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On the McKay correspondence for a finite small subgroup of GL(2, $\mathbb{C}$)

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Abstract. We consider the McKay correspondence for a general quotient surface singularity via Hilbert scheme of G-orbits. By reformulating Wunram's results [W] in terms of derived functors, we shall generalize results by Ito and Nakamura [IN2] in the form conjectured by Riemenschneider [R2].

1. Introduction

Ito and Nakamura [IN2] introduced the notion of the Hilbert scheme of G-orbits and gave another formulation of the McKay correspondence [M] for simple singularities. In this paper, we consider the case of general quotient surface singularities. We shall obtain a correspondence in terms of a derived functor discussed in [BKR] and we shall generalize the results of Ito and Nakamura.

Before going into details, let us review the history of the problem. The McKay correspondence [M] for a finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$ gives a bijective correspondence of non-trivial irreducible representations of G to irreducible exceptional curves of the minimal resolution of the simple singularity $\mathbb{C}^2/G$. Since this observation depends on explicit computations on group representations, efforts were made to understand this phenomenon geometrically. Gonzalez-Sprinberg and Verdier [GV] described the McKay correspondence as an isomorphism of Grothendieck groups with case-by-case checking. Then Artin and Verdier [AV] gave a short proof using properties of reflexive modules over rational surface singularities. Hilbert scheme of G-orbits (G-Hilb), introduced by Nakamura, allows us a moduli-theoretic approach to the problem. Ito and Nakamura [IN2], noting that G-Hilb is the minimal resolution, gave another way to associate a representation to an irreducible component of the exceptional locus. They studied I/mI as representations of G for ideals I in the exceptional locus of G-Hilb. Their arguments were based on case-by-case analysis. On the other hand, the McKay correspondence was understood as an equivalence of derived categories by Kapranov and Vasserot [KV] and generalized to higher dimensions including all the cases of finite subgroups of $\mathrm{SL}(3,\mathbb{C})$ by Bridgeland, King and Reid [BKR].

In this paper, we shall generalize the original situation to the case of a finite small subgroup G of $\mathrm{GL}(2,\mathbb{C})$. In general, there are less irreducible exceptional curves than non-trivial irreducible representations and we cannot expect equivalence of derived categories nor isomorphism of Grothendieck groups. But Wunram [W] defined a subclass, the class of *special* irreducible representations (see Section 4), in terms of reflexive modules; he gave a bijective correspondence of non-trivial special representations to irreducible exceptional curves. On the other hand, when G is a cyclic group, Kidoh [K] gave a concrete description of G-Hilb in terms of continued fraction expansions; in particular, she proved that G-Hilb is the minimal resolution. Riemenschneider [R2], looking at her result, pointed out that we can generalize Ito-Nakamura's result to the cyclic group case by considering special representations. In this paper, we shall give such a generalization for an arbitrary finite small subgroup. Our proof is based on: the adjointness of two derived functors discussed by Bridgeland, King and Reid [BKR]; and a reformulation of Wunram's results [W], which gives the McKay correspondence on derived category level.

Let us state the results more precisely. Let $G \subset \mathrm{GL}(2,\mathbb{C})$ be a finite small subgroup, that is, a finite subgroup acting freely on $\mathbb{C}^2 \setminus \{0\}$. Recall that a *$G$-cluster Z* is a finite subscheme of $\mathbb{C}^2$ such that Z is G-invariant and that $H^0(\mathcal{O}_Z)$ is the regular representation of G. Let G-Hilb$(\mathbb{C}^2)$ be the moduli space of G-clusters on $\mathbb{C}^2$; this is a finite union of connected components of the G-fixed point locus $\left(\mathrm{Hilb}^{\# G}(\mathbb{C}^2)\right)^G$ of $\mathrm{Hilb}^{\# G}(\mathbb{C}^2)$. Then G-Hilb$(\mathbb{C}^2)$ is smooth since so is $\mathrm{Hilb}^{\# G}(\mathbb{C}^2)$ (Fogarty [F]) and since the G-action can be locally linearized. We don't know the connectedness of G-Hilb$(\mathbb{C}^2)$ (which will be proved in section 8) at this stage and so let Y denote the connected component dominating $\mathbb{C}^2/G$. In the case $G \subset \mathrm{SL}(2,\mathbb{C})$, Ito and Nakamura proved the following. (Crawley-Boevey [C] gave a proof of this result without case-by-case checking, by using the theory of preprojective algebras.)

Theorem (Ito-Nakamura, [IN2]). *When $G \subset \mathrm{SL}(2,\mathbb{C})$, the following hold.*

(1) *Y is the minimal resolution of $\mathbb{C}^2/G$.*

(2) *There is a bijection of irreducible exceptional curves $\{E_1, \dots, E_n\}$ with non-trivial irreducible representations $\{\rho_1, \dots, \rho_n\}$ such that*

$$I_y/(mI_y + \mathfrak{n}) \cong \begin{cases} \rho_i & \text{if } y \in E_i, \text{ and } y \notin E_j \ (j \neq i), \\ \rho_i \oplus \rho_j & \text{if } y \in E_i \cap E_j, \end{cases}$$

where $\mathfrak{n} = (m^G)\mathcal{O}_{\mathbb{C}^2}$. Moreover this bijection coincides with original McKay's correspondence.

For general finite small $G \subset \mathrm{GL}(2,\mathbb{C})$, Riemenschneider [R2] conjectured that both the assertions hold, where in (2) we should choose *special* representations; his conjecture is verified explicitly in the case of cyclic groups by virtue of the work of Kidoh [K].

