1)

Infinite dimensional automorphism groups of algebraic varieties, multiple transitivity, and unirationality

V. L. Popov

(July 15, 2014, Longyearbyen).

$\bar{k} = k$, any char k.

Guiding statement:

Thm. G algebraic group

$\varphi_i : T_i \to G$, $i \in I$, morphisms s.t.
T_i irred. var
$e \in X_i := \varphi_i(T_i)$

A - a subgroup of G generated, as abstract group, by $M = \bigcup_{i \in I} X_i$

Then:

- M = intersection of all closed subgroups of G containing M
- A connected
- $\exists\, (i_1, \ldots, i_n) \in I^n$ s.t.

$$A = X_{i_1}^{e_1} \ldots X_{i_n}^{e_n}, \quad e_i = \pm 1 \ \forall i$$

It appears that the same holds if G is replaced by $\operatorname{Aut} X$, X irred alg. var, & φ_i is defined in a proper way. Such group $\operatorname{Aut} X$ shares many important properties of alg. groups concerning orbits, quotients, and invariant fields

Definition X irred alg. var.

T irred. alg. var. Consider a map

$$\varphi : T \to \operatorname{Aut} X, \quad t \mapsto \varphi_t$$

then

2)

- $\{\varphi_t\}_{t\in T}$ is called a *family in Aut X parametrized by T*

Put $\varphi_T := \varphi(T)$

- Let $\underline{I}$ be a nonempty collection of families in Aut X.

The subgroup of Aut X generated by $\bigcup_{\{\varphi_t\}_{t\in T}\in \underline{I}} \varphi_T$ is called *a group generated by $\underline{I}$*

- $\{\varphi_t\}_{t\in T}$ is called
  - *injective* if $\varphi_t = \varphi_s$ for $t \neq s$,
  - *unital* if $id_X \in \varphi_T$,
  - *algebraic* if $\tilde{\varphi}: T\times X \to X$, $(t, x) \mapsto \varphi_t(x)$ is *a morphism*

- $\{\varphi_t^{-1}\}_{t\in T}$ is called *the inverse of $\{\varphi_t\}_{t\in T}$*

- Given $\{\varphi_t\}_{t\in T}, \dots, \{\psi_s\}_{s\in S}$,

$\{\varphi_t\circ \dots \circ \psi_s\}_{(t,\dots,s)\in T\times\dots\times S}$ is called *the product of these families*.

Properties of being algebraic and unital are preserved under taking inverses and products.

- Let $\underline{I}$ be a collection of families in Aut X then $\{\varphi_t\}_{t\in T}$ in Aut X is called *derived from $\underline{I}$* if $\{\varphi_t\}_{t\in T}$ is a product of families each of which is either a family from $\underline{I}$ or its inverse.

3)

- $G < \operatorname{Aut} X$ is called finite dimensional if $\exists\, n$ s.t. $\dim T \le n$ $\forall$ injective algebraic family $\{\varphi_t\}_{t\in T}$ in G. Smallest n with this property is called $\dim G$. If such an n does not exist, then G is called infinite dimensional.

- $G < \operatorname{Aut} X$ is called connected if $\forall\, g \in G$ $\exists$ a unital algebraic family $\{\varphi_t\}_{t\in T}$ in G s.t. $g \in \varphi_T$.

Description of all finite dimensional connected subgroups in $\operatorname{Aut} X$

- If $\{\varphi_t\}_{t\in T}$ an algebraic family in $\operatorname{Aut} X$ s.t. T is a connected algebraic group and

$$\tilde{\varphi} : T \times X \to X \ , \ (t, x) \mapsto \varphi_t(x)$$

is an action, then φ_T is a connected finite dimensional subgroup in $\operatorname{Aut} X$

Ramanujam '64: Every connected finite dimensional subgroup in $\operatorname{Aut} X$ is obtained in this way.

Lemma (connected subgroups in $\operatorname{Aut} X$)

$G < \operatorname{Aut} X$. FAE

- G connected,
- G generated by a collection I of unital algebraic families in $\operatorname{Aut} X$

Examples

- Take in Lemma I = <u>all</u> unital alg. families in Aut X. Then G is called <u>connected component of Aut X</u>. Notation: $(\mathrm{Aut}\, X)^{\circ}$.

this group may be ∞-dimensional:

<u>Example-theorem.</u> $(\mathrm{Aut}\, \mathbb{A}^n)^{\circ} = \mathrm{Aut}\, \mathbb{A}^n$.

∞-dimensionality of $\mathrm{Aut}\, \mathbb{A}^n$, $n \geq 2$ follows from

<u>thm.</u> Char $k = 0$, X affine, $\dim X \geq 2$. If Aut X contains $\mathbb{G}_a$, then $\dim X = \infty$.

- $(\mathrm{Aut}\, X)/(\mathrm{Aut}\, X)^{\circ}$ may be infinite.

<u>thm.</u> If Aut X is countable, then $(\mathrm{Aut}\, X)^{\circ} = \{id_X\}$.

<u>Examples.</u>

(1) $X \subset \mathbb{A}^3$, $X: x_1^2 + x_2^2 + x_3^2 = x_1 x_2 x_3 + a$, $a \in k$

For a generic, Aut X contains
$\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ as a subgroup of finite index.
free product

(2) X smooth quartic in $\mathbb{P}^3$

<u>Matsumura '63</u>: $(\mathrm{Aut}\, X)^{\circ} = \{id_X\}$, and if X is generic, there is a bijection between Aut X and solutions (a, b), $a > 0$ of the Pell eqn $x^2 - 7y^2 = 1$, so Aut X is countable.

5)

<u>Thm</u> (Ramanujam '64) 1:
$G < \mathrm{Aut}\, X$, connected. Every G-orbit in X is open in its closure.

