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EXCEPTIONAL QUOTIENT SINGULARITIES

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Abstract. A singularity is said to be exceptional (in the sense of V. Shokurov), if for any log canonical boundary, there is at most one exceptional divisor of discrepancy -1. This notion is important for the inductive treatment of log canonical singularities. The exceptional singularities of dimension 2 are known: they belong to types E_6, E_7, E_8 after Brieskorn. In our previous paper, it was proved that the quotient singularity defined by Klein's simple group in its 3-dimensional representation is exceptional. In the present paper, the classification of all the three-dimensional exceptional quotient singularities is obtained. The main lemma states that the quotient of the affine 3-space by a finite group is exceptional if an only if the group has no semiinvariants of degree 3 or less. It is also proved that for any positive ϵ, there are only finitely many ϵ-log terminal exceptional 3-dimensional quotient singularities.

1. Introduction and statement of main results. The notion of exceptional singularity was introduced by Shokurov [Sh1]. A singularity (X, P) is called *exceptional*, if for any log canonical boundary, there is at most one exceptional divisor of discrepancy -1 over P (see Definition 2.5). The reason for distinguishing these singularities is that they have more complicated multiple anticanonical systems $|-nK_X|$ than the nonexceptional ones. However, Shokurov suggests, and proves in dimension 3, that the exceptional singularities are in a sense bounded (*loc. cit.*, Corollary 7.3). The search for "good" divisors in $|-nK_X|$, or so called *n-complements*, is an essential ingredient of Shokurov's project of the inductive study of log flips, log contractions and of the classification of log canonical singularities [Sh1, Sect. 7].

According to [Sh, 5.2.3, 5.6], [Sh1, 1.5], the exceptional log terminal singularities in dimension 2 are exactly the singularities of types E_6, E_7, E_8 in the sense of Brieskorn [Br]. Shokurov's approach gives a description of the dual graphs of their resolutions (cf. [I], [Ut, Chapter 3]). According to [Br], the exceptional graphs correspond to finite subgroups of $GL_2(\mathbb{C})$ of tetrahedral (E_6), octahedral (E_7) or icosahedral (E_8) types. We found a classification-free approach to the proof which works also in dimension 3. The exceptional groups of types E_6, E_7, E_8 are exactly those finite subgroups of $GL_2(\mathbb{C})$ which have no semiinvariants of degree ≤ 2. Hence the following proposition implies Shokurov's statement.

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PROPOSITION 1.1. *A two-dimensional quotient singularity $X = \mathbb{C}^2/G$ by a finite group G without reflections is exceptional if and only if G has no semiinvariants of degree ≤ 2.*

In dimension 3, we started the study of exceptional quotient singularities in our previous publication [MP], where we showed that the quotients of $\mathbb{C}^3$ by Klein's simple group of order 168 and by its unique central extension contained in $SL_3(\mathbb{C})$ (of order 504) are exceptional, in using the configuration of the action of Klein's group on $\mathbb{P}^2$ [Kl], [W]. The following Theorem is the main result of the present paper.

THEOREM 1.2. *A three-dimensional quotient singularity $X = \mathbb{C}^3/G$ by a finite group G without reflections is exceptional if and only if G has no semiinvariants of degree ≤ 3.*

Using Miller–Blichfeldt–Dickson classification [MBD] of finite subgroups of $GL_3(\mathbb{C})$, we obtain a complete list of such subgroups yielding exceptional singularities (Theorem 3.13).

Section 2 contains basic definitions and preliminary results. In particular, we show that the exceptional divisor with discrepancy -1 for an exceptional singularity is birationally unique and that its image is a point, independently of the choice of the boundary and in any dimension. Section 3 contains the proofs of Proposition 1.1, Theorem 1.2, and the list of finite subgroups of $GL_3(\mathbb{C})$ with exceptional quotients.

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2. Basics on exceptional singularities. For the reader's convenience, we reproduce here some basic facts about log canonical singularities. We are using the terminology and notation of [MP], [Ut], [Sh] and [Sh1] (see also [Ko] for a nice introduction to the subject).