Both parts of his conjecture will be proved in this paper. We shall prove that Y is the minimal resolution in Theorem 3.1, where a basic property of reflexive modules is essential. The correspondence as in (2) will be given in Theorem 7.1; we shall determine the representation I_y/mI_y for y in the exceptional locus. We shall also determine the socle of $\mathcal{O}_Z$ in Theorem 7.2. For these, we shall consider the derived categories $D_c(Y)$ of coherent sheaves with compact support on Y and $D_c^G(\mathbb{C}^2)$ of G-equivariant coherent sheaves with compact support on $\mathbb{C}^2$. By using the universal family of G-clusters, we can define an adjoint pair of functors between these derived categories as in [KV] and [BKR]. The most essential part is Theorem 5.1, which is a reformulation of Wunram's 'multiplication formula' on reflexive modules. It can be seen to be the McKay correspondence on derived category level (though not an equivalence) and in fact is a generalization of a result in [KV]. The connectedness of G-Hilb will also be proved by using the adjointness of these functors in Theorem 8.1.

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2. Representations, reflexive modules and full sheaves

Let $G \subset \mathrm{GL}(2,\mathbb{C})$ be a finite small subgroup, where G is said to be small if it acts freely on $\mathbb{C}^2 \setminus \{0\}$. We denote by $X = \mathbb{C}^2/G$ the associated quotient singularity with quotient map

$$\pi\colon \mathbb{C}^2 \to X,$$

and by

$$\phi\colon \tilde{X} \to X$$

the minimal resolution with fundamental cycle F. In this section, we review the correspondence among representations of G, reflexive modules on X and full sheaves on $\tilde{X}$. To establish these correspondences as bijections without inessential arguments, we should consider the germ of X at 0 and its fiber instead of X and $\tilde{X}$ respectively. But by abuse of terminology, we say 'reflexive modules on X' and 'full sheaves on $\tilde{X}$' even when we should consider the germs.

Let ρ be a (finite dimensional) representation of G. We define the associated $\mathcal{O}_X$-module as

$$M := (\mathcal{O}_{\mathbb{C}^2} \otimes_{\mathbb{C}} \rho^*)^G,$$

where we use the contragradient ρ^* of ρ according to [GV]. Then M is a reflexive module on X; M is said to be reflexive if the canonical map of M to its double dual $M^{\vee\vee}$ is an isomorphism. This is equivalent to the condition $M \cong H^0(X \setminus \{0\}; M|_{X\setminus\{0\}})$ since X is normal and 2-dimensional. In the above way, we have a bijective correspondence of representations of G to reflexive modules on X.

For a reflexive module M on X, we define a coherent sheaf $\tilde{M} = \phi^* M/\text{torsion}$ on $\tilde{X}$. Then we have the following lemma (see [E]).

Lemma 2.1. *$\tilde{M}$ has the following properties.*

(1) *$\tilde{M}$ is a locally free sheaf.*

(2) *$\tilde{M}$ is generated by global sections.*

(3) $R^1\phi_*\big((\tilde{M})^\vee \otimes K_{\tilde{X}}\big) = 0.$

This motivates the following definition.

Definition 2.2 ([E]). A locally free sheaf $\mathscr{M}$ on $\tilde{X}$ with the above properties is called a *full sheaf*.

By associating $\tilde{M}$ to M, we obtain a bijective correspondence of reflexive modules to full sheaves on $\tilde{X}$ (see [E]).

Thus we have bijective correspondences of representations, reflexive modules and full sheaves. In this correspondence, an irreducible representation corresponds to an indecomposable reflexive module and to an indecomposable full sheaf.

3. Minimality of Y

Let $G \subset \mathrm{GL}(2,\mathbb{C})$ be a small finite subgroup and $Y \subseteqq G\text{-Hilb}(\mathbb{C}^2)$ be as in the introduction. In this section, we show the following.

Theorem 3.1. *Y is the minimal resolution of $X = \mathbb{C}^2/G$.*

Proof. Let $\pi\colon \mathbb{C}^2 \to X = \mathbb{C}^2/G$ be the quotient map and $\phi\colon \tilde{X} \to X$ the minimal resolution. If we identify $\mathbb{C}^2$ with the graph of π in $\mathbb{C}^2 \times X$, then the second projection makes $\pi_*\mathcal{O}_{\mathbb{C}^2}$ a quotient of the $\mathcal{O}_X$-algebra $\mathcal{O}_X \otimes_{\mathbb{C}} H^0(\mathcal{O}_{\mathbb{C}^2}) \cong \mathcal{O}_X[x_1, x_2]$ by its G-invariant ideal (sheaf). Since ϕ^* is right exact, $\phi^*\pi_*\mathcal{O}_{\mathbb{C}^2}$ is a quotient of $\mathcal{O}_{\tilde{X}}[x_1, x_2]$ by its ideal sheaf. The torsion part (as an $\mathcal{O}_{\tilde{X}}$-module) of $\phi^*\pi_*\mathcal{O}_{\mathbb{C}^2}$ must be preserved by its local endomorphisms as an $\mathcal{O}_{\tilde{X}}$-module and in particular by multiplications by local sections of $\mathcal{O}_{\tilde{X}}[x_1, x_2]$. This shows that the torsion part is an $\mathcal{O}_{\tilde{X}}[x_1, x_2]$-submodule and hence $\widetilde{\pi_*\mathcal{O}_{\mathbb{C}^2}}$ is again a quotient of $\mathcal{O}_{\tilde{X}}[x_1, x_2]$ by its G-invariant ideal sheaf. $\widetilde{\pi_*\mathcal{O}_{\mathbb{C}^2}}$ is locally free by Lemma 2.1 and therefore it gives rise to a flat family of G-clusters parameterized by $\tilde{X}$. Thus we have a morphism $\tilde{X} \to G\text{-Hilb}(\mathbb{C}^2)$ which induces an isomorphism away from the fibers over $0 \in \mathbb{C}^2/G$. Since $\tilde{X}$ is the mininal resolution and Y is non-singular, we obtain an isomorphism $\tilde{X} \cong Y$. $\square$

