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On toric log Fano varieties

<u>Main Thm</u> (X,B) toric log Fano, $\mathrm{mld}(X,B) \geq \frac{1}{q}$ $(q \in \mathbb{Z}_{>0})$

$$\Rightarrow \inf\{\mathrm{lct}(X,B;D) \,;\, 0 \leq D \sim_{\mathbb{Q}} -K_X - B\} \geq \frac{q}{u_{d+1,q}}$$

$\overset{\text{def}}{=}\gamma$ "<u>α-invariant</u>"

$d = \dim X$

$(u_{p,q})_{p,q \geq 1}$: $u_{1,q} = q$, $u_{p+1,q} = u_{p,q}(1 + u_{p,q})$.

<u>Conj</u> "toric" not needed.

1. Background
2. Toric formula for α-invariant
3. Dioph. Approx^n ; $\Rightarrow$ proof Main Thm.

1. Background

- X^d proj, normal lc. sing's

$-K_X$ ample	$K_X \sim_{\mathbb{Q}} 0$	K_X ample
Fano	CY	Can. Polar.
mld$(X) \in [0,1]$	$\exists\, r_d K_X \sim 0$ ("Beauville #")	$\sqrt[d]{(K_X^d)} = \mathrm{vol}$; vol $= c \Rightarrow \exists\, \mathcal{M}_{d,c}$ moduli sp. (Viehweg, HMX)

$$\boxed{\text{mld}(X) \geq \varepsilon > 0 \overset{\circledast}{\Longrightarrow} X \in \text{Bound. Family}_{d,\varepsilon}}$$

log version: $X = (X, B=0) \rightsquigarrow (X,B)$.

- <u>mld</u>. (X,B) log variety (K_X+B $\mathbb{Q}$-Cartier)

$E \subset \tilde{X}$ prime div, $\downarrow \mu$ resln, X

$\mu^*(K_X+B) = K_{\tilde{X}} + \sum (1-a_i)E_i$, $\mu_* K_{\tilde X} = K_X$.

$a_i = a(E_i; X,B)$ log discrep. of (X,B) at E_i

$$\text{mld}(X,B) = \inf\{a(E;X,B);\ E \text{ geom. valn of } X\}.$$

eg: $E \subset X$ prime $\rightsquigarrow a(E;X,B) = 1 - \text{mult}_E B \leq 1$ $\therefore$ mld $\in \{-\infty\} \cup [0,1]$

(X,B) has lc sing's $\iff$ mld$(X,B) \geq 0$.

- $\circledast$ true for: X toric (Borisov brothers '92)

dim $X = 2$ (Alexeev '94)

($\exists$ other: X smooth, or $\underline{r}K_X$ Cartier, dim $X=3$).

- <u>α-invariant</u>: (X,B) log var, lc sing's; H $\mathbb{Q}$-div, ample.

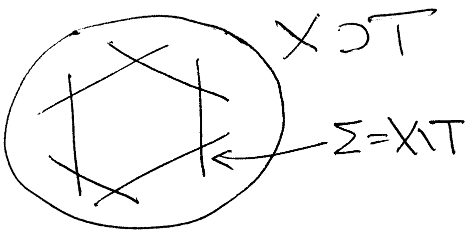
$$\gamma := \inf\{\text{lct}(X,B;D);\ 0 \leq D \sim_{\mathbb{Q}} H\}$$
$$= \sup\{t \geq 0;\ (X, B+tD) \text{ has lc sing's } \forall\, 0 \leq D \sim_{\mathbb{Q}} H\}$$

<u>Siegel lemma</u> $\gamma \cdot \sqrt[d]{(H^d)} \leq d$.

- <u>Meaning of $\circledast$</u>: $|-r_\varepsilon K_X| : X \hookrightarrow \mathbb{P}^{N_\varepsilon}$, deg $\leq D(d,\varepsilon)$.

enough:
- rK_X Cartier, $\exists\, 1 \leq r \leq R(d,\varepsilon)$
- $\gamma \geq \gamma(d,\varepsilon) > 0$.

2. Formula for α-inv. of toric log var's



(X, B) toric log var H $\mathbb{Q}$-ample.
X toric, B supp by Σ.

- **Prop^n** (X,B) toric $\Rightarrow \gamma(X,B;H) = \min$, att^d by T-invar $\mathbb{Q}$-div.

$\Uparrow$

Lemma $\left(\mathbb{C}^n, \sum_{i=1}^{n}(1-a_i)H_i + \gamma D\right)$ has lc sing's, where
$H_i = \mathrm{div}(z_i)$, $D = \mathrm{div}(P(z_1,\dots,z_n))$, $\gamma \le \min_i \dfrac{a_i}{\deg_{z_i} P}$

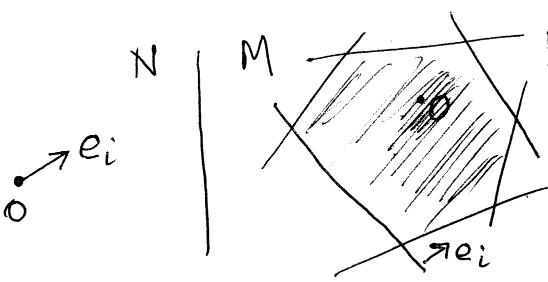
$\Uparrow$

Thm (cf. Viehweg) $(X_i, B_i; H_i)$ $i=1,2$ polarized log var's.

