On polynomial automorphisms which commute with a simple derivation

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- Notice: $e^{a\Delta}$ is the translation $x_i + a$ if and only if $\Delta =$ the partial derivative $\partial/\partial x_i$.

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- **Remark.** $\mathrm{Aut}(D)$ is closed in $\mathrm{Aut}(\Bbbk[x_1, \ldots, x_n])$. "Algebraic" means the degree of polynomials defining elements in that subgroup is bounded (Kambayashi, 1979).

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END

THANKS!