Real plane sextic curves without real singular points

Plan

1. Algebraic curves in $\mathbb{R}P^2$.
2. Singular real sextic curves.
3. Empty real plane curves.

1. Algebraic curves in $\mathbb{RP}^2$

Questions related to Hilbert's 16th problem.

$F \in \mathbb{R}[X_0, X_1, X_2]$ homogeneous polynomial of degree d

$\mathbb{R}F \subset \mathbb{RP}^2$ zero locus of F

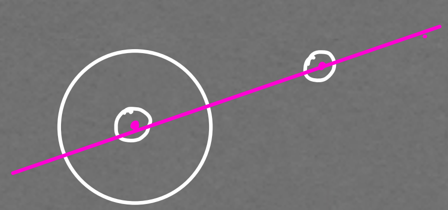
$$\begin{array}{ccc}\mathbb{R}F & \subset & \mathbb{RP}^2\\ \cap & & \cap \\ \mathbb{C}F & \subset & \mathbb{CP}^2\end{array}$$

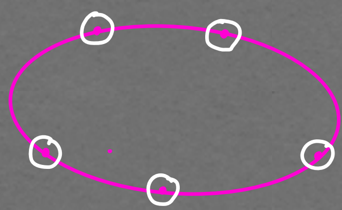
Topology of $\mathbb{R}F$?

Topology of $(\mathbb{RP}^2, \mathbb{R}F)$?

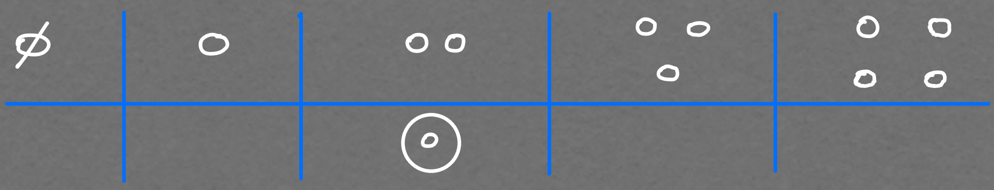
Non-singular curves : F does not have critical points in $\mathbb{C}^3 \setminus \{0\}$

Case $d = 4$





Isotopy classification in degree 4 $\quad (\mathbb{RP}^2, \mathbb{R}F)$

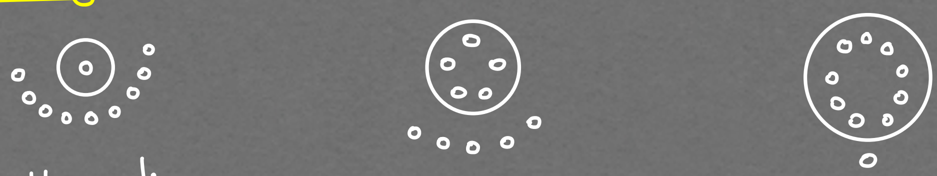


Harnack inequality (1876)

$$\#\text{ c.c. of } \mathbb{R}F \leq \frac{(d-1)(d-2)}{2} + 1$$

Maximal curves. $\#\text{ c.c. of } \mathbb{R}F = \dfrac{(d-1)(d-2)}{2} + 1.$

Isotopy classification of maximal non-singular sextics



Harnack — $p = 10,\ n = 1$

Gudkov — $p = 6,\ n = 5$

Hilbert — $p = 2,\ n = 9$

Gudkov - Rokhlin congruence

For maximal non-singular curves of degree $d = 2k$ in $\mathbb{R}P^2$:

$$p - n = k^2 \bmod 8$$

p # even ovals

n # odd ovals

Halves $\mathbb{R}P^2_+ = \{(x_0 : x_1 : x_2) \in \mathbb{R}P^2 \mid F(x_0, x_1, x_2) \geqslant 0\}$

$\mathbb{R}P^2_- = \{(x_0 : x_1 : x_2) \in \mathbb{R}P^2 \mid F(x_0, x_1, x_2) \leq 0\}$

