On Foliation Adjunction

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Algebraic Geometry, Lipschitz Geometry and Singularities

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Introduction

Joint with C. Spicer.

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Aim: We want to study the birational geometry of $(X, \mathcal{F})$: i.e. does there exists a birational map $X \dashrightarrow Y$ such that the induced foliation $\mathcal{F}_Y$ on Y is either such that $K_{\mathcal{F}_Y}$ is nef or it admits a MFS?

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- Construct a moduli space for foliations.

Singularities

Let $\mathcal{F}$ be a foliation of rank r on X. The inclusion $T_{\mathcal{F}} \hookrightarrow T_X$ induced a morphism

$$\phi\colon (\Omega_X^r)^{**} \otimes \mathcal{O}_X(-K_{\mathcal{F}}) \to \mathcal{O}_X.$$

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By **Frobenius theorem**, for any $x \in X \setminus (\mathrm{Sing}\mathcal{F} \cup \mathrm{Sing}\mathrm{X})$

$$\exists \quad \phi\colon x \in U \subset X \to \mathbb{C}^{n-r}$$

such that $\mathcal{N}_{\mathcal{F}}|_U = \phi^* T_{\mathbb{C}^{n-r}}$ and $T_{\mathcal{F}}|_U$ is the relative tangent bundle.

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A **leaf** of $\mathcal{F}$ is an analytic subvariety of X which is locally a fibre of ϕ.

Algebraically Integrable Foliations

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These foliations are called **algebraically integrable**.

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Def. $(\mathcal{F}, \Delta)$ is **log canonical** (resp. **canonical**) if for any birational morphism $f: Y \to X$ we can write

$$K_{\mathcal{F}_Y} + f_*^{-1}\Delta = f^*(K_{\mathcal{F}} + \Delta) + \sum a_i E_i$$

where the sum runs over the exceptional divisor of f and $a_i \geq -\epsilon(E_i)$ (resp. ≥ 0).

Adjunction Formula

C. - Spicer '23: Let $(\mathcal{F}, \Delta)$ be a foliated pair on X, let $D \subset X$ be a prime divisor which is not contained in the support of Δ and let $D^\nu \to D$ be its normalisation. Then

$$(K_{\mathcal{F}} + \Delta + \epsilon(D)D)|_{D^\nu} = K_{\mathcal{F}_{D^\nu}} + \Delta_D$$

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If $D \subset X$ is a $\mathcal{F}$-invariant subvariety such that D is not contained in $\mathrm{Sing}X \cup \mathrm{Sing}\mathcal{F} \cup \mathrm{Supp}\,\Delta$ then the same result holds, i.e.

$$(K_{\mathcal{F}} + \Delta)|_{D^\nu} = K_{\mathcal{F}_{D^\nu}} + \Delta_D$$

$\mathcal{F}_{D^\nu}$ is a foliation of rank equal to $\mathrm{rk}\,\mathcal{F}$ and $\Delta_D \geq 0$.

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C. - Spicer '23: Assume that $\mathrm{rk}\mathcal{F} = 1$ and $D \subset X$ is a $\mathcal{F}$-invariant divisor. Let $Z \subset X$ be a non $\mathcal{F}$-invarant subvariety such that Z is not contained in $\mathrm{Sing}\mathcal{F}$ and such that $(\mathcal{F}, \Delta)$ is log canonical around the generic point of Z.
Then $(\mathcal{F}_{D^\nu}, \Delta_D)$ is log canonical around the generic point of Z.

Cone Theorem and Base Point Free Theorem

Conjecture (Cone Theorem): Assume that $(\mathcal{F}, \Delta)$ is a log canonical foliated log pair. Then there exist rational curves $C_1, C_2 \ldots$ which are tangent to $\mathcal{F}$ and such that

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4. **Bogomolov-McQuillan '16, C. - Spicer '23:** The Cone Theorem holds for foliations of rank one.

