Asymptotically log Fano varieties (case of log del Pezzos)

Vanya Cheltsov (Edinburgh) :)
joint with Yanir Rubinstein

1) Remind: X = algebraic variety is Fano if $-K_X$ = ample (mild) ample)

Recall $|-nK|: X \hookrightarrow \mathbb{P}^N$

dim 2
smooth
$\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$
Blow up of $\mathbb{P}^2$ $n \leq 8$ points

dim 3, smooth
Iskovskikh, Mori, Mukai
105 families

dim ≥ 4
smooth
OPEN

2) Log Fano variety (X, Δ) Δ = effective $\mathbb{Q}$-divisor

$\Delta = \sum a_i \Delta_i$ $a_i \geq 0$
$a_i \in \mathbb{Q}$

Def: (X, Δ) is log Fano $-(\underline{K_X + \Delta})$ = ample
log anticanonical

KLT log term
(X, Δ) = Kawamata log

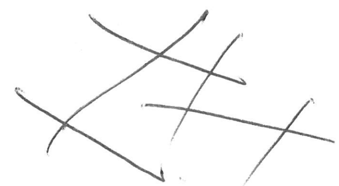
Log FANO (X,Δ) s.t. (X,Δ) = KLT
$-(K+\Delta)$ = ample

$\Delta = 0$
X = KLT

→ dim 2
quotient
sing.

OPEN (∃ some Results)

Def: (X,Δ) = log smooth
if X = smooth
Δ = s.n.c ~~Sup~~
Supp Δ = s.n.c.



GOOD SINGULARITY

TestA ← "Big RATIONAL SURFACES" more or less
the same problem $-(K_X+\Delta)$ = ample

Ex: X = smooth surface, $\Delta = \sum a_i C_i$
C_i = smooth cur
Supp Δ = s.n.c.
$a_i \in \mathbb{Q}_{\geq 0}$

↳ $-(K+\Delta)$ = a

Ex: Kollár : 5-dim real manifold compact
$\hookrightarrow$ Einstein metric on it with positive curvatur

$\swarrow$

(S, Δ) = log Fano (log del Pezzo)

S = ~~Kø~~ smooth ~~surface~~

nice classification
$\Downarrow$
"gives answer"

local $\mathrm{Supp}\,\Delta$ = s.n.c. divisor

$\Delta = \sum a_i C_i$

standard $a_i = \frac{m_i - 1}{m_i}$ $m_i \in \mathbb{N}$

$a_i = \frac{m_i - 1}{m_i}$ $\rightsquigarrow$ orbifold setting

$(X, \Delta = \sum a_i D_i)$

Supp Δ = s.n.c.

STANDARD (Shokurov)

$\forall a_i \le 1$

$\Downarrow$

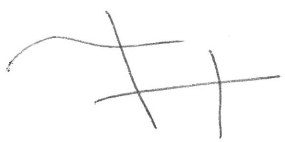
KLT

$a_i \in \left\{ \frac{m-1}{m} \right\}_{\mathbb{Q}} \cup \{ a_i \geq$ some number depends on dim $X \}$

dim $X = 2 \Rightarrow 6/7$

Donaldson + Tian on K.E. edge metrics

$(X, \sum (1-\beta_i) D_i)$ $\quad 0 \le \beta_i \le 1$ $\quad 2\pi\beta_i$ = angle



If $\beta_i = 1/m_i \implies$ orbifold metrics

Ex: X = smooth Fano, $D \in |-K_X|$ smooth

$(X, (1-\beta)D)$ log Fano

$0 < \beta \le 1$

small angles $\rightsquigarrow$ $\exists$ KE metric exists ex for $\beta \ll 1$.

Def: (X, D) = Asymptotically log Fano

$D = \sum^r D_i$ if $(X, (K - \sum(1-\beta_i) D_i)$ is log Fano

for small $(\beta_1 \dots \beta_r)$

$-(K + \sum(1-\beta_i) D_i)$ is ample fo

for ALL small

$(\beta_1 \dots \beta_r) \in (0, 1]^r$

STRONGLY

for some sequence

$\downarrow$

0.

Kollar paper + LOG SMOOTH

Describe them in low dimensions

dim = 2   S = smooth surface

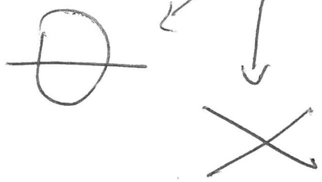
$D = \sum_{i=1}^{r} D_i$ = s.n.c.

$\Rightarrow$ S = rational by Castelnuovo

★ $-(K_S + \sum_{i=1}^{r} (1-\beta_i) D_i)$ ample for ALL small $\beta_1 \dots \beta_r$

ρ = rk Pic S = small

$\rho = 1$ $S = \mathbb{P}^2$ D $\to$ line, conic, cubic



$\rho = 2$ $S = \mathbb{F}_n$ $\to$ $D = (-n)$-section.

$\rho > 2$ By M.M.P / Log M.M.P.

$\to$ $\exists$ (-1)-curve $E \not\subset \mathrm{Supp}\, D$, $E \cap D \neq \emptyset$, $E \cdot D = 1$

$E^2 = -1$, $E = \mathbb{P}^1$

all (-1) curves $E \cap D \neq \emptyset$

$\rho \leq 2$ $\Leftarrow$ MINIMAL

S = smooth.
D = s.n.c.

Inf. many families → $(\mathbb{F}_n, f_n)$ – ~~exi~~ such)

↳ subclass iff $-(K_S + D)$ ample

then (S.D) is STRONGLY A.L. FANO

Maeda

Fujita

Theorem (–, Rubinstein) (S, D) log del Pezzo A.S.

$\rho = 1, 2$
$\exists$ lost

MINIMAL
(no (-1) curve
disjoint
from D)

you take one of these minimal
TAKE ANY number
of points on a boundary D
AND blow them up

Ex: $(\mathbb{P}^2, \text{line } L)$ $p_1 \ldots p_n$ $S \to \mathbb{P}^2$ $\tilde{L}$ $(S, \tilde{L})$

K.E. edge metric on them

$(S, D) = $ (S.A.L.F

log smooth

$-(K_S+D) =$ nef. $\Rightarrow$ semiample

$D \in |-K_S|$

$-(K_S+D) = 0$

$S =$ del Pezzo

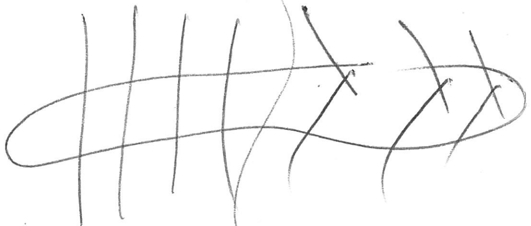
∃ K.E. edge metric exists for small β.

conic bundle

$-(K_S+D) \neq 0$

$-(K_S+D)^2 = 0$

$S \to \mathbb{P}^1$



D

∃ example with KEE in every family

$-(K_S+D)$ nef + big

NO KEE metric

$-(K_S+D) =$ ample

(MAEDA)

∄ KEE metric does not exist for $\beta \ll 1$

$\beta = 1/m$
Ross
Thomas

Ex ($\mathbb{P}^2$, Conic)

∃ KEB $\Leftrightarrow \beta > 1/4$