$p - n = \chi(\mathbb{R}P^2_+)$

Petrovsky inequalities

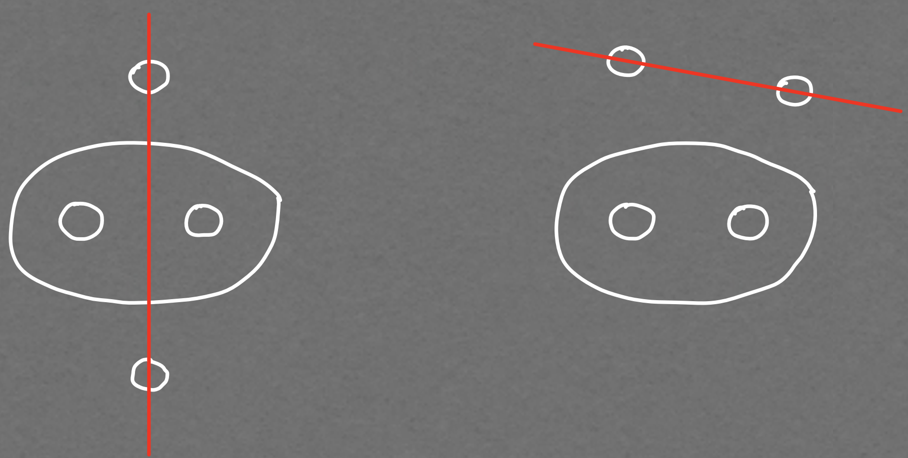
For non-singular curves of degree $d = 2k$ in $\mathbb{R}P^2$:

$$-\frac{3}{2}k(k-1) \leq p - n \leq \frac{3}{2}k(k-1) + 1$$

Rigid isotopy classification

Discriminant $\mathcal{D} \subset \mathbb{R}C_d \simeq \mathbb{R}P^N$, where $N = \frac{d(d+3)}{2}$

Connected components of $\mathbb{R}C_d \setminus \mathcal{D}$



More generally, one can speak about

equivariant equisingular deformation classification.

For sextic curves:

- non-singular (Nikulin)
- with one non-degenerate double point (I)
- with empty real part (joint work with A. Degtyarev)

2. Singular real sextic curves

Consider a simple (that is, with A-D-E singularities) sextic curve C given by a polynomial F.
The minimal resolution $X = X_C$ of the singularities of the double covering $X' \to \mathbb{C}P^2$ ramified at $\mathbb{C}F$ is a K3-surface.

The lattice $L_X = H_2(X)$ is isomorphic to $2E_8 \oplus 3U$.
The classes of exceptional divisors span a root lattice $S \subset L_X$. It can be identified with the (abstract) set of singularities of C.

Denote by $h = h_C$ the polarization, i.e., the class of the pullback of a line. One has $h^2 = 2$. Put $S_h = S \oplus \langle h \rangle \subset L_X$, and let $\tilde{S}_h$ be the primitive hull of S_h.

In the real case, two real structures σ^+ and σ^- on X.

For a real sextic C without real singular points, one can define its real homological type:

the isomorphism class of the quadruple

$$(L_X \supset \tilde{S}_h \ni h, \sigma_*^+).$$

Theorem (Degtyarev - I)

The real homological type of a simple sextic without real singular points determines its equivariant equisingular deformation class.

3. Empty real plane curves

Study up to equivariant equisingular deformations.

For an empty real curve the degree should be even.

- If the degree is 2, empty real curves are non-singular.

- If the degree is 4, the only possible options are
  - non-singular,
  - two non-degenerate double points ($2A_1$),
  - four non-degenerate double points ($4A_1$, reducible, two families: either real, or complex conjugate conics),
  - two cusps ($2A_2$),
  - two tangency points ($2A_3$, reducible, two families: either real, or complex conjugate conics).

- Degree 6   Equivariante equisingular deformation classification (joint work with A. Degtyarev)

  169 classes