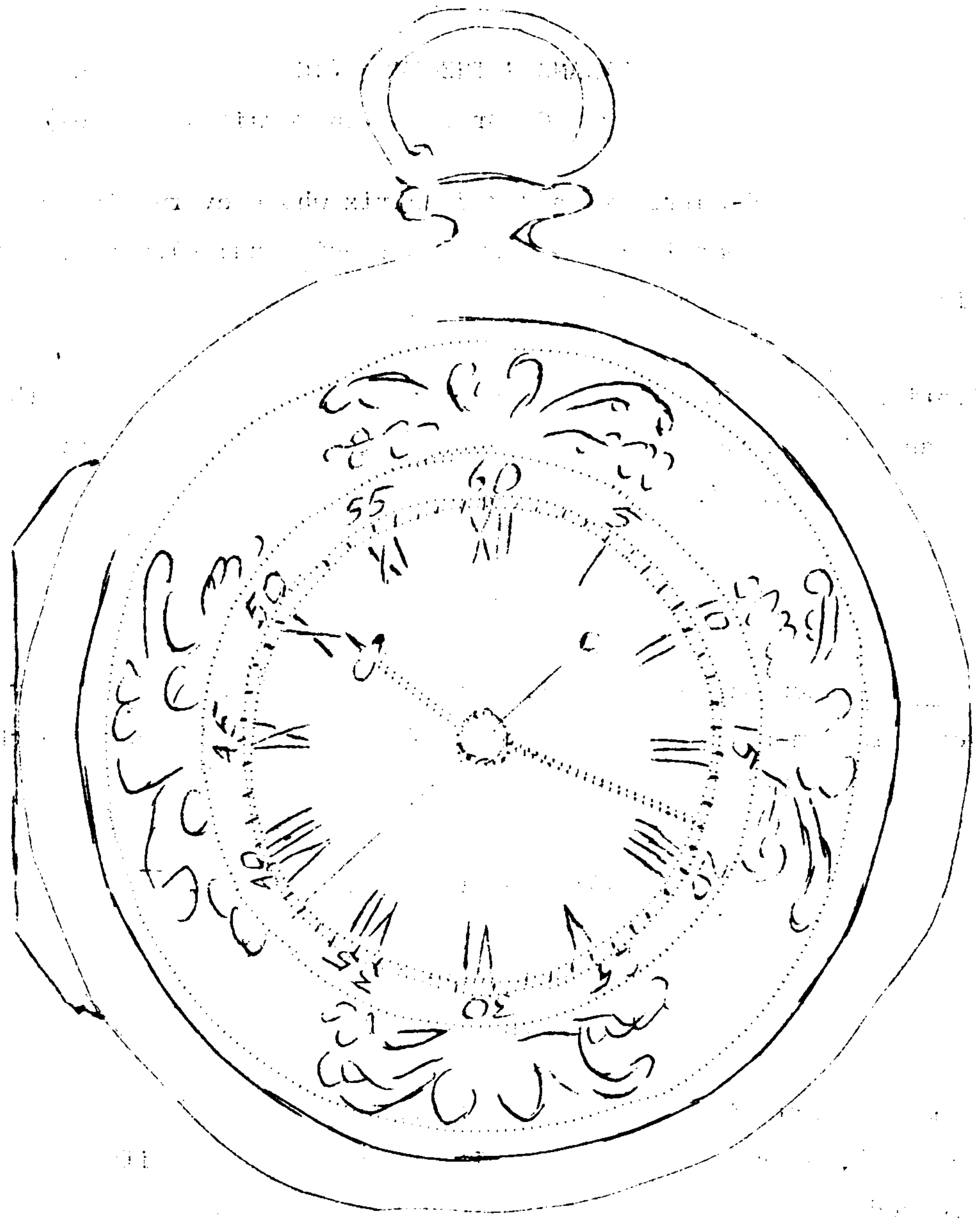

The clock known as K1 made by Larcum Kendal in 1769 as a replica of John Harrison's H4. The picture is about full size. Captain Cook used K1 on his second voyage (1772-1775) in the Resolution.
(See the article by J.B. Parker on Page 7.)