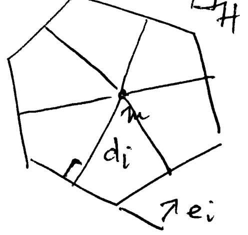
$$(X_1,B_1)\times(X_2,B_2) = (X_1\times X_2,\ X_1\times B_2 + B_1\times X_2)$$
$$H_1 \boxtimes H_2 = H_1\times X_2 + X_1\times H_2$$
$$\gamma\left(\prod (X_i,B_i);\ \boxtimes H_i\right) = \min_i \gamma(X_i,B_i;H_i)$$

proof: $\prod_{i=1}^{n}(\mathbb{P}^1, (1-a_i)\infty)$, $\boxtimes_{i=1}^{n}\mathcal{O}_{\mathbb{P}^1}(d_i)$ $\quad (d_i = \deg_{z_i} P)$

- *explicit formula*:

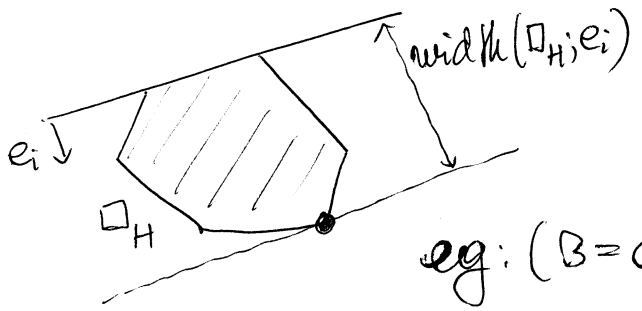


$\square_H$ $\quad \Gamma(nH) = \bigoplus_{m\in M\cap n\square_H} \mathbb{C}x^m$



$m \in \square_H \rightsquigarrow D(m) = \sum d_i E_i$
$\mathrm{lct}(X,B;D(m)) = \min \dfrac{a_i}{d_i}$

$\Rightarrow \gamma$ att^d by some vertex m.



$$\boxed{\gamma = \min_{e_i \in \Delta(1)} \frac{a_i}{\mathrm{width}(\square_H; e_i)}}$$

eg: $(B=0)$ γ^{-1} = largest width of $\square_H$ wrt top face.

3. Diophantine Approx^n

$(u_{p,q})$ $\quad u_{1,q} = q, \quad u_{p+1,q} = u_{pq}(1+u_{pq})$.

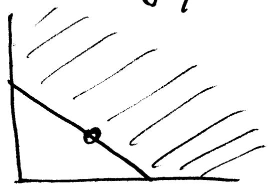
$\Rightarrow \sum_{i=1}^{p} \frac{q}{1+u_{i,q}} = 1 - \frac{q}{u_{p+1,q}}, \quad \prod_{i=1}^{p}(1+u_{i,q}) = \text{lcm} = \frac{u_{p+1,q}}{q}$.

Thm (Hensley ineffective; Averkov $q=1$)

$x_1 \geq \cdots \geq x_d > 0, \quad \sum_{i=1}^{d} x_i \geq \sum_{i=1}^{d} \frac{q}{1+u_{i,q}}, \quad \exists j, \ x_j \neq \frac{q}{1+u_{j,q}}$.

$\Rightarrow \exists\ 0 \neq z \in \mathbb{N}^d$ st

$$\frac{z_i}{\frac{1}{q} + z_1 + \cdots + z_d} < x_i \quad \forall i.$$

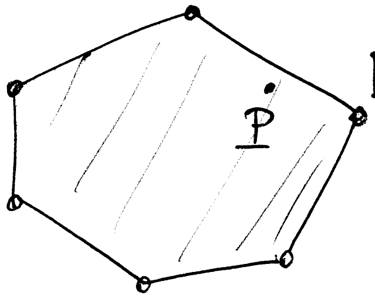


Proof (idea): essentially Minkowski's 1st thm.

Cor. $\square \subset \mathbb{R}^d$ lattice polytope, dim d.

$\#(\mathbb{Z}^d \cap \text{Int}\,\square) = q \geq 1$.

$\Rightarrow \gamma(P \in \square) \geq \frac{q}{u_{d+1,q}} \quad \forall P \in \mathbb{Z}^d \cap \text{Int}\,\square$.

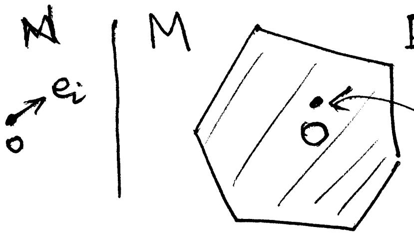


$\square \quad \gamma(P \in \square) = \sup\{\gamma > 0 \mid \gamma \cdot (\square - \square) \subset \square - P\}$.

Cor: main thm

(X, B) toric log Fano

$\gamma < \frac{q}{u_{d+1,q}} \Rightarrow \text{mld} < \frac{1}{q}$



$\square - (K+B)$

$K_X + \Sigma \sim 0$.

$\gamma = \gamma(0 \in \square)$!

reduce to simplex; simplex case $\Leftarrow$ dioph. apx^n.

- also clarifies original proof of Borisovs.

Cor. $\sqrt[d]{(-K_X - B)^d} \leq d \cdot \frac{u_{d+1,q}}{q}$.