



Canonical Quotient Singularities in Dimension Three

Author(s): David R. Morrison

Source: *Proceedings of the American Mathematical Society*, Vol. 93, No. 3 (Mar., 1985), pp. 393-396

Published by: American Mathematical Society

Stable URL: <http://www.jstor.org/stable/2045598>

Accessed: 24/12/2009 00:12

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/action/showPublisher?publisherCode=ams>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



American Mathematical Society is collaborating with JSTOR to digitize, preserve and extend access to *Proceedings of the American Mathematical Society*.

<http://www.jstor.org>

CANONICAL QUOTIENT SINGULARITIES IN DIMENSION THREE

DAVID R. MORRISON¹

ABSTRACT. We classify isolated canonical cyclic quotient singularities in dimension three, showing that, with two exceptions, they are all either Gorenstein or terminal. The proof uses the solution of a combinatorial problem which arose in the study of algebraic cycles on Fermat varieties.

This note is a continuation of our previous paper [7] (written jointly with Glenn Stevens), which studied certain canonical quotient singularities in dimensions three and four. The classification of canonical quotient singularities can be reduced to a certain combinatorial problem, which we partially solved in [7]. After [7] went to press, we learned that a similar (although not quite identical) combinatorial problem occurs in the study of algebraic cycles on Fermat varieties (cf. [6, 8, 10]). In this note, we apply the recent solution of that problem (due to Aoki and Shioda [1, 2, 10]) to the classification of canonical quotient singularities in dimension three.

If x is a rational number, we let $\langle x \rangle$ denote the rational number such that $x \equiv \langle x \rangle \pmod{\mathbf{Z}}$, and $0 \leq \langle x \rangle < 1$.

DEFINITION. A *Fermat quadruple* is a quadruple of rational numbers $(a_1/N, a_2/N, a_3/N, a_4/N)$ such that $a_i \not\equiv 0 \pmod{N}$, $\text{g.c.d.}\{a_i\} = 1$, and, for every integer k with $(k, N) = 1$, $\sum_i \langle a_i k/N \rangle = 2$.

The following theorem was conjectured by Meyer and Neutsch [6] and Shioda [10], and proved by Aoki and Shioda [1, 2, 10].

THEOREM 1. *Let $(a_1/N, a_2/N, a_3/N, a_4/N)$ be a Fermat quadruple. Then there is some k with $(k, N) = 1$ such that $(\langle a_1 k/N \rangle, \langle a_2 k/N \rangle, \langle a_3 k/N \rangle, \langle a_4 k/N \rangle)$ is, after reordering, one of the following:*

- (i) $(\alpha/N, 1 - (\alpha/N), \beta/N, 1 - (\beta/N))$ for some $0 < \alpha, \beta < N$,
- (ii) $(1/2K, K/2K, (K+1)/2K, (2K-2)/2K)$ where $N = 2K$,
- (iii) $(1/2K, (K+1)/2K, (K+2)/2K, (2K-4)/2K)$ where $N = 2K$,
- (iv) $(1/3K, (K+1)/3K, (2K+1)/3K, (3K-3)/3K)$ where $N = 3K$,
- (v) one of 101 "exceptional quadruples" listed in [6]: each has $(N, 6) \neq 1$ and $N \leq 180$.

Received by the editors February 9, 1984.

1980 *Mathematics Subject Classification.* Primary 14B05; Secondary 14L30, 32B30.

Key words and phrases. Canonical singularity, Fermat variety, Gorenstein ring, quotient singularity, terminal singularity.

¹National Science Foundation Postdoctoral Fellow.

The combinatorial fact which we used in [7] to classify terminal quotient singularities is

THEOREM 2 [13, 3, 4 AND 7]. *Let a, b, c be integers relatively prime to N . Suppose, for all $k \in \mathbf{Z} - N\mathbf{Z}$,*

$$\langle ak/N \rangle + \langle bk/N \rangle + \langle ck/N \rangle > 1.$$

Then, after reordering a, b, c , we have $a + b \equiv 0 \pmod{N}$.

Note that under the hypotheses of Theorem 2, if we let $d = -(a + b + c)$, then $(a/N, b/N, c/N, d/N)$ is a Fermat quadruple. For N prime, Theorems 1 and 2 are equivalent; this case of these theorems was also proved by W. Parry (cf. [6 and 8]).