If we denote by $\mathscr{Z} \subset Y \times \mathbb{C}^2$ the universal subscheme, we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathscr{Z} & \xrightarrow{q} & \mathbb{C}^2 \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
Y & \xrightarrow{f} & X = \mathbb{C}^2/G.
\end{array}
\tag{3.1}
$$

Put $\mathscr{R} = p_*\mathcal{O}_{\mathscr{Z}}$. Then $\mathscr{R}$ is a locally free sheaf on Y since p is flat and finite.

Corollary 3.2. *$\mathscr{R}$ is a full sheaf. Moreover, if ρ is a representation of G, then $\left(p_*\left(q^*(\mathcal{O}_{\mathbb{C}^2} \otimes \rho^*)\right)\right)^G$ is also a full sheaf.*

Proof. In the proof of the theorem, we saw that $\mathscr{R} \cong \widetilde{\pi_*\mathcal{O}_{\mathbb{C}^2}}$ and hence is a full sheaf. We can decompose $\mathscr{R}$ with respect to the G-action as

$$\mathscr{R} = \bigoplus_{\rho} R_\rho \otimes \rho$$

where ρ runs through all the irreducible representations. If ρ is an irreducible representation, then $\big(p_*\big(q^*(\mathcal{O}_{\mathbb{C}^2} \otimes \rho^*)\big)\big)^G \cong \mathcal{R}_\rho$ is a direct summand of the full sheaf $\mathcal{R}$ and also is a full sheaf. $\square$

4. Wunram's special representations and the multiplication formula

Generalizing the original McKay correspondence to the case of a general finite small subgroup $G \subset \mathrm{GL}(2, \mathbb{C})$, Wunram [W] associated to each irreducible exceptional curve an indecomposable reflexive module, which are said to be *special* by Riemenschneider [R1], [R3]. Moreover, Wunram generalized the 'multiplication formula' by Esnault and Knörrer [EK] which describes the tensor property of the original McKay correspondence in terms of reflexive modules and full sheaves. In this section, we recall these results of Wunram which are essential to our argument. Let $f\colon Y \to X = \mathbb{C}^2/G$ be the minimal resolution and F the fundamental cycle.

Definition 4.1 ([R1]). A full sheaf $\mathcal{M}$ on Y is said to be *special* if

$$R^1 f_*(\mathcal{M}^\vee)(=H^1(\mathcal{M}^\vee)) = 0.$$

A reflexive module on X (or representation of G) is said to be *special* if the corresponding full sheaf is special.

Note that if $G \subset \mathrm{SL}(2, \mathbb{C})$, then all the reflexive modules are special (see Lemma 2.1). Note also that the trivial representation is always special.

Theorem 4.2 (Wunram, [W]). *There is a bijective correspondence of* (*isomorphism classes of*) *non-trivial indecomposable special reflexive modules M_i to irreducible exceptional curves E_i such that $c_1(\widetilde{M_i}).E_j = \delta_{ij}$. The rank of M_i is $r_i = c_1(\widetilde{M_i}).F$, the coefficient of E_i in the fundamental cycle F.*

Here we recall the construction of M_i in the above theorem. Let D_i be a curve on Y which intersects the exceptional locus transversely at one point in E_i. Take a minimal set of generators $\varphi_1, \dots, \varphi_r$ of $f_*\mathcal{O}_D$ as an $\mathcal{O}_X$-module and let $\mathcal{N}_i$ be the kernel of the surjection $\mathcal{O}_Y^{\oplus r} \to \mathcal{O}_D$ given by $(\varphi_1, \dots, \varphi_r)$. Then it is easy to see that $\mathcal{M}_i := \mathcal{N}_i^\vee$ is the desired special full sheaf. As a consequence, M_i is constructed via an exact sequence

$$0 \to (\widetilde{M_i})^\vee \to \mathcal{O}_Y^{\oplus r} \to \mathcal{O}_{D_i} \to 0. \tag{4.1}$$

Notation 4.3. We denote by $\{\rho_0, \rho_1, \dots, \rho_n\}$ the set of the irreducible representations of G. We assume that ρ_0 is the trivial representation and that ρ_i is special if and only if $i \leqq m$ (therefore $m+1$ is the number of the special irreducible representations). M_i denotes the reflexive module corresponding to ρ_i for $0 \leqq i \leqq n$. Theorem 4.2 associates an irreducible exceptional curve E_i to M_i for $1 \leqq i \leqq m$.

