Complex curves in nilmanifolds

Algebraic Geometry, Lipschitz Geometry and Singularities, Pipa, Brazil

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Motivation

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Main result A general twistor deformation of a hypercomplex nilmanifold admits no complex curves.

Nilmanifolds

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Remark: We consider left action of a lattice Γ on G and G acts on the nilmanifold $\Gamma \backslash G$ from the right.

Relations with Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. Define $\mathfrak{g}_0 := \mathfrak{g}$, $\mathfrak{g}_k := [\mathfrak{g}; \mathfrak{g}_{k-1}]$, $k \in \mathbb{Z}_{>0}$. Then

$$\mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots$$

is called **the lower central series** of $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is called **nilpotent** if $\mathfrak{g}_k = 0$ for some $k \in \mathbb{Z}_{>0}$.

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A connected, simply connected, nilpotent Lie group G admits a co-compact lattice if and only if the Lie algebra of G has integer structure constants relative to some chosen basis.

Geometry of nilmanifolds

In **smooth category** any nilmanifold $\Gamma\backslash G$ is diffeomorphic to an iterated tower of toric bundles:

$$\Gamma\backslash G \longrightarrow \frac{G/Z}{\Gamma/(Z\cap\Gamma)} \longrightarrow \cdots \longrightarrow \mathrm{pt},$$

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Example: Let $\mathfrak{g} = \langle x, y, z, t\rangle$ be a Lie algebra with the only relation $[x, y] = z$ and a complex structure $Ix = y, Iz = t$. Kodaira–Thurston surface is a complex surface which admits a principal fibration over an elliptic curve with elliptic fibers:

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The complex geometry of nilmanifolds is much vaster. It is not always possible to choose toric bundles to be **holomorphic**.

Complex nilmanifolds

Let G be a Lie group, $\mathfrak{g} = Lie(G)$. Every $X \in \mathfrak{g}$ defines **the left-invariant** vector field X^L on G and $X \mapsto X^L \in T_e G$ is an isomorphism of Lie algebras.

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Let $I \in \mathrm{End}(\mathfrak{g})$ be an almost complex structure on a Lie algebra, $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ the eigenspace decomposition. It induces a left-invariant almost complex structure I^L on G.

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The almost complex structure I^L is **integrable** iff $\mathfrak{g}^{1,0}$ is a Lie *subalgebra* of $\mathfrak{g} \otimes \mathbb{C}$.

A complex nilmanifold is a pair (N, I), where $N = \Gamma \backslash G$ and I an integrable left-invariant almost complex structure on G.

A left-invariant complex structure I on G makes G into a complex manifold but in general not into a complex Lie group. A Lie group G is a complex Lie group iff $\mathfrak{g}^{1,0}$ is an ideal of $\mathfrak{g} \otimes \mathbb{C}$.

Hypercomplex nilmanifolds

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Let G be a nilpotent Lie group with a left-invariant hypercomplex structure I, J, K and $\Gamma \subset G$ a cocompact lattice. Then the quadruple $(N = \Gamma \backslash G, I, J, K)$ is called **a hypercomplex nilmanifold**.

Twistors

Consider the product $N \times \mathbb{C}\mathrm{P}^1$, where N is a hypercomplex manifold. **The twistor space** $\mathrm{Tw(N)}$ of the hypercomplex manifold N is $N \times \mathbb{C}\mathrm{P}^1$ with the complex structure is defined as follows.

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For any point $(n, L) \in N \times \mathbb{C}\mathrm{P}^1$ the complex structure on $\mathrm{Tw(N)}$ is L on TN and the standard complex structure $I_{\mathbb{C}P^1}$ on $T\mathbb{C}\mathrm{P}^1$. This almost complex structure is integrable. The space $\mathrm{Tw(N)}$ equipped with the canonical holomorphic projection $\pi : \mathrm{Tw(N)} \longrightarrow \mathbb{C}\mathrm{P}^1$.