We can now state our main combinatorial result.

THEOREM 3. *Let a, b, c be integers relatively prime to N . Suppose, for all $k \in \mathbf{Z} - N\mathbf{Z}$,*

$$(*) \quad \langle ak/N \rangle + \langle bk/N \rangle + \langle ck/N \rangle \geq 1.$$

Then either

- (i) $\langle ak/N \rangle + \langle bk/N \rangle + \langle ck/N \rangle \in \mathbf{Z}$ for all k ,
- (ii) after reordering a, b, c , we have $a + b \equiv 0 \pmod{N}$, or
- (iii) there is some integer k with $(k, N) = 1$ such that, after reordering,
 $(\langle ak/N \rangle, \langle bk/N \rangle, \langle ck/N \rangle) = (1/9, 4/9, 7/9)$ or $(1/14, 9/14, 11/14)$.

Conversely, if either (i), (ii), or (iii) holds, then () is true for all $k \in \mathbf{Z} - N\mathbf{Z}$.*

PROOF. Let $d = -(a + b + c)$. If $d \equiv 0 \pmod{N}$ then we have case (i). If $(d, N) = 1$ then, for each $k \in \mathbf{Z} - N\mathbf{Z}$, $\langle dk/N \rangle > 0$, so that

$$\langle ak/N \rangle + \langle bk/N \rangle + \langle ck/N \rangle + \langle dk/N \rangle \geq 1 + \langle dk/N \rangle > 1.$$

We may then apply Theorem 2 to conclude that we have case (ii).

Thus, we may assume that $d \not\equiv 0 \pmod{N}$ and $(d, N) \neq 1$. In this case, $(a/N, b/N, c/N, d/N)$ is a Fermat quadruple in which exactly three of the numerators are relatively prime to N . We apply Theorem 1 to find such Fermat quadruples. In case (i) of Theorem 1, either 0, 2, or 4 of the numerators are relatively prime to N , and, in cases (ii) and (iii), at most two of the numerators are relatively prime to N . Thus, any Fermat quadruples satisfying our hypothesis will be found in cases (iv) and (v).

Suppose $(1/3K, (K + 1)/3K, (2K + 1)/3K, (3K - 3)/3K)$ is a Fermat quadruple (with $N = 3K$) from case (iv) of Theorem 1. In order to satisfy our hypothesis, 1, $K + 1$, and $2K + 1$ must all be relatively prime to $3K$; in particular, $K \equiv 0 \pmod{3}$. On the other hand,

$$\langle 3/3K \rangle + \langle 3(K + 1)/3K \rangle + \langle 3(2K + 1)/3K \rangle = 3/K \geq 1$$

so that $K \leq 3$. We thus find that the only solution in this case is $K = 3$ and $(\langle ak/N \rangle, \langle bk/N \rangle, \langle ck/N \rangle) = (1/9, 4/9, 7/9)$.

To analyse case (v) of Theorem 1, we have listed in Table 1 all exceptional Fermat quadruples in which exactly three of the numerators are relatively prime to N . We

have also (in all but the first line of the table), listed a value of k and the corresponding triple $(\langle ak/N \rangle, \langle bk/N \rangle, \langle ck/N \rangle)$; since the sum in each case is less than 1, none of these satisfy our hypothesis. On the other hand, it is easily verified that the remaining case $(1/14, 9/14, 11/14)$ satisfies the hypothesis for each k . Q.E.D.

$(a/N, b/N, c/N, d/N)$	k	$(\langle ak/N \rangle, \langle bk/N \rangle, \langle ck/N \rangle)$
$(1/14, 9/14, 11/14, 7/14)$	—	— —
$(1/21, 4/21, 19/21, 18/21)$	6	$(2/7, 1/7, 3/7)$
$(1/28, 9/28, 25/28, 21/28)$	7	$(1/4, 1/4, 1/4)$
$(1/42, 31/42, 37/42, 15/42)$	7	$(1/6, 1/6, 1/6)$
$(1/42, 25/42, 37/42, 21/42)$	7	$(1/6, 1/6, 1/6)$
$(1/78, 55/78, 61/78, 39/78)$	13	$(1/6, 1/6, 1/6)$

TABLE 1

From Theorem 3, we deduce a classification of isolated canonical cyclic quotient singularities in dimension three. Recall that a cyclic group $G \subset \text{Gl}(3, \mathbf{C})$ of order N gives an isolated singularity X/G exactly when all eigenvalues of a generator g are primitive N th roots of unity.

