Global bi-Lipschitz classification of semialgebraic surfaces

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Outline

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- In the first part of this talk, we consider semialgebraic surfaces S in $\mathbb{R}^n$ (with isolated singularities) equipped with the inner distance

  $$d_{S,inn}(x_1, x_2) = \inf\{length(\gamma) : \gamma \text{ is a path on } S \text{ connecting } x_1, x_2 \in S\}$$

  and we classify those surfaces up to bi-Lipschitz homeomorphisms with respect to the inner distance, the so-called *inner lipeomorphims*.

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  and we classify those surfaces up to bi-Lipschitz homeomorphisms with respect to the inner distance, the so-called *inner lipeomorphims*.
- For example, associated to each Nash surface S, we present a list of symbols, $\theta_S \in \{-1, 1\}$, $g_S \in \mathbb{N} \cup \{0\}$, $e_S \in \mathbb{N} \cup \{0\}$ and β_1, ...,β_{e_S}, where β_i's (≤ 1) are rational numbers associated to the ends of S; which determines S up to inner lipeomorphisms.

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- Present several relations between Local and global Lipschitz geometry.

Classification of smooth compact surfaces

Theorem

Let X and Y be two connected smooth (without boundary) compact surfaces. Then the following statements are equivalent:

(1) X and Y are homeomorphic;

(2) X and Y are diffeomorphic;

(3) X and Y are inner lipeomorphic;

(4) $\theta(X) = \theta(Y)$ and $g(X) = g(Y)$,

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Non-compact surfaces

Example

Let $X = \mathbb{R}^2$, $Y = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = 1\}$ and $Z = \{(x, y, z) \in \mathbb{R}^3; z = x^2 + y^2\}$.

a) $\theta(X) = \theta(Y)$, $g(X) = g(Y)$, but X and Y are not homeomorphic;

(b) X and Z are diffeomorphic, but they are not inner lipeomorphic;

Non-smooth surfaces

Example

Let $Y = \{(x,y,z) \in \mathbb{R}^3; (x^2 + \frac{9}{2}y^2 + z^2 - 1)^3 - x^2z^3 - \frac{9}{200}y^2z^3 = 0\}$ and $X = \mathbb{S}^2$. Then X and Y are homeomorphic, but they are not inner lipeomorphic.

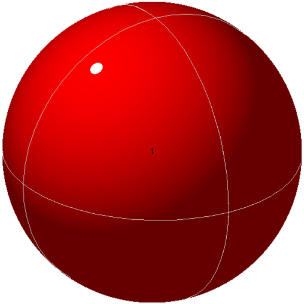


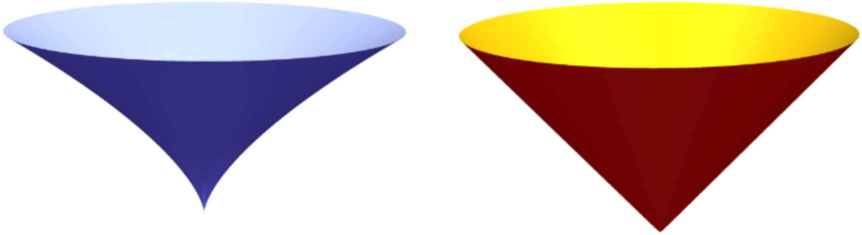
Figure: *The heart surface and the sphere*

Local classification of surfaces

Theorem of Birbrair

Given the germ of a semialgebraic set, (X, a), with isolated singularity and connected link, there is a unique rational number $\beta \geq 1$ such that (X, a) is inner homeomorphic to the germ at $0 \in \mathbb{R}^3$ of the β-horn

$$\{(x, y, z) \in \mathbb{R}^3 \; : \; x^2 + y^2 = z^{2\beta} \text{ and } z \geq 0\}.$$



Preliminary invariants I

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Let $X \subset \mathbb{R}^n$ be a semialgebraic surface with isolated inner Lipschitz singularities. Let us consider the following symbols:

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i) For $p \in \mathrm{Sing}_{inLip}(X)$, $\ell(X,p)$ denotes the number of connected components of the link of X at p;

Preliminary invariants I

Definition

Let $X \subset \mathbb{R}^n$ be a semialgebraic surface with isolated inner Lipschitz singularities. Let us consider the following symbols:

i) For $p \in \mathrm{Sing}_{inLip}(X)$, $\ell(X,p)$ denotes the number of connected components of the link of X at p;

ii) We can consider a sufficient large radius $R > 0$ (and $\rho = 1/R$) such that

$$X' = (X \cap \overline{B(0,R)}) \setminus \left\{ B(x_1,\rho) \cup \cdots \cup B(x_s,\rho) \right\}$$

is a topological surface with boundary and its topological type does not depend on R. Thus, we define

$$\theta(X) = \begin{cases} 1, & \text{if } X' \text{ is orientable} \\ -1, & \text{if } X' \text{ is not orientable.} \end{cases}$$

Preliminary invariants II

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iv) For each $p \in X$, there is $r > 0$ such that

$$X \cap B(p,r) = \bigcup_{i=1}^{\ell(X,p)} X_i$$

and each X_i is a topological surface. Let β_i be the horn exponent of X_i at p (given by Theorem of Birbrair). By reordering the indices, if necessary, we assume that $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{\ell(X,p)}$. In this way, we define $\beta(X,p) = (\beta_1, \beta_2, \cdots, \beta_{\ell(X,p)})$.

