Classifying canonical threefolds with small volume

joint work (very much in progress) with S. Coughlan, Y. Hu, T. Zhang

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Overview

1. Volume and genus
2. Fibrations
3. Geography and Gorenstein index
4. Moduli

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Volume and genus: what they are

Let X be a complex projective variety whose singularities are at worst canonical. Let n be the dimension of X.

Definition

The genus of X is $p_g(X) := h^0(X, K_X)$.
The volume of X is

$$\mathrm{vol}(X) := n! \limsup_{m\to\infty} \frac{h^0(X, mK_X)}{m^n}$$

A variety is of general type if and only if its volume is positive.
X is a canonical model if the singularities of X are canonical and K_X is ample.

Both genus and volume are birational invariants.
If X is a canonical model then

$$\mathrm{vol}(X) = K_X^n.$$

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Volume and genus: Noether inequality

If X is a curve ($n = 1$) then $2\mathbb{N} \ni \mathrm{vol}(X) = 2p_g(X) - 2$.
If X is a surface ($n = 2$) we have the Noether Inequality

$$\mathbb{N} \ni \mathrm{vol}(X) \geq 2p_g(X) - 4$$

If X is a threefold ($n = 3$) we have the Noether inequality[1]

Theorem 1. *Let X be a projective 3-fold of general type and either $p_g(X) \leq 4$ or $p_g(X) \geq 11$. Then*

$$\mathrm{vol}(X) \geq \frac{4}{3}p_g(X) - \frac{10}{3}.$$

Moreover $\mathrm{vol}(X) \in \mathbb{Q}$ may be not integral if the canonical model of X is not Gorenstein.

[1] Chen, Jungkai A.; Chen, Meng; Jiang, Chen *The Noether inequality for algebraic 3-folds*. With an appendix by János Kollár, Duke Math. J.169 (2020), no.9, 1603–1645.

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Fibrations in (1,2)-surfaces

Definition

A $(1,2)$-surface is a canonical model of dimension 2 with $K^2 = 1$ and $p_g = 2$.
They are all hypersurfaces of degree 10 in $\mathbb{P}(1,1,2,5)$.

The first step of the proof of the Noether inequality for threefolds is the proof that

$$\mathrm{vol}(X) < 2p_g(X) - 6 \Rightarrow \text{ there exists a pencil } f\colon X \dashrightarrow \mathbb{P}^1$$

such that the general fibre of f is a $(1,2)-$surface.

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Horikawa surfaces

The canonical models of surfaces X with $\mathrm{vol}(X) = 2p_g(X) - 4$, probably known already to M. Noether and Enriques, are usually referred to as Horikawa surfaces because Horikawa described[2] their moduli space.

> In Sections 3 and 5 we shall prove that minimal algebraic surfaces with given p_g and c_1^2 satisfying $c_1^2 = 2p_g - 4$ and $p_g \geqq 3$ have one and the same deformation type provided that c_1^2 is not divisible by 8.

> In Section 7 we shall study the case in which c_1^2 is divisible by 8. If we fix c_1^2, these surfaces are divided into two deformation types. They are homo-

By this classification, if $p_g \geq 7$, then there is a fibration $f\colon X \to \mathbb{P}^1$ with fibres of genus 2.

[2] Horikawa, Eiji, *Algebraic surfaces of general type with small C_1^2. I.* Ann. of Math. (2) 104 (1976), no.2, 357–387.

By the theory of genus 2 fibrations $f\colon X \to \mathbb{P}^1$, if X is a canonical model

$$\mathrm{vol}(X) = 2\chi(\mathcal{O}_X) - 6 + \deg\tau = 2p_g(X) - 4 + \deg\tau - 2h^1(X, \mathcal{O}_X)$$

where τ is an effective divisor on $\mathbb{P}^1$ supported on the *2-disconnected* fibres, those of the form $A + B$ with $AB = 1$.
Algebraically these special fibres are those whose canonical ring is not of the form[3]

$$\frac{\mathbb{C}[x_0, x_1, z]}{f_6(x_0, x_1, z)}$$

Moreover $\mathrm{vol}(X) = 2p_g(X) - 4 \Leftrightarrow \tau = 0$.