<u>Thm 2</u> $G < \mathrm{Aut}\, X$ connected, Y. G-stable irreducible locally closed subset in X. Then there is an integer $m_{G,Y}$ and a dense open subset $\mathcal{U}$ of Y s.t.

$$\dim \underbrace{G(y)}_{G\text{-orbit of } y} = m_{G,Y} \quad \forall\, y \in \mathcal{U}$$

<u>Thm 3.</u> (generalization of Rosenlicht '56):
G and Y as in Thm 2. Then there is a G-stable open dense subset $\mathcal{U}$ of Y that admits a geometric quotient, i.e.
$\exists$ an irreducible variety Z and a morphism

$$\rho: \mathcal{U} \to Z \text{ s.t.}$$

- ρ is <u>surjective, open, and its fibers are G-orbits in $\mathcal{U}$</u>
- if V is an open subset of $\mathcal{U}$, then

$$\rho^*: k[\rho(V)] \to \{ f \in k[V] \mid f \text{ constant on every fiber of } \rho|_V \}$$

is an <u>isomorphism of k-algebras.</u>

Application

Cor 1. G and Y as in Thm 2. Then there is a finite subset of $k(Y)^G$ separating G-orbits of points of a dense open subset of Y.

Cor 2 $\quad \operatorname{tr\,deg}_k k(Y)^G = \dim Y - m_{G,Y}$

In particular,

$$k(Y)^G = k \iff \exists \text{ an open dense } G\text{-orbit in } Y$$

Thm 4. X nonuniruled. Then there is a nonconstant rational function on X that is G-invariant for every connected affine algebraic subgroup of $\operatorname{Aut} X$.

This then shows that for algebraic actions of affine alg. groups on nonuniruled varieties there is a certain rigidity for orbits: every such orbit lies in a level variety of a certain rational function not depending on the group and the action.

Remark "nonuniruled" cannot be replaced by "nonrational": there are examples of stably nonrational varieties with transitive actions of connected algebraic groups

Def. Aut X is called generically n-transitive if $\exists$ dense open $X_n \subset X$ s.t.

$\forall$ $x, y \in (X_n^*)^n$ lying off the union of the "diagonals"

$\exists$ $g \in$ Aut X s.t. $g(x) = y$.

There are many examples of generically n-transitive actions for $n \geq 2$. In many cases it is proved that such varieties are unirational and no examples of nonunirational such varieties are known at the moment.

The following is an evidence that such examples do not exist:

Thm 5. X irreducible, Aut X generically 2-transitive. Then at least one of the following holds:

(i) X unirational,

(ii) Aut X contains no nontrivial connected algebraic subgroups.

I know no examples with property (ii).

Moreover:

Cor. X complete. If Aut X is generically 2-transitive, then X is unirational.

Applications of Thm 5:

Cor 4. Every Calogero-Moser space

$$\mathcal{C}_n := \{(A,B) \in (\mathrm{Mat}_2(\mathbb{C}))^2 \mid \mathrm{rk}(AB - BA + I_n) = 1\} /\!\!/ PGL_n$$

is unirational.

The proof utilizes multiple-transitivity of $\mathrm{Aut}\,\mathcal{C}_n$ proved by Berest, A. Eshmatov, F. Eshmatov '14.

In fact, one can show that $\mathcal{C}_n$ is rational

Cor 5. char $k = 0$, $n \geq 3$. Every $Q_{m,n}(\tau)$ is unirational.

Here $Q_{m,n}(\tau)$ is described as follows:

Let $F_m := k\{t_1, \ldots, t_m\}$ be free associative k-algebra with free generators $t_1, \ldots, t_m$. Its n-dimensional representations are determined by m-tuples $(A_1, \ldots, A_m) \in (\mathrm{Mat}_n(k))^m$ by

$$t_i \mapsto A_i \quad \forall i$$

Representations are equivalent $\Longleftrightarrow$ corresponding m-tuples are PGL_n-conjugate for the diagonal action of PGL_n on $(\mathrm{Mat}_n(k))^m$.

Consider the categorial quotient:

$$Q_{m,n} := (\mathrm{Mat}_n(k))^m /\!/ PGL_n := \mathrm{Spec}\, k[(\mathrm{Mat}_n(k))^m]^{PGL_n}$$

the natural morphism

$$\pi : (\mathrm{Mat}_n(k))^m \longrightarrow Q_{m,n}$$

is surjective, every fiber $\pi^{-1}(x)$, $x \in Q_{m,n}$ contains a unique closed orbit. The latter is characterised by the property that it consists of ~~such~~ $(A_1, \dots, A_m) \in \pi^{-1}(x)$ in their fiber such that the representation $t_i \mapsto A_i\ \forall i$ is completely reducible, i.e. of the form

$$(\underbrace{\rho_1 \oplus \dots \oplus \rho_1}_{e_1 \text{ times}}) \oplus \dots \oplus (\underbrace{\rho_r \oplus \dots \oplus \rho_r}_{e_r \text{ times}}),$$

ρ_i irreducible $\forall i$

the tuple

$$(e_1, \dim\rho_1, \dots, e_r, \dim\rho_r) = \tau$$

is called the type of $x \in Q_{m,n}$.

By definition,

$$Q_{m,n}(\tau) = \{x \in Q_{m,n} \mid \text{type of } x \text{ is } \tau\}$$

Example: If $m, n \geq 2$, $(m,n) \neq 2,2$, then

$Q_{m,n}(1,n) =$ smooth locus of $Q_{m,n}$ ($\Leftarrow$ Procesi, Le Bruyn).

Generic $n\geq 2$-transitivity of $\mathrm{Aut}\, Q_{m,n}$ on $Q_{m,n}(\tau)$ was proved by Reichstein.