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Theorem (Fernandes and S. (2022))

Given an end E of X (i.e., a connected component of $X \setminus B(0,R)$), there is a unique rational number $0 \leq \beta \leq 1$ such that E is inner lipeomorphic to the β-tube

$$P_\beta = \{(x,y,z) \in \mathbb{R}^3 \, : \, x^2 + y^2 = z^{2\beta} \text{ and } z \geq a > 0\}.$$

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Definition

v) $e(X)$ is the number of ends of X, and if $E_1, \ldots, E_{e(X)}$ are the ends of X, then denote by β_i, the tube exponent of E_i, the only rational number smaller than or equal to 1 such that E_i is a β_i-tube. By reordering the indices, if necessary, we assume that $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{e(X)}$. In this way, we define $\beta(X,\infty) = (\beta_1, \beta_2, ..., \beta_{e(X)})$.

The invariant

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Definition (Inner Lipschitz code)

Let $X \subset \mathbb{R}^n$ be a semialgebraic surface with isolated inner Lipschitz singularities. If $\mathrm{Reg}_{inLip}(X)$ is a connected set, then the collection of symbols

$$\left\{\theta(X), g(X), \beta(X,\infty), \{\beta(X,p)\}_{p\in \mathrm{Sing}_{inLip}(X)}\right\}$$

is called the **inner Lipschitz code of X** and we denote it by $\mathrm{Code}_{inLip}(X)$.

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Remark

We can also define $\mathrm{Code}_{inLip}(X)$ when $\mathrm{Reg}_{inLip}(X)$ is not connected, but we will not consider this case in our talk.

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b) Unbounded Moebius band
$\{(x, y, u, v) \in \mathbb{R}^4 : \; x^2 + y^2 = 1, \; (u^2 - v^2)y = 2uvx\}$: $\{-1, 0, 1, \emptyset\}$;

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d) $\{(z, w) \in \mathbb{C}^2 \ : \ z^2 = w(w - a)(w - b)\}$; $a, b \neq 0$ and $a \neq b$: $\{1, 1, (1, 1, 1), \emptyset\}$;

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f) Paraboloid in $\mathbb{R}^3$: $\{1, 0, 1/2, \emptyset\}$

g) Torus: $\{1, 1, \emptyset, \emptyset\}$

h) Klein bottle: $\{-1, 1, \emptyset, \emptyset\}$

Outline

Global classification of semialgebraic surfaces

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Theorem (Fernandes and S. (2022))

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be semialgebraic surfaces with isolated inner Lipschitz singularities. Then, X and Y are inner lipeomorphic if, and only if, their inner Lipschitz code are equal.

Outline

Nash surfaces

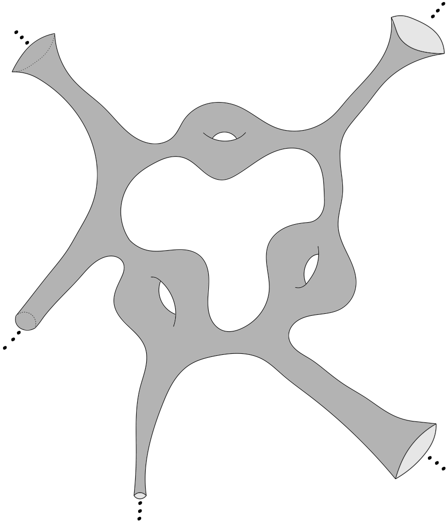

Figure: *An oriented Nash surface with 5 ends and genus 4.*

Classification of compact semialgebraic surfaces

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Theorem (Fernandes and S. (2022))

Let $N_1, N_2 \subset \mathbb{R}^n$ be two Nash surfaces. Then, the following statements are equivalent:

(1) N_1 and N_2 are homeomorphic and $\beta(N_1,\infty) = \beta(N_2,\infty)$;

(2) N_1 and N_2 are diffeomorphic and $\beta(N_1,\infty) = \beta(N_2,\infty)$;

(3) N_1 and N_2 are inner lipeomorphic;

(4) $\theta(N_1) = \theta(N_2)$, $g(N_1) = g(N_2)$ and $\beta(N_1,\infty) = \beta(N_2,\infty)$.