Definition 2.1. Let $(X \ni P)$ be a normal singularity (not necessarily isolated) and let $D = \sum d_i D_i$ be a divisor on X with real coefficients. D is called a boundary if $0 \leq d_i \leq 1$ for all i. It is called a subboundary if it is majorated by a boundary. A proper birational morphism $f\colon Y \longrightarrow X$ is called a log resolution of (X, D) at P if Y is nonsingular near $f^{-1}(P)$ and $\operatorname{Supp}(D) \cup E$ is a normal crossing divisor on Y near $f^{-1}(P)$, where D is used to denote both the subboundary on X and its proper transform on Y, and $E = \cup E_i$ is the exceptional divisor of f. The pair (X, D) or, by abuse of language, the divisor $K_X + D$ is called *terminal, canonical,*

Kawamata log terminal, purely log terminal, and, respectively, *log canonical* near P, if the following conditions are verified:

(i) $K_X + D$ is $\mathbb{R}$-Cartier.

(ii) Let us write for any proper birational morphism $f: Y \dashrightarrow X$

$$K_Y \equiv f^*(K_X + D) + \sum a(E, X, D)E,$$

where E runs over prime divisors on Y, $a(E, X, D) \in \mathbb{R}$, and $a(D_i, X, D) = -d_i$ for each component D_i of D. Then, for some log resolution of (X, D) at P and for all prime divisors E on Y near P, we have: $a(E, X, D) > 0$ (for terminal), $a(E, X, D) \geq 0$ (for canonical), $a(E, X, D) > -1$ and no $d_i = 1$ (for Kawamata log terminal), $a(E, X, D) > -1$ (for purely log terminal, without any restriction on the subboundary D), and, respectively, $a(E, X, D) \geq -1$ (for log canonical).

The coefficients $a(E, X, D)$ are called the *discrepancies* of f, or of (X, D); they depend on the discrete valuations of the function field of X associated to the prime divisors E and on D, but not on the choice of f. We will identify prime divisors with corresponding discrete valuations when speaking about "divisors E over X" without indicating on which birational model E is realized. The conditions given by the inequalities in part (ii) of the above definition do not depend on the choice of a log resolution.

Definition 2.2. Let V be a normal variety and let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on V such that $K_V + D$ is $\mathbb{Q}$-Cartier. A subvariety $W \subset V$ is said to be a *center of log canonical singularities* for (V, D) if there exists a birational morphism from a normal variety $g: \tilde{V} \to V$ and a prime divisor E on $\tilde{V}$ with discrepancy $a(E, V, D) \leq -1$ such that $g(E) = W$. (The case $d_i = 1$, $W = D_i$ is also possible.) The union of all centers of log canonical singularities is called the *locus of log canonical singularities* and is denoted by $LCS(V, D)$ [Sh, 3.14].

The following statement is a weak form of [Ut, 17.4] (in dimension 2 it was proved earlier by Shokurov [Sh]).

THEOREM 2.3. *Let X be a normal projective variety and let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. If $-(K_X + D)$ is nef and big, then the locus of log canonical singularities is connected.*

THEOREM 2.4. [Sh, 6.9] *Let X be a normal projective surface and let $D = \sum d_i D_i$ be a boundary on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. If $K_X + D \equiv 0$, then the locus of log canonical singularities has one or two connected components.*

Shokurov informed us that the previous theorem is valid in any dimension modulo Log Minimal Model Program.

Definition 2.5. [Sh1, 1.5] Let $(X \ni P)$ be a normal singularity and let $D = \sum d_i D_i$ be a boundary on X such that $K_X + D$ is log canonical. The pair (X, D) is said to be *exceptional* if there exists at most one exceptional divisor E over X with discrepancy $a(E, X, D) = -1$. The singularity (X, P) is said to be *exceptional* if (X, D) is exceptional for any D whenever $K_X + D$ is log canonical.