THEOREM 4. *Let $G \subset \text{Gl}(3, \mathbf{C})$ be a cyclic group of order N such that all eigenvalues of a generator g are primitive N th roots of unity. Then the isolated singularity X/G is canonical if and only if one of the following holds:*

- (i) X/G is Gorenstein,
- (ii) X/G is terminal,
- (iii) G is conjugate in $\text{Gl}(3, \mathbf{C})$ to the group generated by

$$\text{diag}(e^{2\pi i/9}, e^{8\pi i/9}, e^{14\pi i/9}),$$

or to that generated by

$$\text{diag}(e^{\pi i/7}, e^{9\pi i/7}, e^{11\pi i/7}).$$

PROOF. Let $\zeta = e^{2\pi i/N}$, and choose coordinates so that $g = \text{diag}(\zeta^a, \zeta^b, \zeta^c)$, with $(a, N) = (b, N) = (c, N) = 1$. By the criterion of Reid [9], Shepherd-Barron, and Tai [11], X/G is canonical if and only if

$$\langle ak/N \rangle + \langle bk/N \rangle + \langle ck/N \rangle \geq 1$$

for all $k \in \mathbf{Z} - N\mathbf{Z}$. On the other hand, by the theorem of Khinich [5] and Watanabe [12], X/G is Gorenstein if and only if $\det g = 1$ (which holds if and only if $\langle ak/N \rangle + \langle bk/N \rangle + \langle ck/N \rangle \in \mathbf{Z}$ for all k). Since we proved in [7] that X/G is terminal if and only if case (ii) in Theorem 3 holds, this theorem now follows immediately from Theorem 3. Q.E.D.

REFERENCES

1. N. Aoki, *On some arithmetic problems related to the Hodge cycles on the Fermat varieties*, Math. Ann. **266** (1983), 23–54.
2. N. Aoki and T. Shioda, *Generators of the Néron-Severi group of a Fermat surface*, Arithmetic and Geometry, Vol. I: Arithmetic, Progress in Math., Vol. 35, Birkhauser, Boston, Basel and Stuttgart, 1983, pp. 1–12.
3. V. I. Danilov, *The birational geometry of toric threefolds*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 972–981 = Math. USSR-Izv. **21** (1983), 269–280.
4. M. A. Frumkin, *Description of the elementary three-dimensional polyhedra*, First All-Union Conference on Statistical and Discrete Analysis of Non-numerical Information, Expert Estimation and Discrete Optimization (abstract of conference reports), Moscow-Alma-Ata, 1981. (Russian)
5. V. A. Khinich, *On the Gorenstein property of the ring of invariants of a Gorenstein ring*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), 50–56 = Math. USSR-Izv. **10** (1976), 47–53.
6. W. Meyer and W. Neutsch, *Fermatquadrupel*, Math. Ann. **256** (1981), 51–62.
7. D. R. Morrison and G. Stevens, *Terminal quotient singularities in dimensions three and four*, Proc. Amer. Math. Soc. **90** (1984), 15–20.
8. Z. Ran, *Cycles on Fermat hypersurfaces*, Compositio Math. **42** (1981), 121–142.
9. M. Reid, *Canonical 3-folds*, Géométrie Algébrique d'Angers (A. Beauville, ed.), Sijthoff and Noordhoff: Alphen aan den Rijn, 1980, pp. 273–310.
10. T. Shioda, *On the Picard number of a Fermat surface*, J. Fac. Sci. Univ. Tokyo Sect. 1A Math. **28** (1982), 725–734.
11. Y. S. Tai, *On the Kodaira dimension of the moduli space of Abelian varieties*, Invent. Math. **68** (1982), 425–439.
12. K. Watanabe, *Certain invariant subrings are Gorenstein*. I, II, Osaka J. Math. **11** (1974), 1–8, 379–388.
13. G. K. White, *Lattice tetrahedra*, Canad. J. Math. **16** (1964), 389–397.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08544

Current address: Research Institute for Mathematical Sciences, Kyoto University, Kitashirakawa, Sakyo-ku, Kyoto 606, Japan