NORMS (JCMN 11)

The norm of a real vector being undefined except that it has to satisfy the usual axioms, we use it to define the norm of a square matrix A as $\max \| Ax \| / \| x \|$. The problem of Kestelman was to show that the norm of

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$ must be strictly greater than one.

A proof came in from C.A. Davis. For brevity use the notation and ideas from the first page of JCMN 11. Put $A = I + B$, then $B^2 = 0$ and $A^n = I + nB$. By the triangle inequality $n\| Bx \| = \| A^n x - x \| \le \|A^n x\| + \|x\| \le (\|A\|^n + 1) \| x \|$. If $\| A \| \le 1$ then the inequality would lead to a contradiction when n increases and x is any vector for which $Bx \neq 0$.

GUINAND'S THEOREM (JCMN 3, 4, 5 and 6)

by C.G. Broyden (University of Essex)

Definitions. A G-matrix is a 3 × 3 matrix whose every diagonal element is zero. It is a proper G-matrix if also every off-diagonal element is non-zero, otherwise it is an improper G-matrix.

Editorial note: In JCMN 3 it was set as a problem to prove that if ABC = I and if A and B are proper G-matrices and if two diagonal elements of C are zero then so is the third. J.B. Parker produced an ingenious geometrical proof using the theorem of Pappus, but A.P. Guinand later told me that it all started with Pappus' theorem, he thought there ought to be a neat algebraic proof.

Lemma 1. A non-singular G-matrix is proper if and only if every element of its inverse is non-zero. The proof is left to the reader because this issue is short of space ---- Editor.

The inverse of a non-singular G-matrix G must be

$$G^{-1} = \begin{bmatrix} a & b & c \\ \lambda a & \lambda b & \mu c \\ \nu a & \lambda\nu b/\mu & \nu c \end{bmatrix}$$

because the principal 2 × 2 sub-matrices must be singular. None of the parameters a, b, c, λ, μ or ν can be zero. It is now readily verified that G^{-1} may be expressed as

$$G^{-1} = \Lambda\ B\ A \qquad (1)$$

where $\Lambda = \text{diag}\ (1, \lambda, \nu)$

$A = \text{diag}\ (a, b, c)$

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & \beta \\ 1 & 1/\beta & 1 \end{bmatrix} \qquad (2)$$

and $\beta = \mu/\lambda \neq 1$ (for nonsingularity)

We consider now the necessary and sufficient conditions for the product of the inverses of the two proper G-matrices G_1 and G_2 itself to be a G-matrix. From equation (1) we have, with obvious notation,

$$(G_1G_2)^{-1} = G_2^{-1}G_1^{-1} = \Lambda_2B_2A_2\Lambda_1B_1A_1$$

and since we are interested only in whether the diagonal elements of the product are zero we need only, since A_2 and A_1 are both diagonal and nonsingular, consider the product B_2RB_1, where $R = A_2\Lambda_1$. If

$$R = \text{diag}\ (r, s, t) \qquad (3)$$

it follows from equation (2) that

$$B_2RB_1 = \begin{bmatrix} r+s+t & x_{12} & x_{13} \\ x_{21} & r+s+t\ \beta_2/\beta_1 & x_{23} \\ x_{31} & x_{32} & r+s\ \beta_1/\beta_2 + t \end{bmatrix} \tag{4}$$

where the values of the elements denoted by x_{ij} are irrelevant, so that the necessary and sufficient conditions for B_2RB_1 to be a G-matrix are

$$r + s + t = 0 \tag{5}$$

and $$\beta_1 = \beta_2 \tag{6}$$

Note also that the vanishing of any two diagonal elements of B_2RB_1 implies the vanishing of the third (Guinand's Theorem).

It is now possible to demonstrate directly, using equations (4) - (6), that if B_2RB_1 is a G-matrix then it is a proper G-matrix. A simpler and more rewarding proof follows from

Lemma 2. The product of a G-matrix with an improper G-matrix cannot be the inverse of a proper G-matrix.