LEMMA 2.6. *Assume that there exists a reduced divisor $S = \sum S_i$ passing through P such that $K_X + S$ is log canonical. Then $(X \ni P)$ is nonexceptional.*

For the proof, see Lemma 1.7 in [MP]. This implies, in particular, that any three-dimensional cDV-singularity [R], and any three-dimensional terminal singularity is nonexceptional (see [MP, 1.9]).

PROPOSITION 2.7. *Let (X, P) be a $\mathbb{Q}$-factorial exceptional singularity. Then the divisor E from Definition 2.5 with discrepancy $a(E, X, D) = -1$ is birationally unique. This means that there exists a unique discrete valuation ν of the field of rational functions $k(X)$, such that for any pair (X, D), exceptional at P, and for any log resolution $f : \tilde{X} \longrightarrow X$, if there is an exceptional divisor E in $\tilde{X}$ with $a(E, X, D) = -1$, then the corresponding discrete valuation $\nu_E = \nu$.*

Proof. Let $\{D_1, \dots, D_r\}$ be a finite set of irreducible divisors on X, and $f : \tilde{X} \longrightarrow X$ a log resolution of $(X, \sum_{i=1}^r D_i)$ at P. We can represent the boundaries $D = \sum_i d_i D_i$ with components from $\{D_1, \dots, D_r\}$ as the unit cube $I^r \subset \mathbb{R}^r$ of vectors (d_i). Then the subset $\Lambda \subset I^r$ corresponding to log canonical pairs (X, D) is given by a finite number of linear inequalities in d_i's

$$a(E, X, D) = a(E, X, 0) - \sum_i d_i \operatorname{mult}_E (D_i) \geq -1,$$

where E runs over the exceptional divisors such that $f(E) \ni P$. Let $\partial_+\Lambda$ be the closure of $\{(d_i) \in \partial\Lambda \mid 0 < d_i < 1 \ \forall\ i = 1, \dots, r\}$. Then the exceptionality of (X, P) implies that there are no points in $\partial_+\Lambda$ satisfying the equality $a(E, X, 0) - \sum_i \operatorname{mult}_E (D_i) = -1$ for two different E's. So, $\partial_+\Lambda$ (if nonempty) is an open convex polyhedron lying in exactly one hyperplane Π_E with equation $a(E, X, 0) - \sum_i d_i \operatorname{mult}_E (D_i) = -1$.

Now, let (X, P) be an exceptional singularity, $(X, D^{(1)}), (X, D^{(2)})$ two exceptional pairs, $f_i : X_i \longrightarrow X$ two log resolutions with divisors E_i such that $a(E_i, X, D^{(i)}) = -1$. Then we can apply the previous argument, taking the union of all the components of $D^{(1)}, D^{(2)}$ as $\{D_1, \dots, D_r\}$, and some log resolution dominating f_1, f_2 as f. Then we will have: firstly, by Lemma 2.6, $D^{(1)}, D^{(2)}$ are represented by points of $\partial_+\Lambda$, and secondly, $\Pi_{E_1} = \Pi_{E_2}$. But then, both of the E_i are of discrepancy -1 with respect to any D represented by a point of $\partial_+\Lambda$. As $D^{(1)} \in \partial_+\Lambda$, we have $a(E_1, X, D^{(1)}) = a(E_2, X, D^{(1)}) = -1$. By exceptionality, $E_1 = E_2$. This proves the assertion of the proposition. $\square$

PROPOSITION 2.8. *Let (X,P) be a $\mathbb{Q}$-factorial exceptional singularity. Then for any pair (X,D) exceptional at P and such that $LCS(X,D) \neq \varnothing$, we have $LCS(X,D) = \{P\}$.*

Proof. By Proposition 2.7, the exceptional divisor E with discrepancy -1 is biratonally unique. Suppose, $LCS(X,D) \supsetneqq \{P\}$. Then for a generic hyperplane section H of (X,P), we have $\mathrm{mult}_E(H) = 0$. This implies that $a(E,X,D+\epsilon H) = a(E,X,D) = -1$, and, by exceptionality, $a(E',X,D+\epsilon H) > -1$ for any sufficiently small positive ϵ and any exceptional divisor $E' \neq E$. Let $\epsilon_* = \sup\{\epsilon \mid (X,D+\epsilon H) \text{ is log canonical}\}$. Then $(X,D+\epsilon_* H)$ has two different divisors with discrepancy -1: E and either one of the $E' \neq E$ with minimal discrepancy, or H in the case when $\epsilon_* = 1$. This contradicts the exceptionality of (X,P). □