Next we shall review the 'multiplication formula', Theorem 1.3 in [W]. Let

$$0 \to K_{\mathbb{C}^2} \to \Omega^1_{\mathbb{C}^2} \to \mathcal{O}_{\mathbb{C}^2} \to \mathcal{O}_0 \to 0 \tag{4.2}$$

be the Koszul complex of $m = m_{0,\mathbb{C}^2}$, which is G-equivariant. For an irreducible representation ρ of G and the corresponding reflexive module M, tensor the Koszul complex (4.2) with ρ^* and take the G-invariant part. Then we obtain a complex (the Auslander-Reiten sequence)

$$0 \to \tau(M) \to N_M \to M \to 0 \tag{4.3}$$

of reflexive modules on X, where $\tau(M) = (M \otimes \omega_X)^{\vee\vee}$ and $N_M = (M \otimes \Omega_X^1)^{\vee\vee}$. Note that (4.3) is exact unless $M \cong \mathcal{O}_X$. In the case $M = \mathcal{O}_X$, we have an exact sequence

$$0 \to \tau(\mathcal{O}_X)(=\omega_X) \to N_{\mathcal{O}_X} \to \mathcal{O}_X \to \mathcal{O}_0 \to 0. \tag{4.4}$$

The multiplication formula is stated as follows.

Theorem 4.4 (Wunram, [W]). *For an indecomposable reflexive module M_i, we have*

$$c_1(\widetilde{N_{M_i}}) - c_1(\widetilde{M_i}) - c_1\big(\widetilde{\tau(M_i)}\big) = \begin{cases} -F & \text{if } i = 0, \\ E_i & \text{if } 1 \leqq i \leqq m, \\ 0 & \text{if } m < i \leqq n. \end{cases}$$

We shall give the proof of this after restating the theorem in terms of a derived functor in the next section.

5. Correspondence of sheaves with compact support

In this section, we translate the multiplication formula into derived category language and moreover give a proof of it because it is essential to our arguments. Let $D_c(Y)$ denote the derived category of coherent sheaves with compact support on Y and $D_c^G(\mathbb{C}^2)$ the derived category of G-equivariant coherent sheaves with compact support on $\mathbb{C}^2$ (see [BKR]). We define a functor Ψ: $D_c^G(\mathbb{C}^2) \to D_c(Y)$ by

$$\Psi(-) := [p_*\mathbb{L}q^*(-)]^G.$$

Note p is finite and hence p_* is exact. Then the refined version of multiplication formula can be stated as below. This is a generalization of [KV], Theorem 2.3.

Theorem 5.1. *Let ρ_i be an irreducible representation of G indexed as in Notation 4.3. Denote by $\mathcal{O}_0$ the skyscraper sheaf $\mathbb{C}$ at $0 \in \mathbb{C}^2$ and by F the fundamental cycle. Then, we have*

$$\Psi(\mathcal{O}_0 \otimes \rho_i^*) \cong \begin{cases} \mathcal{O}_{E_i}(-1)[1] & \text{if } 1 \leqq i \leqq m, \\ \mathcal{O}_F & \text{if } i = 0, \\ 0 & \text{if } m < i \leqq n, \end{cases}$$

where $\mathcal{O}_{E_i}(-1)$ is the line bundle of degree -1 on $E_i \cong \mathbb{P}^1$ and $[\cdot]$ means the shift of a complex.

Proof. $\mathbb{L}q^*(\mathcal{O}_0 \otimes \rho_i^*)$ is defined by replacing $\mathcal{O}_0 \otimes \rho_i^*$ by its (equivariant) locally free resolution. We can calculate this by taking the Koszul complex (4.2) tensored with ρ_i^*; thus $\Psi(\mathcal{O}_o \otimes \rho_i^*)$ is given by the complex

$$0 \to \big(p_*q^*(K_{\mathbb{C}^2} \otimes \rho_i^*)\big)^G \to \big(p_*q^*(\Omega^1_{\mathbb{C}^2} \otimes \rho_i^*)\big)^G \to \big(p_*q^*(\mathcal{O}_{\mathbb{C}^2} \otimes \rho_i^*)\big)^G \to 0.$$

By virtue of Corollary 3.2, this is a complex of full sheaves and hence coincides with the complex

$$(5.1) \qquad 0 \to \widetilde{\tau(M_i)} \xrightarrow{\varphi} \widetilde{N_{M_i}} \xrightarrow{\psi} \widetilde{M_i} \to 0,$$

the torsion-free pull back of (4.3). Since φ is always injective, we have

$$(5.2) \qquad \Psi^j(\mathcal{O}_0 \otimes \rho_i^*) = 0 \quad (j \neq 0, -1)$$

for arbitrary i, where $\Psi^j(-)$ denotes the j-th cohomology sheaf of the complex $\Psi(-)$. From the exactness of (4.3) (when $i \neq 0$) and (4.4) (when $i = 0$), we deduce

$$(5.3) \qquad \Psi^0(\mathcal{O}_0 \otimes \rho_i^*)(=\operatorname{coker}\psi) \cong \begin{cases} 0 & \text{if } i \neq 0, \\ \mathcal{O}_F & \text{if } i = 0. \end{cases}$$

Putting

$$\mathscr{F}_i = \Psi^{(-1)}(\mathcal{O}_0 \otimes \rho_i^*)(=\ker\psi/\operatorname{Im}\varphi),$$

we shall determine these sheaves. Note that

$$(5.4) \qquad \operatorname{supp}(\mathscr{F}_i) \subseteqq F(\text{set-theoritically}),$$

$$(5.5) \qquad H^0(\mathscr{F}_i) = H^1(\mathscr{F}_i) = 0,$$

$$(5.6) \qquad \mathscr{F}_i \text{ is purely of dimension 1 (if it is non-zero).}$$

(5.4) and (5.5) follow from the exactness of (4.3) (or (4.4)) and the rationality of the singularity. (5.6) is a consequence of (5.5) and also of the definition of $\mathscr{F}_i$ as a quotient of a locally free sheaf by its locally free subsheaf of the same rank.