Proof. Let G_1 and G_2 be G-matrices with G_1 improper. Then at least one row of G_1 has but a single nonzero element so that at least one element of G_1G_2 is zero, and the proof then follows from Lemma 1.

The result leads immediately to the following

Theorem. If G_1, G_2 and G_3 are G-matrices satisfying

$$G_1G_2G_3 = I \tag{7}$$

then either every factor G_i is proper, or no factor G_i is proper.

Proof. Let one factor, G_1 say, be improper. Then $G_2^{-1} = G_3G_1$ and $G_3^{-1} = G_1G_2$ so that, from Lemma 2, both G_2 and G_3 are improper. Similar arguments in respect of G_2 and G_3 establish the theorem.

Assume now that G_1, G_2 and G_3 are proper G-matrices satisfying equation (4). It follows from equation (1) that

$$G_i = \Lambda_i B A_i \tag{8}$$

where, since each factor G_i is the product of the inverses of the two other factors, the matrix B is the same for all values of i (equation (6)). Inverting equation (7) thus yields

$$\Lambda_3 BA_3 \Lambda_2 BA_2 \Lambda_1 BA_1 = I$$

and this may be rearranged to give

$$BR_1 BRB = R_2^{-1} \tag{9}$$

where $\quad A_3\Lambda_2 = R_1$ (10a)

$$A_2\Lambda_1 = R \tag{10b}$$

and $\quad A_1\Lambda_3 = R_2$ (10c)

Now R, R_1 and R_2 are diagonal matrices and expansion of the left-hand side of equation (9) using equations (2) and (5) shows that the product BR_1BRB is diagonal if, and only if,

$$R_1 = k_1 R \tag{11a}$$

where k_1 is an arbitrary scalar. Similarly

$$R_2 = k_2 R \tag{11b}$$

so that equations (9) may be written

$$K^3 = L^3 I \tag{12}$$

where $\quad K = BR$

and $\quad L^3 = (k_1k_2)^{-1}$ (14)

Finally, evaluation of the diagonal elements of $K_1K_2(BR)^3$ shows that these are equal to unity if, and only if

$$k_1k_2 \; rst(\beta + \beta^{-1}-2) = 1 \tag{15}$$

where β is defined in equation (2) and R = diag (r, s, t).

The preceding analysis shows that if G_1, G_2 and G_3 are proper G-matrices satisfying equation (7) they are each related, through equations (8) and (10) - (13), to the single matrix K. This matrix is not arbitrary since it is required to satisfy both equation (12) and the condition that K^2 is a proper G-matrix. It is trivial to show, though, that any matrix that does satisfy these conditions may be used to generate three Guinand factors satisfying equation (7).

We now consider the eigenvalues of K. From equation (12), it follows that, with the usual notation, the eigenvalues of K are L, $L\omega$ and $L\omega^2$. Finally we note that, since $\det(B) = \beta + \beta^{-1}-2$ and $\det(R) = rst$, the cube of equation (15) may be obtained by taking determinants of the equation

$$K_1K_2(BR)^3 = I.$$

In order to investigate the possibility of an extension of this theory to 4 × 4 matrices it is instructive first to consider the case of 2 × 2 matrices. It is readily verified that the product of three matrices, each having zeros on the diagonal cannot be diagonal but that the product of two such matrices always is. This, together with the eigenvalues of K deduced above, suggests that if an analogue of Guinand's theorem exists for 4 × 4 matrices it should refer to the product of four factor matrices. That improper 4 × 4 Guinand factors exist is deduced immediately from the fourth order cyclic permutation matrix

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

for which $K^4 = I$ and the diagonal elements of K^3 are all zero. If it were possible to construct a matrix K such that $K^4 = I$, the diagonal elements of K^3 were all zero and the off-diagonal elements of K^3 were all nonzero it would be possible to generate four "proper" G-matrices G_i, $i = 1, 2, 3, 4$ such that

$$G_1 G_2 G_3 G_4 = I$$

Whether or not then a theorem analogous to Guinand's Theorem could then be discovered, though, is at the moment a matter of conjecture.