PROPOSITION 2.9. *Let (X_i,P_i), $i = 1,2$, be two germs of normal $\mathbb{Q}$-factorial varieties of dimension > 0. Then $(X_1 \times X_2, (P_1,P_2))$ is nonexceptional.*

Proof. If at least one of X_i is not log canonical with zero boundary, we are done, for $X_1 \times X_2$ is not log canonical with any boundary. If $(X_i,0)$ are log canonical, then, starting from any nonzero boundaries $D^{(i)}$ at P_i, we can find $\epsilon_i \geq 0$ such that both pairs $(X_i,\epsilon_i D^{(i)})$ are log canonical and possess at least one discrete valuation of discrepancy -1 for $i = 1,2$. Let W_i be their centers. Then $LCS(X_1 \times X_2, \epsilon_1 D^{(1)} \times X_2)$ contains $W_1 \times X_2$, and $LCS(X_1 \times X_2, X_1 \times \epsilon_2 D^{(2)})$ contains $X_1 \times W_2$. Hence $(X_1 \times X_2, (P_1,P_2))$ is nonexceptional by Proposition 2.8. □

3. Quotients $\mathbb{C}^m/G$. In the first part of this section we work in arbitrary dimension. The following Lemma was proved in [MP, 2.1] in the 3-dimensional case; now we generalize it to arbitrary dimension.

LEMMA 3.1. *Let $\pi: V \to X$ be a finite morphism of normal varieties, étale in codimension 1. Let D be a boundary on X, and $D' = \pi^* D$. Assume that $K_X + D$ is log canonical (and then automatically is $K_V + D'$). Then (X,D) is exceptional if and only if (V,D') is.*

Proof. For every proper birational morphism $f: Y \to X$ consider the following commutative diagram

$$\begin{tikzcd} W \arrow[r, "\varphi"] \arrow[d, "g"'] & Y \arrow[d, "f"'] \\ V \arrow[r, "\pi"] & X. \end{tikzcd} \tag{1}$$

where W is the normalization of the dominant component of $V \times_X Y$. By the ramification formula (cf. [Ko, proof of 3.16]) for every exceptional divisor F on Y and every exceptional divisor E on W which dominates F we have

$$a(E, V, D') + 1 = r(a(F, X, D) + 1), \tag{2}$$

where r is the ramification index at the generic point of E. Therefore

$$a(E, V, D') = -1 \quad \text{if and only if} \quad a(F, X, D) = -1. \tag{3}$$

Thus if (X, D) is nonexceptional, then (V, D') is so.

Conversely, assume that (V, D') is nonexceptional. Then there are infinitely many exceptional divisors over V with discrepancy $a(\ \ , V, D') = -1$. As in [Ko, proof of 3.16] we can see that all these divisors appear in some commutative square (1) (for suitable f), whence (X, D) is also nonexceptional by (3). □

Warning. The lemma does not imply that the singularity of X is exceptional if and only if V is. The assertion concerns *pairs* (V, D') with D' a pullback of a boundary from X.

3.2. Let $\pi: V \to X$ be the quotient morphism, where $V = \mathbb{C}^m$, $X = \mathbb{C}^m/G$, and $G \subset GL_m(\mathbb{C})$ is a finite subgroup without reflections. Let D be a boundary on X and let $D' := \pi^* D$. By [Sh, 2.2], (see also [Ut, 20.3], [Ko, 3.16]) $K_X + D$ is log canonical (resp., purely log terminal, Kawamata log terminal) if and only if $K_V + D'$ is as well.

Question 3.3. In the notation of 3.2, assume that G has a semiinvariant of degree $\leq m$. Does this imply that $(X \ni P)$ is nonexceptional?