As a first step, we shall show the statement of Theorem 4.4:

$$(5.7) \qquad c_1(\mathscr{F}_i) = \begin{cases} E_i & \text{if } 1 \leqq i \leqq m, \\ 0 & \text{otherwise.} \end{cases}$$

To do this along Wunram's idea, we calculate $p_*\mathbb{L}q^*(\mathcal{O}_0)$ by using the Koszul complex (4.2); $p_*\mathbb{L}q^*(\mathcal{O}_0)$ is given by

$$(5.8) \qquad 0 \to \mathscr{R} \otimes_{\mathbb{C}} \det(\rho_{\mathrm{nat}}^*) \to \mathscr{R} \otimes_{\mathbb{C}} \rho_{\mathrm{nat}}^* \to \mathscr{R} \to 0,$$

where $\mathscr{R} = p_*\mathcal{O}_{\mathscr{Z}}$ and ρ_{nat} denotes the 2-dimensional representation given by the inclusion $G \subset \mathrm{GL}(2, \mathbb{C})$. On the other hand, we have a decomposition

$$(5.9) \qquad p_*\mathbb{L}q^*(\mathcal{O}_0) = \bigoplus_{i=0}^{n} \Psi(\mathcal{O}_0 \otimes \rho_i^*) \otimes_{\mathbb{C}} \rho_i$$

with respect to the G-action. Since the alternating sum of the first Chern classes of sheaves in (5.8) is 0, we can deduce from (5.2), (5.3) and (5.9) that

$$\sum_{i=0}^{n} (\dim \rho_i) c_1(\mathscr{F}_i) = c_1\bigl(\Psi^0(\mathcal{O}_0 \otimes \rho_0)\bigr) = F = \sum_{i=1}^{m} r_i E_i.$$

In this equation, each $c_1(\mathscr{F}_i)$ $(0 \leqq i \leqq n)$ is represented by an effective divisor supported by the exceptional locus by virtue of (5.4), and we have $r_i = \dim \rho_i$ for $1 \leqq i \leqq m$ as in Theorem 4.2. Thus (5.7) is proved if we show $\mathrm{supp}(\mathscr{F}_i) \supseteqq E_i$ for $1 \leqq i \leqq m$.

We note that (4.3) is non-split by virtue of the non-splitness of the Koszul complex (4.2). On the other hand, for the special reflexive module M_i $(1 \leqq i \leqq m)$, we have

$$\mathrm{Ext}^1_{\mathscr{O}_Y}\big(\widetilde{M_i}, \widetilde{\tau(M_i)}\big) = 0$$

since $H^1(\widetilde{M_i})^\vee = 0$ (speciality) and since $\widetilde{\tau(M_i)}$ is generated by global sections. It follows that (5.1) is non-exact for $1 \leqq i \leqq m$ and we have

$$0 \neq \mathrm{Ext}^1_{\mathscr{O}_Y}(\widetilde{M_i}, \ker \psi) \cong \mathrm{Ext}^1_{\mathscr{O}_Y}(\widetilde{M_i}, \mathscr{F}_i).$$

From the construction (4.1) of M_i $(1 \leqq i \leqq m)$, we obtain an exact sequence

$$0 \to (\widetilde{M_i})^\vee \otimes \mathscr{F}_i \to \mathscr{F}_i^{\oplus r} \to \mathscr{F}_i \otimes \mathscr{O}_{D_i} \to 0.$$

We have $H^1(\mathscr{F}_i) = 0$ by (5.5), $H^1\big((\widetilde{M_i})^\vee \otimes \mathscr{F}_i\big) \neq 0$ and hence $H^0(\mathscr{F}_i \otimes \mathscr{O}_{D_i}) \neq 0$ by the above exact sequence. This implies $\mathrm{supp}(\mathscr{F}_i) \cap D_i \neq \emptyset$ and hence $\mathrm{supp}(\mathscr{F}_i) \supseteqq E_i$. (See (5.4) and (5.6).) Thus we have proved (5.7).

Now (5.5), (5.6) and (5.7) imply

$$\mathscr{F}_i \cong \begin{cases} \mathscr{O}_{E_i}(-1) & \text{if } 1 \leqq i \leqq m, \\ 0 & \text{otherwise.} \end{cases}$$

This, together with (5.2) and (5.3), proves our assertion. □

6. An adjoint functor

In this section, we consider a correspondence in the opposite direction. We have a commutative diagram

$$\begin{array}{ccc} Y \times \mathbb{C}^2 & \xrightarrow{\pi_{\mathbb{C}^2}} & \mathbb{C}^2 \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X = \mathbb{C}^2/G \end{array}$$

where π_Y and $\pi_{\mathbb{C}^2}$ denote the projections. We defined the functor $\Psi: D_c^G(\mathbb{C}^2) \to D_c(Y)$ so that

$$\Psi(-) := [p_* \mathbb{L} q^*(-)]^G = \big[\mathbb{R}\pi_{Y*}\big(\mathscr{O}_{\mathscr{Z}} \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_{Y \times \mathbb{C}^2}} \pi^*_{\mathbb{C}^2}(-)\big)\big]^G.$$

