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Arcs and singularities

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Arcs and jets

$\mathscr{L}_\infty(\mathbb{A}^n) = \mathrm{Hom}_{\mathbb{C}-alg}(\mathbb{C}[x_1,\ldots,x_n], \mathbb{C}[\![t]\!]) \simeq \mathbb{A}^\infty$
$\mathscr{L}_m(\mathbb{A}^n) = \mathrm{Hom}_{\mathbb{C}-alg}(\mathbb{C}[x_1,\ldots,x_n], \mathbb{C}[\![t]\!]/t^{m+1}) \simeq \mathbb{A}^{n(m+1)}$
Truncation: $\pi_m : \mathscr{L}_\infty(\mathbb{A}^n) \to \mathscr{L}_m(\mathbb{A}^m)$ trivial fibration.

Contact loci of $f \in \mathbb{C}[x_1,\ldots,x_n]\backslash\mathbb{C}$

$\mathscr{L}_\infty(\mathbb{A}^n) \supset \mathscr{X}_m^\infty = \{\gamma \mid f(\gamma)(t) = *t^m + h.o.t.,\ * \neq 0\}$
$\mathscr{L}_m(\mathbb{A}^n) \supset \mathscr{X}_m = \{\gamma \mid f(\gamma)(t) \equiv *t^m,\ * \neq 0\}$
$\mathscr{X}_m^\infty = \pi_m^{-1}\mathscr{X}_m$, so $\mathscr{X}_m^\infty$ and $\mathscr{X}_m$ are essentially the same.

Encode: singularities of f.

Good: have explicit equations.

Bad: their basic topology is not understood.

Closely related: $\mathscr{X}_m^{res} := \{\gamma \in \mathscr{X}_m \mid * = 1\}$, the restricted contact loci.

Motivic zeta function

$Z_f(T) = \sum_m [\mathscr{X}_m] T^m$ is rational (Denef-Loeser).

Monodromy Conjecture (Igusa, D-L): The poles of $Z_f(t)$ give eigenvalues of monodromy on the cohomology of Milnor fibers of f.

Example $f = y^2 - x^3$

$\gamma = (x + x_1 t + x_2 t^2 + \ldots, y + y_1 t + y_2 t^2 + \ldots)$

$f(\gamma) = f + f_1 t + f_2 t^2 + \ldots =$

$= f + (2yy_1 - 3x^2 x_1)t + (2(y_1^2 + yy_2) - 3(2xx_1^2 + x^2 x_2))t^2 + \ldots$

$\mathscr{X}_3^\infty = \{f = f_1 = f_2 = 0\} \setminus \{f_3 = 0\}$

Example f hyperplane arrangement $\Rightarrow$ cohomology rings $H^*(\mathscr{X}_m, \mathbb{Z})$ are explicit combinatorial invariants (B.-Tue)

General setup X smooth $\mathbb{C}$ variety, D effective divisor, Σ closed subset of D. Consider $\mathscr{X}_m(X, D, \Sigma)$ and $\mathscr{X}_m^{res}(X, D, \Sigma)$.

Fix $\mu : Y \to X$ an m-separating log resolution, $\mu^{-1}D = \sum_{i \in S} N_i E_i$. Get partition

$$\mathscr{X}_m^{\infty} = \sqcup_{i \in S_m} \mathscr{X}_{m,i}$$

where $S_m = \{i \in S \mid N_i \text{ divides } m,\ \mu(E_i) \subset \Sigma\}$ and

$$\mathscr{X}_{m,i} = \{\gamma \in \mathscr{X}_m^{\infty} \mid \gamma \text{ lifts with center on } E_i^{\circ}\}$$

are non-empty, irreducible, smooth, locally closed, homotopy type related to E_i° (Ein-Lazarsfeld-Mustata). Similarly for $\mathscr{X}_m^{res}$.

Theorem (B.-Bobadilla-Le-Nguyen) An explicit spectral sequence with $\mathbb{E}_1$ in terms of E_i° computes $H_c^*(\mathscr{X}_m^{res}, \mathbb{Z})$.