SOME HARD ANALYSIS

Let n be a positive integer, and let y be defined in terms of x by the equation $\sin(x/2) = \sinh(y/2)$. Prove or disprove

$$\int_0^{\pi} \cot(x/2) \sin(nx)\, e^{-ny}\, dx = \pi/2$$

and

$$\int_0^{\pi} \cot(x/2) \sin(nx)\, e^{ny}\, dx = 3\pi/2 - (-1)^n \pi$$

ANECDOTE ONE (JCMN 5, 6 and 7)
MORE TRIGONOMETRY (JCMN 10 and 11)

There is a connection between these contributions. From the formula

$$\sum_1^n \tan^2 r\pi/(2n+1) = n\,(2n+1)$$

we may obtain (replacing r by n+1-r)

$$\pi^2 = \sum_1^n \frac{\pi^2}{n(2n+1)} \cot^2 \frac{2r-1}{4n+2}\pi$$

and taking the limit as n tends to infinity (Weierstrass M-test)

$$\pi^2 = \sum_1^{\infty} 8/(2r-1)^2$$

(JCMN 12)

MATRIX NORMS AGAIN

In n-dimensional complex vector space take any norm satisfying the usual axioms, that $\|x\| \geq 0$ with equality only when $x = 0$, $\|tx\| = |t|\ \|x\|$ and the triangle inequality. The norm of a square matrix A is $\|A\| = \max \|Ax\|/\|x\|$. Prove or disprove that if $\lim A^r = C$ then $\Sigma \|A^r - C\|$ is finite.

H. Kestelman.

EQUATIONS (JCMN 11)

If $a_1, \ldots a_n$ are the complex zeros of a polynomial f(z) and are distinct, show that $\sum a_r^k/f'(a_r)$ is zero if k is 0, 1, ... n-2 and is 1 if k = n-1. (Apologies for omission of the condition that the leading term of the polynomial must be z^n.)

The four solutions that have come in can be classified into three equivalence classes, *Vichian Laohakosol* and *John Parker* both use complex integration, as follows. The given sum is the sum of the residues of the function $z^k/f(z)$ at all its poles. The integral round a large circle tends to zero when $k < n-1$, but if $k = n-1$ the function is approximately $1/z$ on a large circle and so the integral tends to $2\pi i$, and in fact equals $2\pi i$.

G.M.L. Gladwell uses first the partial fraction expansion of $1/f(z)$

$$\frac{1}{f(z)} = \sum \frac{1}{f'(a_r)(z-a_r)}$$

for $|z| > \max |a_r|$ expand both sides of this equation in powers of $1/z$.

$$z^{-n} + \sum a_r z^{-n-1} + \ldots = \sum_{k=0}^{\infty} \left\{ \sum_{r=1}^{n} a_r^k/f'(a_r) \right\} z^{-k-1}$$

Equating coefficients of z^{-1}, z^{-2}, etc. gives the required result. In fact this method also gives the sum in the case k = n, and, with a little bit more work, for larger values of k.

C.A. Davis expresses $z^{k+1}/f(z)$ in partial fractions and lets z tend to zero.

TWO POINTS IN A TRIANGLE (JCMN 1, 2 and 3)

Given any triangle ABC, thinking of it as in the complex plane, the two points L and N may be defined as the stationary values of a cubic that vanishes at the vertices A, B and C. *C.A. Davis* has written to say that L and N are the foci of the ellipse that touches the sides of the triangle at their mid-points, which is the inscribed ellipse of maximal area. Can anyone supply a proof?

CAPTAIN COOK AND THE DETERMINATION OF LONGITUDE

Determination of a ship's longitude was an unsolved problem until the second half of the eighteenth century. The vital nature of the problem had, however, been recognized much earlier, and in 1714 (a few years after a navigational disaster to the English fleet off the Scilly Isles), a Board of Longitude was set up, a maximum prize of £20,000 being offered for determining longitude with an error of not more than half a degree.