Let $\Phi: D_c(Y) \to D_c^G(\mathbb{C}^2)$ denote the functor in the other direction defined as

$$\Phi(-) := \mathbb{R}\pi_{\mathbb{C}^2 *}\big(\mathscr{O}^\vee_{\mathscr{Z}} \overset{\mathbb{L}}{\otimes} \pi^*_Y(- \otimes \rho_0) \otimes (\pi^*_{\mathbb{C}^2} K_{\mathbb{C}^2})\big)[2],$$

where

$$\mathcal{O}_{\mathcal{Z}}^{\vee} = \mathbb{R}\,\mathcal{H}om_{\mathcal{O}_{Y\times\mathbb{C}^2}}(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}_{Y\times\mathbb{C}^2})$$

is the derived dual of $\mathcal{O}_{\mathcal{Z}}$ and where $-\otimes\rho_0$ denotes the functor

$$D_c(Y) \to D_c^G(Y)$$

attaching the trivial G-action. (In the notation $D_c^G(Y)$, we consider G-equivariant coherent sheaves with respect to the trivial G-action on Y.) Then we have

Lemma 6.1. *Ψ is a left adjoint of Φ.*

Proof. This can be proved as in [BKR]:

$$\begin{aligned}
\mathrm{Hom}_{D_c(Y)}\big(\Psi(\clubsuit), \spadesuit\big) &\cong \mathrm{Hom}_{D_c(Y)}\big(\big[\mathbb{R}\pi_{Y*}\big(\mathcal{O}_{\mathcal{Z}} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{Y\times\mathbb{C}^2}} \pi_{\mathbb{C}^2}^*(\clubsuit)\big)\big]^G, \spadesuit\big)\\
&\cong \mathrm{Hom}_{D_c^G(Y)}\big(\mathbb{R}\pi_{Y*}\big(\mathcal{O}_{\mathcal{Z}} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{Y\times\mathbb{C}^2}} \pi_{\mathbb{C}^2}^*(\clubsuit)\big), \spadesuit\otimes\rho_0\big)\\
&\cong \mathrm{Hom}_{D^G(Y\times\mathbb{C}^2)}\big(\mathcal{O}_{\mathcal{Z}} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{Y\times\mathbb{C}^2}} \pi_{\mathbb{C}^2}^*(\clubsuit), \pi_Y^*(\spadesuit\otimes\rho_0)\otimes(\pi_{\mathbb{C}^2}^*K_{\mathbb{C}^2})[2]\big)\\
&\cong \mathrm{Hom}_{D^G(Y\times\mathbb{C}^2)}\big(\pi_{\mathbb{C}^2}^*(\clubsuit), \mathcal{O}_{\mathcal{Z}}^{\vee} \overset{\mathbb{L}}{\otimes} \pi_Y^*(\spadesuit\otimes\rho_0)\otimes(\pi_{\mathbb{C}^2}^*K_{\mathbb{C}^2})[2]\big)\\
&\cong \mathrm{Hom}_{D_c^G(\mathbb{C}^2)}\big(\clubsuit, \mathbb{R}\pi_{\mathbb{C}^2*}\big(\mathcal{O}_{\mathcal{Z}}^{\vee} \overset{\mathbb{L}}{\otimes} \pi_Y^*(\spadesuit\otimes\rho_0)\otimes(\pi_{\mathbb{C}^2}^*K_{\mathbb{C}^2})[2]\big)\big)\\
&\cong \mathrm{Hom}_{D_c^G(\mathbb{C}^2)}\big(\clubsuit, \Phi(\spadesuit)\big).
\end{aligned}$$

Here, the third isomorphism is the equivariant Grothendieck duality (see [BKR]) for π_Y, which can be applied since $\mathcal{O}_{\mathcal{Z}} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{Y\times\mathbb{C}^2}} \pi_{\mathbb{C}^2}^*(\clubsuit)$ has proper support over Y. The other adjunctions are rather standard ([BKR]). □

In the case $G \subset \mathrm{SL}(2,\mathbb{C})$, Ψ and Φ give equivalence of categories as in [KV] and [BKR]. If $G \not\subset \mathrm{SL}(2,\mathbb{C})$, $\Psi(\mathcal{O}_0\otimes\rho_i^*) \cong 0$ for a non-special irreducible representation ρ_i and hence Ψ is not an equivalence. But we have the following.

Theorem 6.2. *Φ is fully faithful.*

Proof. We can follow the argument in [BKR]. We have the inequality

$$\dim Y \times_X Y \leqq 2,$$

which is sharper than the one assumed in [BKR]. Therefore, we can apply the intersection theorem without using the vanishing of $G\text{-}\mathrm{Ext}^2_{\mathcal{O}_{\mathbb{C}^2}}(\mathcal{O}_{Z_{y_1}}, \mathcal{O}_{Z_{y_2}})$ $(y_1 \neq y_2 \in Y)$. □