Normal forms

Normal forms

- For $\theta \in \{-1, 1\}$ and $g \in \mathbb{N}$, let $N(\theta, g) \subset \mathbb{R}^5$ be a compact Nash surface such that $\theta(N(\theta, g)) = \theta$ and $g(N(\theta, g)) = g$;

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- For a positive integer number e and $\beta = (\beta_1, ..., \beta_e) \in \mathbb{Q}$ such that $\beta_1 \leq \beta_2 \leq .... \leq \beta_e \leq 1$, we remove e distinct points of $N(\theta, g)$, let us say $x_1, ..., x_e \in N(\theta, g)$, and we define $F\colon N(\theta, g) \setminus \{x_1, ..., x_e\} \to \mathbb{R}^{6e}$ given by

$$F(x) = (\frac{x - x_1}{\|x - x_1\|^{1+\beta_1}}, \|x - x_1\|^{-1}, ..., \frac{x - x_e}{\|x - x_e\|^{1+\beta_e}}, \|x - x_e\|^{-1});$$

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- We also define $N(\theta, g, \emptyset) = N(\theta, g)$.

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- Note that $\theta(N(\theta, g, \beta)) = \theta$, $g(N(\theta, g, \beta)) = g$ and $\beta(N(\theta, g, \beta), \infty) = \beta$.

Normal forms

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- Note that $\theta(N(\theta, g, \beta)) = \theta$, $g(N(\theta, g, \beta)) = g$ and $\beta(N(\theta, g, \beta), \infty) = \beta$.

Theorem (Fernandes and S. (2022))

Let $N \subset \mathbb{R}^n$ be a Nash surface. Then, $N(\theta(N), g(N), \beta(N, \infty))$ and N are inner lipeomorphic.

Classification of minimal surfaces

Theorem (Fernandes and S. (2022))

Let $M_1, M_2 \subset \mathbb{R}^3$ be two connected properly embedded minimal surfaces with finite total curvature. Then, the following statements are equivalent:

(1) M_1 and M_2 are homeomorphic;

(2) M_1 and M_2 are inner lipeomorphic;

(3) $g(M_1) = g(M_2)$ and $e(M_1) = e(M_2)$.

Classification of complex algebraic curves

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Theorem (Fernandes and S. (2022))

Let $C_1, C_2 \subset \mathbb{C}^2$ be two complex algebraic curves. Then, the following statements are equivalent:

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(2) C_1 and C_2 are inner lipeomorphic.

Corollary

Let $C_1, C_2 \subset \mathbb{C}^2$ be two LNE complex algebraic curves. Then, the following statements are equivalent:

(1) C_1 and C_2 are homeomorphic;

(2) C_1 and C_2 are outer lipeomorphic.

Classification of compact semialgebraic surfaces

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Definition

Let $X \subset \mathbb{R}^n$ be an unbounded closed subset. Let $\widehat{X} = \rho^{-1}(X) \cup \{e_{n+1}\}$, where $\rho\colon \mathbb{S}^n \setminus \{e_{n+1}\} \to \mathbb{R}^n$ is the stereographic projection of $e_{n+1} = (0, ..., 0, 1) \in \mathbb{S}^n$.

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Theorem (Fernandes and S. (2022))

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be semi-algebraic surfaces with isolated inner Lipschitz singularities. Then, X and Y are inner lipeomorphic if, and only if, the pointed spaces $(\widehat{X}, e_{n+1})$ and $(\widehat{Y}, e_{m+1})$ are inner lipeomorphic.

In the last result, the equivalence as pointed spaces can not be dropped.

Example

Let $P = \{(x, y, z) \in \mathbb{R}^3; z = x^2 + y^2\}$ and $H = \{(x, y, z) \in \mathbb{R}^3; z^3 = x^2 + y^2\}$. Then $\widehat{P} = \{(x, y, z, w) \in \mathbb{S}^3; z(1 - w) = x^2 + y^2\}$ and $\widehat{H} = \{(x, y, z, w) \in \mathbb{S}^3; z^3 = (x^2 + y^2)(1 - w)\}$. Thus, $\mathrm{Code}_{inLip}(P) = \{1, 0, \frac{1}{2}, \emptyset\}$ and $\mathrm{Code}_{inLip}(H) = \{1, 0, 1, \frac{3}{2}\}$. Therefore, by Inner Lip Classification Theorem, P and H are not inner lipeomorphic. Moreover, $\mathrm{Code}_{inLip}(\widehat{P}) = \mathrm{Code}_{inLip}(\widehat{H}) = \{1, 0, \emptyset, \{\frac{3}{2}\}\}$. Therefore, by Inner Lip Classification Theorem, $\widehat{P}$ and $\widehat{H}$ are inner lipeomorphic.