The two main lines of attack were to develop a device for measuring GMT accurately throughout what could be a very lengthy voyage, and the method of Lunar Distances. The affinity between longitude and time had long been recognized (1 degree of longitude corresponding to 4 minutes of time) so that if the positions of the heavenly bodies were known relative to some reference meridian, a single appropriate astronomical sight would yield longitude provided GMT were known. There long remained two snags. First GMT, as opposed to local time, could not be obtained nearly accurately enough and secondly the existing instrumentation for measuring the altitude of a star was inadequate. The mathematicians therefore pooh-poohed this line of attack, even after the appearance of Hadley's octant in 1732, which removed the second difficulty.

The Lunar Distance approach was the mathematicians' favoured route. First one measured the distance of the Moon's limb from the limb of the sun or from a suitable star, using the sextant (a slight refinement of Hadley's octant). The exact <u>local</u> time of this measurement was noted. Next, using Lunar Distance tables, the Greenwich time for which the distance between the moon and the other body was equal to that observed was found. After a foolscap page of calculations, longitude was found. Though long recommended by the mathematicians, it was not until reliable Lunar Distance tables became available (Tobias Mayer, 1755) that the method could be used to yield longitude reasonably accurately.

The necessary complex calculations could be swept aside if accurate <u>GMT</u> could be carried on board ship. In 1763 Harrison's chronometer was put to the test on a long sea voyage and performed very well, the error after 147 days being under two minutes (≡ half a degree in longitude) yet Harrison was only given £10,000, some of the members of the Board of Longitude not being satisfied.

So at the time of Cook's second voyage (1772) there were two viable methods of determining longitude, though neither had by any means gained wide acceptance. It was the thoroughgoing approach of Captain Cook that nailed the matter once and for all. Equipped with modern instrumentation, including two chronometers (one by Kendall, a replica of Harrison's) and a Hadley sextant, astronomical observations were systematically carried out throughout the voyage and longitude determined by both methods. At the close of the second voyage the error in the Kendall Watch was only about 30 seconds (8 minutes in longitude) and the argument was at an end. Harrison (by now a very old man) got his outstanding ten thousand quid.

J.B. Parker

BICENTENNIAL PROBLEM

Every function of the form $p(z) + \exp z$, where p is a polynomial (not identically zero), has a complex zero. C.A. Davis points out that because the zeros of p are bounded the function $\exp(-z)p(z)$ has no zero in a neighbourhood of the isolated essential singularity at infinity. By Picard's second theorem the function takes every value except zero in the neighbourhood, and in particular it takes the value -1, and so $p(z) + \exp(z)$ has a zero.

Dame Mary Cartwright also comments. Your bicentennial problem is a very special case of the extensive theory of exceptional values, Nevanlinna and Picard are the great names. There is a technique used by Hardy about 1904 on a more complicated function to find a zero. Our editor, unable to find Hardy's paper, tried to reconstruct a direct way to find a zero.

Without loss of generality we may take the equation as

$$\exp z = z^n + a_1 z^{n-1} + \dots + a_n .$$

Then for any large positive integer m take $z_0 = n \log (2\pi m) + \pi i(2m+n/2)$ as first approximation. In the circle of unit radius round this point the function $F(z) = \log (z^n + \dots)$ with the imaginary part defined as between $\pi(2m+n/2\pm1)$ is a contraction mapping with contraction ratio $O(1/m)$ because $|F'(z)| = O(1/m)$ uniformly in the circle, and F maps z_0 into a point at distance $O(1/m)$ from z_0. Therefore F has a fixed point.

NUMBER THEORY

Given integers a and b, for what positive integers n does $pq \equiv a \pmod n$ imply $p+q \equiv b \pmod n$?

C.A. Davis

INEQUALITY WANTED (JCMN 10 and 11)

If $0 < a_1 \leq \ldots \leq a_n$ is there an inequality

$$(k/a_n) \sum_{1\leq i<j\leq n} (a_i-a_j)^2 \leq \frac{a_1+\ldots+a_n}{n} - (a_1 \ldots a_n)^{1/n} \leq (k/a_1) \sum_{1\leq i<j\leq n} (a_i-a_j)^2$$

where k is a function of n?