Let ψ be such a homogeneous semiinvariant of minimal degree $d \leq m$ and let D' be its zero locus. By Lemma 3.1 it is sufficient, for the positive answer to 3.3, to prove that $K_V + D'$ is log canonical. It is clear that D' is a cone. The following proposition is an easy particular case of this question.

PROPOSITION 3.4. *Let G be maximally imprimitive, that is, let G contain a normal abelian subgroup A whose character subspaces V_χ are* 1*-dimensional and form one orbit under the action of G. Then V/G is nonexceptional.*

Proof. G/A acts by permutations of the V_χ, so G has a semiinvariant of the form $x_1 \cdots x_m$, where x_i are coordinate linear forms. Since $D' = \{x_1 \cdots x_m = 0\}$ is a simple normal crossing divisor, (V, D') is log canonical. Then (X, D) is also log canonical, and the assertion follows from Lemma 2.6. □

3.5. We cannot treat in general the case of *imprimitive* groups, that is, groups G which permute transitively factors of some direct sum decomposition $V = \oplus V_i$. *Maximally imprimitive* are those for which $\dim V_i = 1$. But if the action of G is not transitive for some direct sum decomposition, the singularity is also nonexceptional. This follows from the next proposition.

A group G is called *reducible* if $V = V_1 \oplus V_2$ with V_1, V_2 invariant under G, $\dim V_i > 0$ $(i = 1, 2)$.

PROPOSITION 3.6. *If G is reducible, then V/G is nonexceptional.*

Proof. The proof is similar to that of Proposition 2.9, though the result is not a corollary of it. Let G_i denote the image of G in $GL(V_i)$. Let D_i $(i = 1, 2)$ be any nonzero boundaries for $(V_i, 0)$, given by G_i-semiinvariant polynomials from $k[V_i]$. We can find $\epsilon_i > 0$ such that both pairs $(V_i, \epsilon_i D_i)$ possess at least one discrete valuation of discrepancy -1 for $i = 1, 2$. Let W_i be their centers. Define $D^{(1)} = D_1 \times V_2, D^{(2)} = V_1 \times D_2$ in V. Then $LCS((V, \epsilon_i D^{(i)})$ contains $W_1 \times V_2$ or $V_1 \times W_2$. Using a commutative square of the same type as (1), we conclude from (2) that the images of $W_1 \times V_2$ and $V_1 \times W_2$ in V/G are also centers of log canonical singularities. Hence V/G is nonexceptional by Proposition 2.8. □

LEMMA 3.7. *If $\dim X = m \leq 3$, the answer to Question 3.3 is affirmative.*

Proof. In [MP, Lemma 2.2], we proved the assertion in dimension 3. The same (and an even easier) argument works in dimension 2. □

This Lemma shows the "if" part of Proposition 1.1 and Theorem 1.2.

3.8. Now we will explain the logic of our proof of the exceptionality of singularities in Proposition 1.1 and Theorem 1.2. Let $(X \ni P)$ be a quotient singularity and D a boundary, such that $K_X + D$ is log canonical near P. Further, we will use the notations of Lemma 3.1. Take the smallest $n \in \mathbb{N}$ such that $F := nD'$ is an integral divisor. Then F locally near 0 can be defined by a semiinvariant function, say ψ. Denote $d := mult_0(\psi)$. Let $\sigma: W \to V = \mathbb{C}^m$ be the blow-up of the origin and let $S \simeq \mathbb{P}^{m-1}$ be the exceptional divisor. Then $K_W = \sigma^* K_V + (m-1)S$ and $\sigma^* F = R + dS$, where R is the proper transform of F.