7. Representations constructed from ideals

In this section, we shall establish a Ito-Nakamura type correspondence using the results in the previous sections. We use Lemma 6.1 in the form

$$\operatorname{Hom}^k_{D_c(Y)}\big(\Psi(\mathcal{O}_0 \otimes \rho_i^*), \mathcal{O}_y\big) \cong \operatorname{Hom}^k_{D_c^G(\mathbb{C}^2)}\big(\mathcal{O}_0 \otimes \rho_i^*, \Phi(\mathcal{O}_y)\big), \tag{7.1}$$

where $\mathcal{O}_0$ and $\mathcal{O}_y$ are the structure sheaves of $\{0\} \subset \mathbb{C}^2$ and $\{y\} \subset Y$ respectively. As for the left hand side of (7.1), we have calculated $\Psi(\mathcal{O}_0 \otimes \rho_i^*)$ in Theorem 5.1. For the right hand side, the definition of Φ implies

$$\Phi(\mathcal{O}_y) = \mathcal{O}_{Z_y}^\vee \otimes K_{\mathbb{C}^2}[2] (\cong \omega_{Z_y}), \tag{7.2}$$

where Z_y is the subscheme of $\mathbb{C}^2$ corresponding to y and $\mathcal{O}_{Z_y}^\vee = \mathbb{R}\mathscr{H}om_{\mathcal{O}_{\mathbb{C}^2}}(\mathcal{O}_{Z_y}, \mathcal{O}_{\mathbb{C}^2})$ is the derived dual on $\mathbb{C}^2$.

From (7.1), Theorem 5.1 and (7.2), we deduce that for $1 \leqq i \leqq m$,

$$\begin{aligned} \operatorname{Ext}^{k-1}_{\mathcal{O}_Y}\big(\mathcal{O}_{E_i}(-1), \mathcal{O}_y\big) &\cong \operatorname{Hom}^k_{D_c^G(\mathbb{C}^2)}(\mathcal{O}_0 \otimes_{\mathbb{C}} \rho_i^*, \mathcal{O}_{Z_y}^\vee \otimes K_{\mathbb{C}^2}[2]) \\ &\cong \operatorname{Hom}^k_{D_c^G(\mathbb{C}^2)}\big(\mathcal{O}_{Z_y}, (\mathcal{O}_0 \otimes_{\mathbb{C}} \rho_i^*)^\vee \otimes K_{\mathbb{C}^2}[2]\big) \\ &\cong \operatorname{Hom}^k_{D_c^G(\mathbb{C}^2)}\big(\mathcal{O}_{Z_y}, (\mathcal{O}_0 \otimes_{\mathbb{C}} \rho_i)\big) \\ &= G\text{-}\operatorname{Ext}^k_{\mathcal{O}_{\mathbb{C}^2}}(\mathcal{O}_{Z_y}, \mathcal{O}_0 \otimes_{\mathbb{C}} \rho_i) \end{aligned} \tag{7.3}$$

where $(\mathcal{O}_0 \otimes_{\mathbb{C}} \rho_i^*)^\vee$ is also the derived dual on $\mathbb{C}^2$. Note that the third isomorphism above is given by the duality for $\{0\} \hookrightarrow \mathbb{C}^2$ (which can easily be verified by using the Koszul complex (4.2)). Similarly, we have

$$\operatorname{Ext}^k_{\mathcal{O}_Y}(\mathcal{O}_F, \mathcal{O}_y) \cong G\text{-}\operatorname{Ext}^k_{\mathcal{O}_{\mathbb{C}^2}}(\mathcal{O}_{Z_y}, \mathcal{O}_0 \otimes_{\mathbb{C}} \rho_0) \tag{7.4}$$

and

$$0 \cong G\text{-}\operatorname{Ext}^k_{\mathcal{O}_{\mathbb{C}^2}}(\mathcal{O}_{Z_y}, \mathcal{O}_0 \otimes_{\mathbb{C}} \rho_i) \quad (m < i \leqq n). \tag{7.5}$$

Put $k = 1$ in (7.3), (7.4) and (7.5). Then we deduce that $G\text{-}\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{C}^2}}(\mathcal{O}_{Z_y}, \mathcal{O}_0 \otimes_{\mathbb{C}} \rho_i)$ is non-zero (then is 1-dimensional) if and only if

$$(i = 0 \text{ and } y \in F) \quad \text{or} \quad (1 \leqq i \leqq m \text{ and } y \in E_i).$$

If we have a non-trivial G-equivariant extension

$$0 \to \mathcal{O}_0 \otimes_{\mathbb{C}} \rho_i \to \mathscr{F} \to \mathcal{O}_{Z_y} \to 0,$$

we can lift $1 \in \mathcal{O}_{Z_y}$ to an invariant section s of $\mathscr{F}$. Then the non-triviality of the extension and the irreducibility of ρ_i induce that $\mathscr{F}$ is generated by s as an $\mathscr{O}_{\mathbb{C}^2}$-module. Therefore $\mathscr{F}$ is of the form

$$\mathscr{F} \cong \mathscr{O}_{\mathbb{C}^2}/J$$

with $mI_y \subseteqq J \subseteqq I_y$ and $I_y/J \cong \mathscr{O}_0 \otimes \rho_i$. Thus we obtain

Theorem 7.1. *Denote by I_y the ideal corresponding to $y \in Y$ and by m the maximal ideal of $\mathscr{O}_{\mathbb{C}^2}$ corresponding to the origin 0. If y is in the exceptional locus, then we have an isomorphism*

$$I_y/mI_y \cong \begin{cases} \rho_i \oplus \rho_0 & \text{if } y \in E_i, \text{ and } y \notin E_j \text{ for } j \neq i, \\ \rho_i \oplus \rho_j \oplus \rho_0 & \text{if } y \in E_i \cap E_j, \end{cases}$$

as representations of G, where ρ_i is the special representation associated with the irreducible exceptional curve E_i.