Solution from *G. Szekeres*:-

The correct constant in Williams' inequality is $1/(2n^2)$. Take the right-hand inequality, the other can be proved similarly. More generally consider (for $0 < k < n$)

$$(1) \quad \ldots \quad \frac{a_1+\ldots+a_k + mb}{n} - (a_1 \ldots a_k b^m)^{1/n} \leq \frac{1}{2n^2 a_1} \sum_{1\leq i<j\leq k} (a_i-a_j)^2 + \frac{m}{2n^2 a_1} \sum_1^k (b-a_i)^2$$

where $m = n-k$ and $0 < a_1 \leq \ldots \leq a_k \leq b$. The original inequality corresponds to to the case where $m = 1$ and $b = a_n$. Inequality (1) follows if we can show:-

(i) The b-derivative of the left hand side of (1) is $\leq$ the b-derivative of the right hand side,

and

(ii) the inequality holds for $b = a_k$.

The second statement (ii) follows trivially by induction on k if we can also prove the first part (i). Now the b-derivatives are

$$(m/n) \left(1-(a_1 \ldots a_n)^{1/n} b^{m/n-1}\right) \quad \text{and} \quad m/(n^2 a_1) \sum_1^k (b-a_i)$$

and we must show:-

$$(2) \quad \ldots \quad 1-(a_1 \ldots a_k)^{1/n} b^{m/n-1} \leq 1/(na_1) \sum_1^k (b-a_i)$$

for all $b \geq a_k$. Again taking b-derivatives, the derivative of the left hand side is

$$(1-m/n) (a_1 \ldots a_k)^{1/n} b^{m/n-2}$$

$$= (k/n) \left((a_1/b) \ldots (a_k/b)\right)^{1/n} (1/b) \leq k/(na_1)$$

which is the derivative of the right hand side.

For $b = a_k$ the inequality (2) becomes

$$1 - (a_1 \ldots a_{k-1})^{1/n} b^{(m+1)/n-1} \leq 1/(na_1) \sum_1^{k-1} (b-a_i)$$

which follows by induction on k because when $k = 1$ both sides are zero. This establishes (2) and therefore (1).

(JCMN 12)

ACCORDING TO COCKER

Seven sciences supremely excellent,
Are the chief stars in Wisdom's firmament?
Whereof Arithmetic is one, whose worth
The beams of profit and delight shine forth:
This crowns the rest, this makes man's mind compleat;
This treats of numbers, and of this we treat.

From Mr. Edward Cocker's Preface to his Arithmetic, published posthumously in 1677.

RATIONAL APPROXIMATIONS (JCMN 11)

Show that $T_n(z) = (z^2-1)^{\frac{1}{2}}\, U_{n-1}(z) + O(z^{-n})$ for large z and hence find the Padé approximation $(1-x^2)^{\frac{1}{2}} = P_n(x)/Q_n(x) + R(x)$ where P_n and Q_n are polynomials of degree n and where $R = O(x^{2n})$ for small x.

Similar solutions from C.A. Davis and J.B. Parker were as follows. The numbers may be thought of as real, but with a little care the complex case is also covered. Given z, take t so that $z = \cosh t$ and $s = \sinh t$ is one of the two square roots of z^2-1. The Chebyshev polynomials may be defined by

$$T_n(z) = \cosh nt \text{ and } s\, U_{n-1}(z) = \sinh nt. \text{ Then}$$

$$T_n(z) - s\, U_{n-1}(z) = e^{-nt} = (z+s)^{-n}$$

If $0<|x|<1$ put $z = 1/x$ and note that $x^n T_n(z) = P_n(x)$ and $x^{n-1}U_{n-1}(z) = Q_n(x)$ are polynomials of degree n or n-1 or n-2.