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[3] $\deg x_i = 1$, $\deg z = 3$, $\deg f_d = d$

Definition (CP 2022)

A simple fibration in $(1,2)-$surfaces is a morphism $f\colon X \to B$ between projective varieties of respective dimension 3 and 1 such that

1. B is smooth;
2. all singularities of X are canonical;
3. K_X is $f-$ample;
4. for *all* $p \in B$, the canonical ring of the fibre $X_p = f^{-1}p$ is of the form[a]

$$\frac{\mathbb{C}[x_0, x_1, y, z]}{f_{10}(x_0, x_1, y, z)}$$

[a] $\deg x_i = 1$, $\deg y = 2$, $\deg z = 5$, $\deg f_{10} = 10$

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Gorenstein motivating example

Choose integers d, d_0 and defines $\mathbb{F} = \mathbb{F}(d, d_0) = \mathbb{C}^6 // (\mathbb{C}^*)^2$ to be the toric 4-fold with weight matrix

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 & y & z \\ 1 & 1 & d - d_0 & d_0 - 2d & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{pmatrix}$$

and irrelevant ideal $(t_0, t_1) \cap (x_0, x_1, y, z)$.
Then (t_0, t_1) defines a morphism $\mathbb{F} \to \mathbb{P}^1$, whose fibres are $\mathbb{P}(1,1,2,5)$s. A bihomogeneous equation of the form

$$z^2 + y^5 + \dots$$

(of bidegree $(0, 10)$) defines a **Gorenstein** threefold $X = X(d; d_0)$ that, if all its singularities are canonical, is a regular[4] simple fibration in $(1,2)-$surfaces with

$$K_X^3 = \frac{4}{3} p_g(X) - \frac{10}{3} \qquad p_g(X) = 3d - 2$$

[4] $h^1(X, \mathcal{O}_X) = 0$

2−Gorenstein motivating example

Choose integers d, d_0, N and define $\mathbb{F} = \mathbb{F}(d, d_0) = \mathbb{C}^6//(\mathbb{C}^*)^2$ to be the toric 4-fold with weight matrix

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 & y & z \\ 1 & 1 & d - d_0 & d_0 - 2d & -N & -2N \\ 0 & 0 & 1 & 1 & 2 & 5 \end{pmatrix}$$

and irrelevant ideal $(t_0, t_1) \cap (x_0, x_1, y, z)$.
Then (t_0, t_1) defines a morphism $\mathbb{F} \to \mathbb{P}^1$, whose fibres are $\mathbb{P}(1, 1, 2, 5)$s. A bihomogeneous equation of the form

$$z^2 + f_N(t_0, t_1)y^5 + \dots$$

(of bidegree $(-4N, 10)$) defines a **2-Gorenstein** threefold $X = X(d; d_0)$ that, if all its singularities are canonical, is a regular simple fibration in $(1, 2)$−surfaces with

$$K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3} + \frac{N}{6} \qquad p_g(X) = 3d - 2 - 2N$$

A conjecture

Theorem (HZ 2022)

If X is a minimal threefold with $p_g(X) \geq 11$ and $K_X^3 \leq \frac{4}{3}p_g(X) - \frac{10}{3} + \frac{2}{6}$ then X is 2-Gorenstein and $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3} + \frac{N}{6}$ with $N = 0, 1, 2$. Moreover X is Gorenstein if and only if $N = 0$.

Then we claimed the following

Conjecture [CP 2022]: *There exists an $\epsilon > 0$ such that if $p_g >> 1$ then all threefolds X with $\mathrm{vol}(X) < \frac{4}{3}p_g(X) - \frac{10}{3} + \epsilon$ admits a regular simple fibrations in $(1,2)-$surfaces $f\colon X \to \mathbb{P}^1$.*