Further

$$K_W + S + \frac{m}{d} R = \sigma^* \left(K_V + \frac{m}{d} F \right). \tag{4}$$

By [Ko, Lemma 3.10] $K_V + \frac{m}{d} F$ is log canonical if and only if $K_W + S + \frac{m}{d} R$ is. In this case the pair $(V, \alpha F)$ is exceptional for all $0 \leq \alpha \leq \frac{m}{d}$ if and only if $K_W + S + \frac{m}{d} R$ is purely log terminal. By the inversion of adjunction (see [Sh, 5.13], [Ut, 17.6]) the purely log terminal condition for $K_W + S + \frac{m}{d} R$ is equivalent to the condition that $K_S + \frac{m}{d} C$ is Kawamata log terminal, where $C = R \cap S$. It is clear that C is given by the equation $\psi_{\min} = 0$, where $\psi_{\min}$ is the homogeneous component of ψ of minimal degree d. Therefore we have:

PROPOSITION 3.9. *In the above notations, if* $K_S + \frac{m}{d}C$ *is Kawamata log terminal, then* $(V, \alpha F)$ *is exceptional for any* $0 \leq \alpha \leq \frac{m}{d}$*, and is not log canonical for any* $\alpha > \frac{m}{d}$.

As $D' = \frac{1}{n}F$, we need to apply this proposition only to the α of the form $\frac{1}{n}$. Whenever $K_X + D$ is log canonical, we have $0 \leq \alpha = \frac{1}{n} \leq \frac{m}{d}$, so the condition that $K_S + \frac{m}{d}C$ be Kawamata log terminal is sufficient for the exceptionality of $K_X + D$.

Remark 3.10. In the above notations, if G has no semiinvariants of degree $\leq m$, then $\lfloor \frac{m}{d}C \rfloor = 0$.

3.11. Two-dimensional case.

Proof of Proposition 1.1. We obtain Proposition 1.1 as an easy corollary of Proposition 3.9. Assume that G has no invariants of degree ≤ 2. Recall that $S \simeq \mathbb{P}^1$ in our case, so C is a finite set. By Proposition 3.9 it is sufficient to prove that $K_S + \frac{2}{d}C$ is Kawamata log terminal, and this is equivalent to $\lfloor \frac{2}{d}C \rfloor = 0$. The last assertion follows by Remark 3.10. □

3.12. Three-dimensional case.

Proof of Theorem 1.2. Assume that G has no invariants of degree ≤ 3. We shall prove that $\mathbb{C}^3/G$ is exceptional. By Proposition 3.9 it is sufficient to prove that $K_S + \frac{3}{d}C$ is Kawamata log terminal. Take c to be the log canonical threshold of (S, C), that is, the maximal α such that $K_S + \alpha C$ is log canonical. If $K_S + \frac{3}{d}C$ is not Kawamata log terminal, then $c \leq 3/d$. First we consider the case when $c < 3/d$. Then $-(K_S + cC)$ is ample. By the connectedness of Theorem 2.3, the locus of log canonical singularities is connected. By Remark 3.10 $\lfloor \frac{3}{d}C \rfloor = 0$, so any divisor of discrepancy -1 should be exceptional. Therefore the locus of log canonical singularities on S is a unique point, which must be invariant under the action of G. As $S = \mathbb{P}^2$, this point corresponds to an invariant vector line in $\mathbb{C}^3$, which has an invariant 2-dimensional complement H. The linear form defining H is a semiinvariant of degree 1, and this contradicts our assumptions. In the case when $c = 3/d$, we can use Theorem 2.4. Similarly to the above, we see that the locus of log canonical singularities is one or two points. In both cases, there is an invariant line in S, giving a semiinvariant of degree 1. This ends the proof of Theorem 1.2. □

The finite subgroups of $G \subset GL_3(\mathbb{C})$ were classified by Miller–Blichfeldt–Dickson [MBD] modulo central extensions (compare with [P]). There are 10 types of such groups, denoted by A,B, ...,J in [MBD]. A stands for abelian, B for reducible, and C, D are imprimitive. The groups of type C are called *tetrahedral*; their images in the symmetric group $\mathfrak{S}_3$ permuting the V_i, $i = 1, 2, 3$ (notations as in 3.5) are cyclic of order 3. The groups of type D are called *general*

