Fast vanishing cycles on pertubations of weighted-homogeneous germs

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This work was suported by Israel Science Foundation grants 1910/18 and 1405/22.

- Sequence of the talk:

1. Historical review of theme;
2. Some definitions, examples and results
3. Fast cycle definition and new results.
4. Examples on deformed weighted homogeneous germs
5. Inside / Behind of criterions / control conditions (and proof)

Notation: $\mathbb{B}_{\epsilon_0} = \{ x \in \mathbb{R}^N ; \ \|x\| \leq \epsilon_0 \}$, $\mathbb{S}_{\epsilon_0} = \{ x \in \mathbb{R}^N ; \ \|x\| = \epsilon_0 \}$

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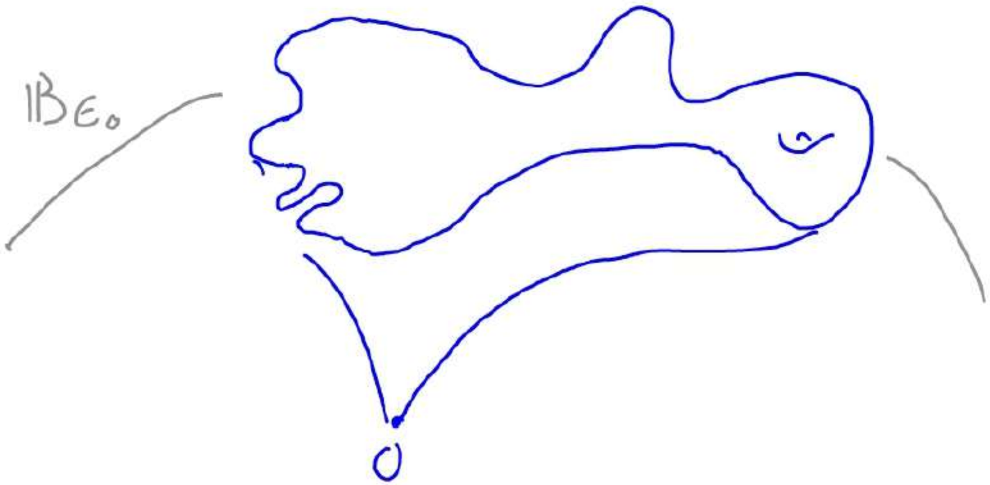

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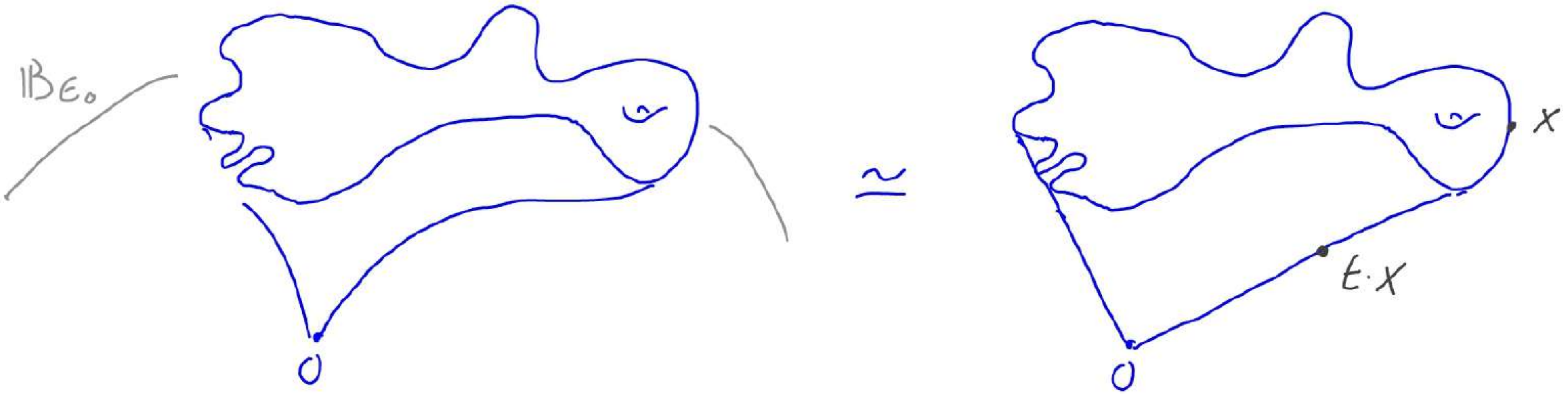