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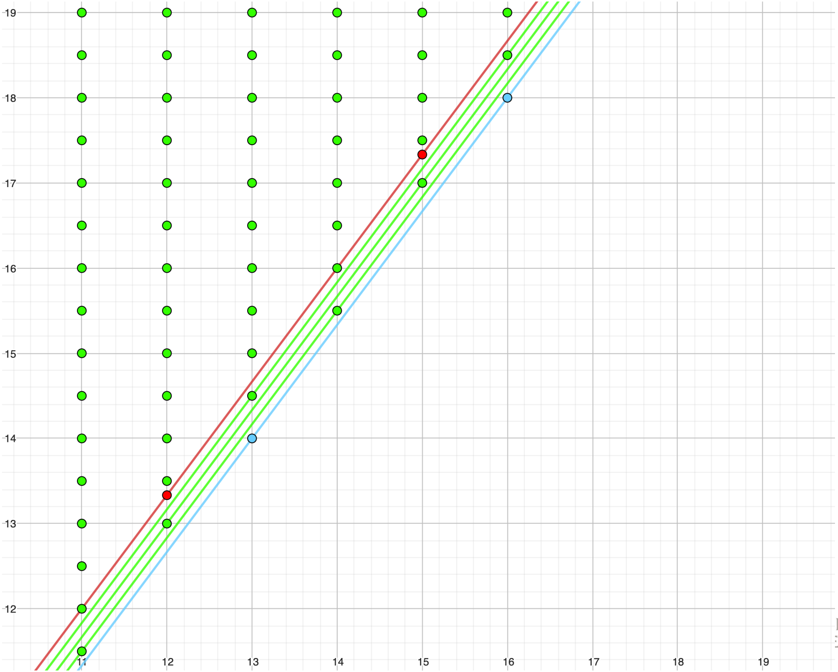

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Theorem (CHPZ 2023)

Suppose that X is a canonical threefold with $p_g \geq 11$. If $K_X^3 = \frac{4}{3}p_g - \frac{10}{3}$ then there is a crepant birational morphism $X_0 \to X$ such that X_0 is a Gorenstein regular simple fibration[a] *in $(1, 2)$-surfaces. If $p_g \geq 23$, then $X_0 \cong X$.*
In particular the conjecture is true for $\epsilon = \frac{1}{6}$.

[a] More precisely they are exactly the Gorenstein examples we gave

Problem: *Determine the maximal ϵ so that, if $p_g >> 1$ then all threefolds X with* $\mathrm{vol}(X) \leq \frac{4}{3}p_g(X) - \frac{10}{3} + \epsilon$ *are regular simple fibrations in $(1, 2)-$surfaces $f \colon X \to \mathbb{P}^1$.*

Theorem (CHPZ 2023)

For each positive integer a there are canonical threefolds with $K^3 = \frac{4}{3}p_g - \frac{10}{3} + \frac{2}{3}$ and $p_g = 3a$ of Gorenstein index 3,
Since all simple fibrations in $(1, 2)$-surfaces are $2-$Gorenstein, then $\epsilon < \frac{2}{3}$.

Please recall that $p_g = 3d - 2$ so d is deformation invariant. From now on we assume for sake of simplicity $d \geq 5$ (equiv. $p_g \geq 11$).

Theorem (CP 2022)

There are $X(d; d_0)$ with canonical singularities if and only if

$$\frac{1}{4}d \leq d_0 \leq \frac{3}{2}d$$

There are $X(d; d_0)$ smooth if and only if

$$d \leq d_0 \leq \frac{3}{2}d \text{ or } d_0 = \frac{7}{8}d$$

The threefolds $X(d; d_0)$ with $d \leq d_0 \leq \frac{3}{2}d$ form an irreducible component, the CHK component[a]*, of the moduli space of canonical threefolds. Those with $d_0 = \frac{7}{8}d$ form a different irreducible component.*

[a] The threefolds in this component had been found by a different construction by Y. Chen and Y. Hu, generalizing a construction of Kobayashi.

Theorem

Assume $\frac{d}{4} \leq d_0 \leq \frac{25d-3}{26}$. Then the threefolds $X(d;d_0)$ form an irreducible component of the moduli space of threefolds of general type.

We are not able to determine if the threefolds $X(d;d_0)$ with $\frac{25d-3}{26} < d_0 < d$ form an irreducible component of the moduli space or are specialisation of threefolds in the CHK component. However

Corollary

The number of irreducible components of the moduli space of canonical threefolds with $p_g \geq 11$, $K^3 = \frac{4}{3}p_g - \frac{10}{3}$ is at most $\left\lfloor \frac{p_g+6}{4} \right\rfloor$ and at least $\left\lfloor \frac{p_g+6}{4} \right\rfloor - \left\lfloor \frac{p_g+8}{78} \right\rfloor$.

The explicit algebraic description of these threefolds allowed us to prove:

Corollary (CP 2022)

The general element of the CHK component is a Mori Dream Space.