Theorem (McLean) A spectral sequence with same $\mathbb{E}_1$ computes Floer cohomology $HF^*(\phi_f^m)$, where ϕ_f is the monodromy on the Milnor fiber of f, if f has an isolated singularity.

Arc-Floer conjecture (BBLN)

$$H_c^*(\mathscr{X}_m^{res}(\mathbb{A}^n, D, 0), \mathbb{Z}) \simeq HF^*(\phi_f^m)$$

where $D = f^{-1}(0)$, $0 \in D$ isolated singularity.

OK: for $m = mult_0 f$. It would recover:

Theorem (Denef-Loeser)

$$\chi(\mathscr{X}_m) = \sum_k (-1)^k Trace\{\phi_f^m \circlearrowright H^k(M_{f,0}, \mathbb{C})\}$$

where $M_{f,0}$ is the Milnor fiber of f at 0.

Theorem (de la Bodega - de Lorenzo Poza) The Arc-Floer conjecture is true for plane curves.

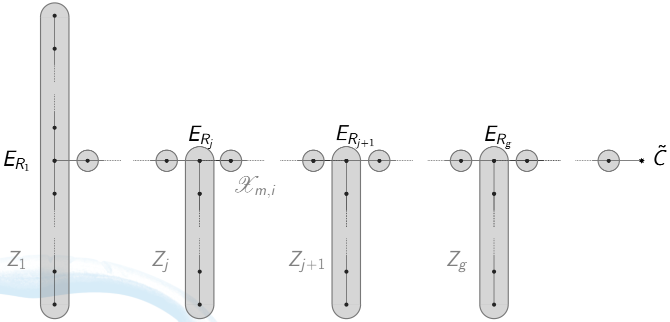
Embedded Nash problem Determine geometrically the irreducible components of $\mathscr{X}_m(X, D, \Sigma)$. That is, which E_i give components $\overline{\mathscr{X}_{m,i}}$ of $\mathscr{X}_m$?

Theorem (B., Bobadilla, de la Bodega, de Lorenzo Poza, Pelka) If E_i is not contracted on a minimal model of $(Y, (\mu^* D_{red}))$ over X, then $\overline{\mathscr{X}_{m,i}}$ is a component of $\mathscr{X}_m$ for any m divisible by N_i.

Theorem (BBBLP) Combinatorial solution to the embedded Nash problem for $(\mathbb{C}^2, C, 0)$ with C an unibranch plane curve singularity.

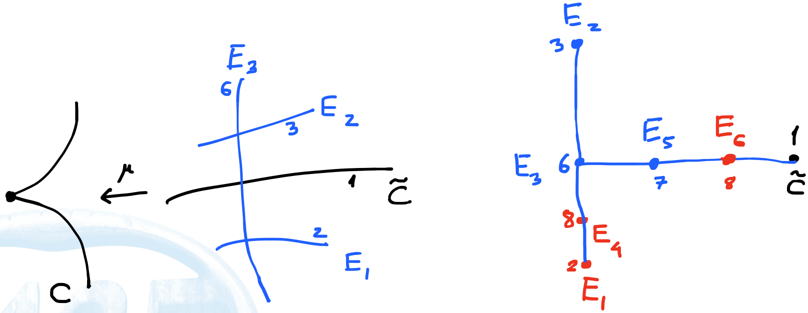
Theorem (de la Bodega - de Lorenzo Poza) All plane curves C.

These are embedded analogs of results for the classical Nash problem of Bobadilla-Pe Pereira, de Fernex-Docampo.



E_{R_1}
E_{R_j}
$E_{R_{j+1}}$
E_{R_g}
$\tilde{C}$
$\mathscr{X}_{m,i}$
Z_1
Z_j
Z_{j+1}
Z_g

Example $f = y^2 - x^3$, $m = 8$



So $\mathscr{X}_8(\mathbb{C}^2, f, 0)$ has two irreducible (and disjoint) components:

$$\mathscr{X}_{8,E_6} \text{ and } \overline{\mathscr{X}_{8,E_4}} = \mathscr{X}_{8,E_4} \sqcup \mathscr{X}_{8,E_1}.$$