monomial; their map to $\mathfrak{S}_3$ is surjective. The primitive subgroups of $GL_3(\mathbb{C})$ belong to the 6 types E, F, G, H, I, J. The orders of the associated collineation groups $PG = G/(G \cap \mathbb{C}^*) \subset PGL_3(\mathbb{C})$ are 36, 72, 216, 60, 360, 168; the first three are solvable, and the last three are simple. The collineation groups from G to J have their names: the Hessian group, the icosahedral one, the alternating group $\mathfrak{A}_6$ of degree 6, and, finally, Klein's simple group. We have also $P\mathrm{E} \triangleleft P\mathrm{F} \triangleleft P\mathrm{G}$.

The following assertion is a consequence of Theorem 1.2.

THEOREM 3.13. *Let G be a finite subgroup of $GL_3(\mathbb{C})$. Then the quotient $\mathbb{C}^3/G$ is exceptional if and only if G belongs to one of the* 4 *types* F, G, I, J.

Proof. We can eliminate the types A–D by Propositions 3.4, 3.6. Further, the icosahedral group in its 3-dimensional representation has an invariant of degree two, which is a semiinvariant of any group of type H. This follows, for example, from the fact that the 3-dimensional representation is the complexification of the standard real one, which has an invariant scalar product.

LEMMA 3.14. *Any group of type* E *has two nonproportional semiinvariants of degree* 3. *Groups of types* F, G *have no semiinvariants of degree* ≤ 3.

Proof. According to [MBD, Sect. 115], a group G of one of the types E, F, G is an extension of a group H of type D which leaves invariant the set of four triangles t_1, t_2, t_3, t_4 defined in appropriate coordinates by the equations

$$x_1x_2x_2 = 0,\ (x_1 + x_2 + x_3)(x_1 + \omega x_2 + \omega^{2+i}x_3)(x_1 + \omega^2 x_2 + \omega^{1+i}x_3) = 0$$
$$\left(\omega = \exp\frac{2\pi\sqrt{-1}}{3},\ i = 0, 1, 2\right),$$

and the associated collineation group PG is completely characterized by its image in the group $\mathfrak{S}_4$ of permutations on the set $\{t_1, t_2, t_3, t_4\}$. It is the subgroup of order 2, conjugate to $\{1, (t_1t_2)(t_3t_4)\}$ for the type E, $\{1, (t_1t_2)(t_3t_4), (t_1t_3)(t_2t_4), (t_1t_4)(t_2t_3)\}$ for the type F, and the full alternating group on four letters for the type G. Moreover, one can easily verify that the t_i belong to one pencil Φ of plane cubics (see [Sp, Lemma 4.7.6]). So, a group of type E acts on the pencil Φ with image $\mathbb{Z}/2\mathbb{Z}$ in $Aut(\mathbb{P}^1)$. Any involution on $\mathbb{P}^1$ has two fixed points, which implies the result for type E.

Since the image of G in $Aut(\mathbb{P}^1)$ has no fixed points for the groups of types F, G, they do not have semiinvariants in the pencil Φ. However, any semiinvariant of G should also be that of H. According to [MBD, Sect. 113], PH is generated by the cycle $c = (x_1x_2x_3)$, transposition $\tau = (x_2x_3)$ and dilatation $\kappa = diag(1, \omega, \omega^2)$. Any semiinvariant of $\mathfrak{S}_3 =< c, \tau >$ is a polynomial in elementary symmetric functions σ_i $(i = 1, 2, 3)$ in x_1, x_2, x_3 and in $\Delta = (x_1 - x_2)(x_2 - x_3)(x_1 - x_3)$. A direct verification shows that in degrees ≤ 3, only the linear combinations of $x_1^3 + x_2^3 + x_3^3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$ and of σ_3 are (skew-)symmetric semiinvariants under κ. This yields exactly the pencil Φ. $\square$

Thus, by Lemma 3.7, the quotients of types E, H are not exceptional. By Theorem 1.2, the quotients of types F, G are exceptional. It remains to verify that the groups of types I, J have no semiinvariants of degree ≤ 3.