Outline

O-minimal structure

Definition

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4) If $\pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates and $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$;

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In this talk, we fix an o-minimal structure $\mathcal{S}$ on $\mathbb{R}$.

Inner distance is conical

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Theorem (S. (2023))

Let $A \subset \mathbb{R}^n$ be a definable set in $\mathcal{S}$. Let $\varphi\colon A \to \mathbb{R}$ be a **radius function**, i.e., φ is a definable outer Lipschitz function such that there is $C \geq 1$ satisfying $\frac{1}{C}\|x\| \leq \|\varphi(x)\| \leq C\|x\|$ for all $x \in A$.

(a) If the link of A at infinity is connected, then there are constants $K, r \geq 1$ such that for each $t \in (r, +\infty)$, we have

$$d_{A,inn}(x,y) \leq d_{A_{\varphi,t},inn}(x,y) \leq K d_{A,inn}(x,y),$$

for all $x, y \in A_{\varphi,t} = \{x \in A; \varphi(x) = t\}$.

(b) If the link of A at 0 is connected, then there are constants $K, r \geq 1$ such that for each $t \in (0, \frac{1}{r})$, we have

$$d_{A,inn}(x,y) \leq d_{A_{\varphi,t},inn}(x,y) \leq K d_{A,inn}(x,y),$$

for all $x, y \in A_{\varphi,t}$.

Definable Lipschitz geometry: Local vs. global

Theorem (S. (2023))

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be definable sets in $\mathcal{S}$ with connected links at infinity. Let $\sigma, \tilde{\sigma} \in \{inn, out\}$. Then, the following statements are equivalent:

1. There is a definable lipeomorphism at infinity $\varphi\colon (X, d_{X,\sigma}) \to (Y, d_{X,\tilde{\sigma}})$ which preserves the outer distance to the origin;
2. There is a germ of definable lipeomorphism $\psi\colon (\widehat{X}, d_{\widehat{X},\sigma}, e_{n+1}) \to (\widehat{Y}, d_{\widehat{Y},\tilde{\sigma}}, e_{m+1})$ which preserves the last coordinate;
3. There is a germ of lipeomorphism $\tilde{\varphi}\colon (\iota(X \setminus \{0\}), d_{\iota(X\setminus\{0\}),\sigma}, 0) \to (\iota(Y \setminus \{0\}), d_{\iota(Y\setminus\{0\}),\tilde{\sigma}}, 0)$ which preserves the outer distance to the origin.

Definable Lipschitz geometry: Local vs. global

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Theorem (S. (2023))

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be definable sets in $\mathcal{S}$. Let $\sigma, \tilde{\sigma} \in \{inn, out\}$. Then, $(X, d_{X,\sigma})$ and $(Y, d_{X,\tilde{\sigma}})$ are definably lipeomorphic if and only if the pointed stereographic modifications $(\widehat{X}, d_{\widehat{X},\sigma}, \infty)$ and $(\widehat{Y}, d_{\widehat{Y},\tilde{\sigma}}, \infty)$ are definably lipeomorphic.

Outline

Outer Lipschitz geometry: Local vs. global

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Theorem

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be sets. Then, the following statements are equivalent:

1. X and Y are outer lipeomorphic at infinity;
2. The germs of the stereographic modifications $(\widehat{X}, e_{n+1})$ and $(\widehat{Y}, e_{m+1})$ are outer lipeomorphic;
3. The germs of the inversions $(\iota(X \setminus \{0\}), 0)$ and $(\iota(Y \setminus \{0\}), 0)$ are outer lipeomorphic.

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This result appeared firstly in the preprint arXiv:2305.07469 [math.MG] written by Grandjean and Oliveira. However, our proofs are different. Their proof is by contradiction and the mine is a direct proof.

Smoothness at infinity

Ahern and Rudin in 1993 defined the notion of a set to be C^1-smooth at infinity.

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A set $V \subset \mathbb{R}^n$ is C^1-**smooth at infinity** if $\iota(V \setminus \{0\}) \cup \{0\}$ is a C^1 submanifold around 0.

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Theorem (Ahern and Rudin (2023))

A complex analytic set $V \subset \mathbb{C}^n$ is C^1-smooth at infinity if and only if V is the union of an affine linear subspace of $\mathbb{C}^n$ and a (possibly empty) finite set.

Lipschitz smoothness at infinity

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Theorem (Fernandes and S. (2020))

A complex analytic set $V \subset \mathbb{C}^n$ is outer lipeomorphic to an Euclidean space (outside of compact sets) if and only if V is the union of an affine linear subspace of $\mathbb{C}^n$ and a (possibly empty) finite set.

Final comment

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An open problem

Classify the semialgebraic surfaces (with isolated singularities) up to outer lipeomorphisms (Local and global).

Thank you!