Curves in nilmanifolds

Theorem: *Let (N, I, J, K) be a hypercomplex nilmanifold and assume that corresponding Lie algebra is $\mathbb{H}$-solvable. Then there are no complex curves in the general fiber of the holomorphic twistor projection $\pi : \mathrm{Tw}(N) \longrightarrow \mathbb{C}\mathrm{P}^1$.*

Cohomology of a nilmanifolds

The Chevalley–Eilenberg differential $d : \mathfrak{g}^* \longrightarrow \Lambda^2\mathfrak{g}^*$ extends to the complex

$$0 \longrightarrow \mathfrak{g}^* \longrightarrow \Lambda^2\mathfrak{g}^* \longrightarrow \cdots \longrightarrow \Lambda^n\mathfrak{g}^* \longrightarrow 0$$

by the Leibniz rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\tilde{\alpha}}\alpha \wedge d\beta$, $\alpha, \beta \in \mathfrak{g}^*$ and $n = \dim_{\mathbb{R}} \mathfrak{g}$.

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Theorem (Nomizu): *Let N be a nilmanifold and $(\Lambda^*\mathfrak{g}^*, d)$ its Chevalley–Eilenberg complex. The natural inclusion of the complex of the left-invariant differential forms $\Omega^{inv}(G)$ on the nilpotent Lie group G into the de Rham algebra on the nilmanifold $\Omega(N)$ is a quasi-isomorphism.*

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The homology $H_*(N, \mathbb{R}) \approx H_*(\Gamma, \mathbb{R})$, hence by the theorem of Nomizu $H_*(\Gamma, \mathbb{R}) \approx H_*(\mathfrak{g}, \mathbb{R})$. Pickel showed that $H^*(\Gamma, \mathbb{Q}) \approx H^*(\mathfrak{g}, \mathbb{Q})$ as well.

Positive bivectors

The elements of the space $\Lambda^{1,1}\mathfrak{g} \subset \Lambda^2\mathfrak{g}$ are called **(1,1)-bivectors**. A non-zero real bivector $\xi \in \Lambda^{1,1}\mathfrak{g}$ is called **positive** if for any nonzero $\alpha \in \Lambda^1\mathfrak{g}^*$ one has $\xi(\alpha, I\alpha) \geq 0$.

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Let $C_I \subset N$ be a complex curve in a complex nilmanifold (N, I) and $\omega \in \Lambda^2\mathfrak{g}^*$ a two-form. We identify $\Lambda^2\mathfrak{g}^*$ with the space of left-invariant 2-forms on the Lie group G, which descends to the space of 2-forms $\Lambda^2(N)$ on the nilmanifold $N = \Gamma\backslash G$.

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Since the homology $H_*(N) = H_*(\mathfrak{g})$ by the theorem of Nomizu a complex curve C_L corresponds to the bivector ξ.

$\mathbb{H}$-solvable Lie algebra

Let $(\mathfrak{g}, I, J, K)$ be a hypercomplex structure on a nilpotent Lie algebra. For any subspace $\mathfrak{h} \subset \mathfrak{g}$, denote by $\mathbb{H}\mathfrak{h}$ the space $\mathfrak{h} + I\mathfrak{h} + J\mathfrak{h} + K\mathfrak{h}$.

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Examples: Abelian complex structures, the quaternionic double of the Kodaira surface (non-abelian).

Foliation

Let Σ_i be a left-invariant foliation in a Lie group G generated by the subalgebra $\mathfrak{g}_i^{\mathbb{H}}$ for each $i \in \mathbb{Z}_{>0}$ and denote by $\mathfrak{L}_{x,i-1}$ the leaf of the corresponding foliation.

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Theorem: *Let C_L be a complex curve in a complex nilmanifold (N, L), where $L \in \mathbb{C}\mathrm{P}^1$ is a generic complex structure. Suppose that C_L is tangent to the foliation Σ_{i-1}. Then it is also tangent to Σ_i.*

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Corollary-Theorem: *Let (N, I, J, K) be a hypercomplex nilmanifold and assume that corresponding Lie algebra is $\mathbb{H}$-solvable. Then there are no complex curves in the general fiber of the holomorphic twistor projection $\pi : \mathrm{Tw}(N) \longrightarrow \mathbb{C}\mathrm{P}^1$.*