Cone Theorem for Rank One Foliations

C.-Spicer '23: Assume that X is $\mathbb{Q}$-factorial and $(\mathcal{F}, \Delta)$ is a log canonical foliated log pair. Then there exist rational curves $C_1, C_2 \dots$ which are tangent to $\mathcal{F}$ and such that

$$0 < -(K_{\mathcal{F}} + \Delta) \cdot C_i \leq 2 \dim X$$

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Let $\mathrm{Null}(H_R) = \bigcup_{H_R|_V \text{ is not big}} V$ be the Null locus of H_R and let $W \subset \mathrm{Null}(H_R)$ a component such that R is contained in the image of $\overline{NE}(W) \to \overline{NE}(X)$.

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Since $(\mathcal{F}, \Delta)$ is log canonical, we may show that W is not contained in $\mathrm{Sing}\mathcal{F}$.

H_R supporting hyperplace for $\overline{NE(X)}$ at R and $W \subset \mathrm{Null}(H_R)$ a component.
We distinguish two cases:

1. W is $\mathcal{F}$-invariant: by adjunction, we may write

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and $H_R|_{W^\nu}$ is not big. By Miyaoka's theorem, it follows that W is covered by rational curves which are tangent to $\mathcal{F}$ and spanning R.
2. W is not $\mathcal{F}$-invariant: through the general point of W, there exists a non $\mathcal{F}$-invariant curve C such that $(K_\mathcal{F} + \Delta) \cdot C < 0$. We may construct a $\mathcal{F}$-invariant (possibly analytic) surface $S \subset X$ containing C. By adjunction, we may write
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   But $\mathrm{rk}\mathcal{F}_{C^\nu} = 0$, i.e. $K_{\mathcal{F}_{C^\nu}} = 0$ and, therefore $(K_\mathcal{F} + \Delta) \cdot C = \deg \Delta_C \geq 0$, a contradiction.

Existence of Flips

Let $(\mathcal{F}, \Delta)$ be a log canonical pair and let $R = (K_{\mathcal{F}} + \Delta)$-negative extremal ray. Let

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C. - Spicer '20, '21: The conjecture holds if $\dim X = 3$.

Contraction Theorem and Existence of Flips for Algebraically Integrable Foliations

C. - Spicer '23: Assume termination of flips for $\mathbb{Q}$-factorial klt pairs in dimension r. Let $(\mathcal{F}, \Delta)$ be an algebraically integrable foliated pair of rank r with log canonical singularities, such that (X, Δ) is klt. Let R=$(K_{\mathcal{F}} + \Delta)$-negative extremal ray such that $\dim \mathrm{locus}(R) \leq \dim X - 2$.
Then both the contraction $h\colon X \to Y$ and the flip $\phi\colon X \dashrightarrow X'$ exist.

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Idea of the proof:

1. We first prove the theorem for a very special class of algebraically integrable foliations.
2. Using some suitable MMP, we construct a flip for any algebraically integrable foliations.

Property $(*)$

Def. Let $f \colon X \to Z$ and let (X, B) be a log pair with $B \geq 0$. Then the pair $(X/Z, B)$ satisfies **Property (*)** if

1. There exists a reduced divisor Σ_Z on Z such that (Z, Σ_Z) is log smooth;
2. the vertical part of B coincides with $f^{-1}(\Sigma_Z)$; and
3. for any $z \in Z$ and for any $P \geq 0$ such that $(Z, \Sigma_Z + P)$ is log smooth around z, we have that $(X, B + f^*(\Sigma - \Sigma_Z))$ is log canonical around $f^{-1}(z)$.

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$$\overline{X} = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_k = \overline{X}'$$

and $0 < \lambda \ll 1$ such that $K_{\overline{\mathcal{F}}'} + \overline{\Delta}' + (1-\lambda)\overline{A}'$ is big and nef, where $\overline{\mathcal{F}}'$, $\overline{\Delta}'$ and $\overline{A}'$ are the induced items on $\overline{X}'$.

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- So far we have

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where $E, F \geq 0$, $(\overline{X}, \Gamma + E)$ is klt and the support of E contains $\mathrm{Exc}\,\pi$. Let Γ' and E' be the strict transform of Γ and E on $\overline{X}'$.

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- Thus $H_R = K_{\mathcal{F}} + \Delta + A$ is also semi-ample and the contraction $c_R \colon X \to Y$ exists.
- The corresponding map $X \dashrightarrow \overline{X}''$ is the desired flip. $\square$