Multiplying the equation above by $x/U_{n-1}(z)$ gives

$$xs = P_n(x)/Q_n(x) - x^{2n}(1+xs)^{-n}/Q_n(x)$$

Now it is necessary to choose the square root. Take $xs = (1-x^2)^{\frac{1}{2}}$ to be the value that has real part positive, then the remainder $R \sim -x^{2n}/2^{2n-1}$ for small x. The proposer, G.M.L. Gladwell, asked also for a generalisation, but none has yet come in.

PROBLEM UNSUITABLE FOR UNDERGRADUATES (JCMN 11)

The real function $f(x)$ is defined for all real x and has finite derivative at every point of the set A, which has Lebesgue measure zero. Does the image set $f(A)$ also have measure zero? The answer is YES, and similar proofs came from *J.M. Hammersley* and *H. Kestelman.*

Let E_{mn} (for positive integers m and n) be the set of x in A such that if $|h| < 1/n$ then $|f(x+h) - f(x)| < m|h|$. The union of all E_{mn} is A which has measure zero, therefore each E_{mn} has measure zero and so can (for any $\varepsilon>0$) be covered by open intervals $I_r = (p_r - q_r, p_r + q_r)$ such that $q_r < 1/n$ and p_r is in E_{mn} and $\sum q_r < \varepsilon$. If x is in such an interval I_r then $|f(x) - f(p_r)| < m|x-p_r| < m\, q_r$ and so $f(I_r)$ is in an interval of length no more than $2mq_r$. Therefore $\mu f(E_{mn}) \leq 2m \sum q_r < 2m\varepsilon$. This is for all $\varepsilon > 0$ and so $\mu f(E_{mn}) = 0$. Finally $f(A)$ is the union of all $f(E_{mn})$ and so has measure zero.

J.M. Hammersley adds the comment that it is not necessary to assume that f is differentiable, it is sufficient if f satisfies a Lipschitz condition of order 1 at each point of A.

TWO QUESTIONS ON BINOMIAL COEFFICIENTS

These come from *C.A. Davis.*

(a) If n is a positive integer and

$$u(k) = \sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} \nu^{k}$$

show that $u(k) = 0$ for $k = 0, 1, \ldots n-1$, and find $u(n)$ and $u(n+1)$.

(b) If k and r are non-negative integers show that

$$\sum_{0}^{k} (-1)^{s} \binom{k}{s} \binom{s-k}{r} = (-1)^{k} \binom{-k}{r-k}$$

(JCMN 12)

CAN YOU SOLVE A QUADRATIC EQUATION?

If a and b are real positive parameters, what conditions must they satisfy for the quadratic

$$z^2 + az + b + ia = 0$$

to have the real parts of both roots negative?

Readers may be interested in the origin of this problem. Prof. J.F. Ward of the Physics Department of JCUNQ showed me an article *"The Motion of a Small Sphere in a Rotating Velocity Field: A Possible Mechanism for Suspending Particles in Turbulence"* by Paul F. Tooby, Gerald L. Wicks and John D. Isaacs in the Journal of Geophysical Research, Volume 82, pages 2096-2100. This work was motivated by interest in the suspension and transport of sediment by turbulent water.

Suppose that a small object when in still water obeys a law of motion: External force = $M\ddot{x} + A\dot{x} + B$, where M is an effective inertia, A is the factor giving the resistance to motion at small Reynolds numbers, x is the position vector and B is the product of the volume and the pressure gradient, which is the buoyancy term. Consider this particle in water rotating uniformly (as if solid) about a fixed horizontal axis. In the plane perpendicular to this axis take coordinates x horizontally and y vertically, with origin on the axis, and let $z = x + iy$.

The equation of motion is of the form

$$M\ddot{z} + A(\dot{z} + i\omega z) + C(\omega^2 z - ig) + mig = 0$$

If M and $C\omega^2$ are small enough to be neglected the equation has a solution giving a circular orbit for the particle, but if these terms are included they will modify the circular orbit and the question arises whether they make the orbit shrink to a point (so that the particle remains suspended in the eddy) or make the orbit expand (so that the particle leaves the eddy and probably pursues a random path tending to settle on the bottom). This leads to the problem given above about the two roots of a quadratic.

Your editor would like to hear from you anything connected with mathematics or with James Cook.

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