We can also calculate the socle of $\mathscr{O}_{Z_y}$. By Serre duality we have

$$\begin{aligned} G\text{-}\mathrm{Ext}^k_{\mathscr{O}_{\mathbb{C}^2}}(\mathscr{O}_{Z_y}, \mathscr{O}_0 \otimes \rho) &\cong G\text{-}\mathrm{Ext}^{2-k}_{\mathscr{O}_{\mathbb{C}^2}}(\mathscr{O}_0 \otimes \rho \otimes K^*_{\mathbb{C}^2}, \mathscr{O}_{Z_y}) \\ &\cong G\text{-}\mathrm{Ext}^{2-k}_{\mathscr{O}_{\mathbb{C}^2}}(\mathscr{O}_0 \otimes \rho \otimes \det \rho_{\mathrm{nat}}, \mathscr{O}_{Z_y}). \end{aligned}$$

Thus by putting $k = 2$ in (7.3), (7.4) and (7.5) we obtain

Theorem 7.2. *Under the same notation as in the previous theorem, the socle of $\mathscr{O}_{Z_y}$ is*

$$(I_y : m)/I_y \cong \begin{cases} \rho_i \otimes \det \rho_{\mathrm{nat}} & \text{if } y \in E_i, \text{ and } y \notin E_j \text{ for } j \neq i, \\ (\rho_i \oplus \rho_j) \otimes \det \rho_{\mathrm{nat}} & \text{if } y \in E_i \cap E_j. \end{cases}$$

8. Connectedness of G-Hilb($\mathbb{C}^2$)

In this section, we prove the connectedness.

Theorem 8.1. G-Hilb$(\mathbb{C}^2)$ *is connected.*

Proof. We shall show that $Y = G$-Hilb. Assume to the contrary that there exists a G-cluster $[Z] \in G\text{-Hilb}(\mathbb{C}^2)\backslash Y$. We consider $\Psi(\mathscr{O}_Z^\vee \otimes K_{\mathbb{C}^2}[2])$ under the notation of the previous sections. Since $\mathscr{O}_Z^\vee \otimes K_{\mathbb{C}^2}[2] \cong \omega_Z$ is a sheaf, $\Psi(\mathscr{O}_Z^\vee \otimes K_{\mathbb{C}^2}[2])$ is given by a complex

$$0 \to \widetilde{N^{-2}} \xrightarrow{\varphi} \widetilde{N^{-1}} \xrightarrow{\psi} \widetilde{N^0} \to 0 \tag{8.1}$$

of full sheaves on Y.

From the assumption $[Z] \notin Y$, we deduce that for any $y \in Y$,

$$\begin{aligned} \mathrm{Hom}_{D_c(Y)}\big(\Psi(\mathscr{O}_Z^\vee \otimes K_{\mathbb{C}^2}[2]), \mathscr{O}_y\big) &\cong \mathrm{Hom}_{D_c^G(\mathbb{C}^2)}\big(\mathscr{O}_Z^\vee \otimes K_{\mathbb{C}^2}[2], \Phi(\mathscr{O}_y)\big) \\ &\cong \mathrm{Hom}_{D_c^G(\mathbb{C}^2)}(\mathscr{O}_Z^\vee \otimes K_{\mathbb{C}^2}[2], \mathscr{O}_{Z_y}^\vee \otimes K_{\mathbb{C}^2}[2]) \\ &\cong \mathrm{Hom}_{D_c^G(\mathbb{C}^2)}(\mathscr{O}_{Z_y}, \mathscr{O}_Z) = 0. \end{aligned}$$

It follows that ψ is surjective and hence that

$$\Psi(\mathcal{O}_Z^\vee \otimes K_{\mathbb{C}^2}[2]) = \mathcal{F}[1],$$

where

$$\mathcal{F} := \ker\psi/\operatorname{im}\varphi.$$

We shall calculate $H^1(\mathcal{F})$. Since (8.1) is a complex of acyclic sheaves, we can calculate the cohomology of $\mathcal{F}$ by taking global sections of (8.1):

$$H^1(\mathcal{F}) \cong \operatorname{coker} f_*\psi\colon N^{-1} \to N^0.$$

Since $H^0(\mathcal{O}_Z)$ is a regular representation of G, so is $H^0(\mathcal{O}_Z^\vee \otimes K_{\mathbb{C}^2}[2]) \cong H^0(\mathcal{O}_Z)^*$ and we see $\operatorname{coker} f_*\psi \cong \big(H^0(\mathcal{O}_Z^\vee \otimes K_{\mathbb{C}^2}[2])\big)^G = \mathbb{C}$. As a consequence, we have $H^1(\mathcal{F}) \neq 0$ and hence $\dim\operatorname{supp}(\mathcal{F}) = 1$. Therefore $c_1(\mathcal{F}) \neq 0$ in the Picard group.

On the other hand, the first Chern class (in the Picard group) of a sheaf (or a complex) can be determined by its class in the Grothendieck group and hence we have

$$\begin{aligned} c_1(\mathcal{F}) &= -c_1\big(\Psi(\mathcal{O}_Z^\vee \otimes K_{\mathbb{C}^2}[2])\big) \\ &= -\sum_{i=0}^{n} (\dim\rho_i) c_1\big(\Psi(\rho_i^* \otimes \mathcal{O}_0)\big) = 0 \end{aligned}$$

by Theorem 5.1. Thus we have met a contradiction. $\square$

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