For the groups of type I, we can take a representative $I_0 \subset SL_3(\mathbb{C})$ of order 1080. The homogeneous semiinvariants of any group G of type I will coincide with those of I_0, because they are central extensions of the same group $PI_0 \simeq \mathfrak{A}_6$. But all the semiinvariants of the group I_0 are indeed invariants. This follows from the fact that it has no normal subgroups with abelian quotient: the only nontrivial normal subgroup of I_0 is its center, isomorphic to $\mathbb{Z}/3\mathbb{Z}$, and its quotient is the simple group $\mathfrak{A}_6$. The algebra of invariants of I_0 was determined by Wiman [Wi] (see also a modern account of invariants of finite subgroups of $SL_3(\mathbb{C})$ in [YY], where I of [MBD] is denoted by L); it is generated by basic invariants of degrees 6, 12, 30 and 45 with one relation of weighted degree 90 between them. Thus, there are no semiinvariants of degree ≤ 3, and we are done.

Klein's simple group has a representation $J_{168} \subset SL_3(\mathbb{C})$. As above, the only semiinvariants are invariants, and they were determined by Klein [Kl], see also [W] or [YY]. The degrees of basic invariants here are 4, 6, 14, 21, and there is one relation of weighted degree 42 between them. Again there are no semiinvariants of degree ≤ 3, and we are done. This ends the proof of Theorem 3.13. □

Using the classification of finite subgroups in $SL_3(\mathbb{C})$ ([P], [YY]), one can get the following assertion:

COROLLARY 3.15. *Let G be a finite subgroup of $GL_3(\mathbb{C})$ such that $\mathbb{C}^3/G$ is an exceptional canonical singularity. Then G is, up to conjugation, one of the following subgroups of $SL_3(\mathbb{C})$:*

(i) *Klein's simple group $J_{168} \subset SL_3(\mathbb{C})$,*

(ii) *the unique central extension J'_{504} of J_{168} contained in $SL_3(\mathbb{C})$,*

(iii) *the Hessian group $G_{648} \subset SL_3(\mathbb{C})$,*

(iv) *the normal subgroup F_{216} of G_{648},*

(v) *a central extension I_{1080} of $\mathfrak{A}_6$.*

Proof. It suffices to show that $G \subset SL_3(\mathbb{C})$. Let r be the order of the center Z of G. Then $\mathbb{C}^3/Z$ is resolved by a single blow-up, giving an exceptional divisor with discrepancy $-1 + \frac{3}{r}$. Construct, as in the proof of Lemma 3.1, a diagram (1) with $\mathbb{C}^3/Z$, $\mathbb{C}^3/G$ in place of V, X respectively. Then the formula (2) implies that the minimal discrepancy of exceptional divisors E over X $a_{\min} \leq -1 + \frac{3}{r}$. For canonical singularities, $a_{\min} \geq 0$, hence $r \leq 3$. If $r = 1$ or 3, $G \subset SL_3(\mathbb{C})$ and we are done.

Assume that $r = 2$. Then $Z = \{\pm 1\}$. Let $G_0 = G \cap SL_3(\mathbb{C})$. The orders of the subgroups of $SL_3(\mathbb{C})$ of types F, G, I, J are even, hence G_0 contains an element g of order 2. Then either g or $-g$ is a reflection. This contradicts our hypotheses. Hence $r = 2$ is impossible. □

Recall that a variety X is said to have ϵ-log terminal singularities if $a(E, X, 0) > -1 + \epsilon$ for any exceptional divisor E over X. Adapting the proof of the previous corollary to the case $a_{\min} > -1 + \epsilon$ instead of $a_{\min} \geq 0$, we obtain the following result:

COROLLARY 3.16. *Fix $\epsilon > 0$. Then the set of subgroups $G \subset GL_3(\mathbb{C})$ without reflections such that $\mathbb{C}^3/G$ is an exceptional ϵ-log terminal singularity is finite up to conjugation.*

A similar finiteness result was obtained by [Bor] for abelian quotients of any dimension (which are never exceptional by Proposition 3.6).

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