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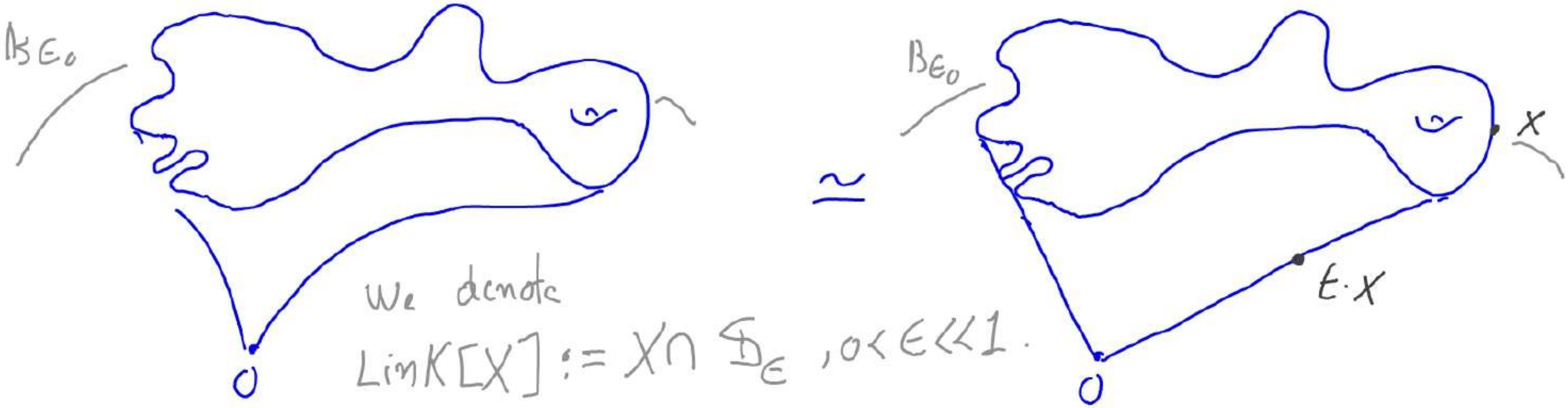

We denote

$\mathrm{Link}[X] := X \cap \mathbb{S}_\epsilon$, $0 < \epsilon \ll 1$.

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Example: Any complex analytic curve is IMC.

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⇒ the singularity at $0 \in \mathbb{C}^3$ of Kleinian surface X defined as $x^2 + y^2 = z^{2k}$, $k > 2$ is not metrically conical.

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↳ the singularity at $0 \in \mathbb{C}^3$ of Kleinian surface X defined as $x^2 + y^2 = z^{2k}$, $k > 2$ is not metrically conical.

↳ A weighted homogeneous germ of the quotient type $\mathbb{C}^2/\mu$ IMC has its lowest weights necessarily equal.

[Birbrair-Fernandez-Neumann, 2009] For Brieskorn-Pham singularities

$V(a,b,b) = \{(x,y,z) \in \mathbb{C}^3;\ x^a + y^b + z^b = 0\}$, $a < b$, the

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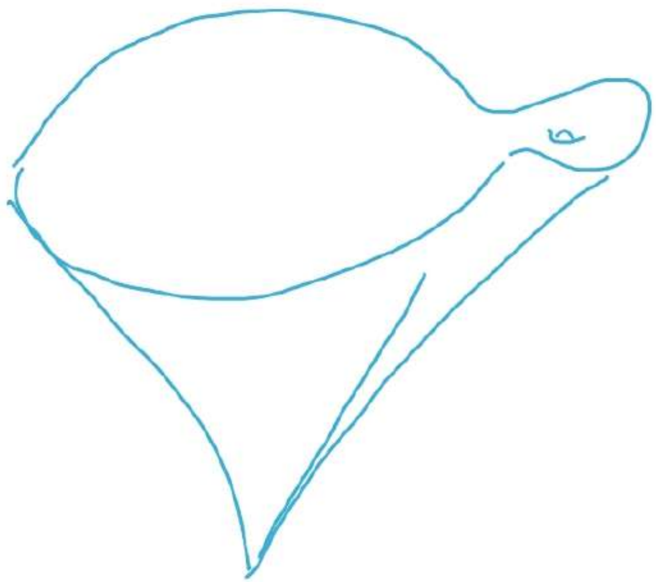

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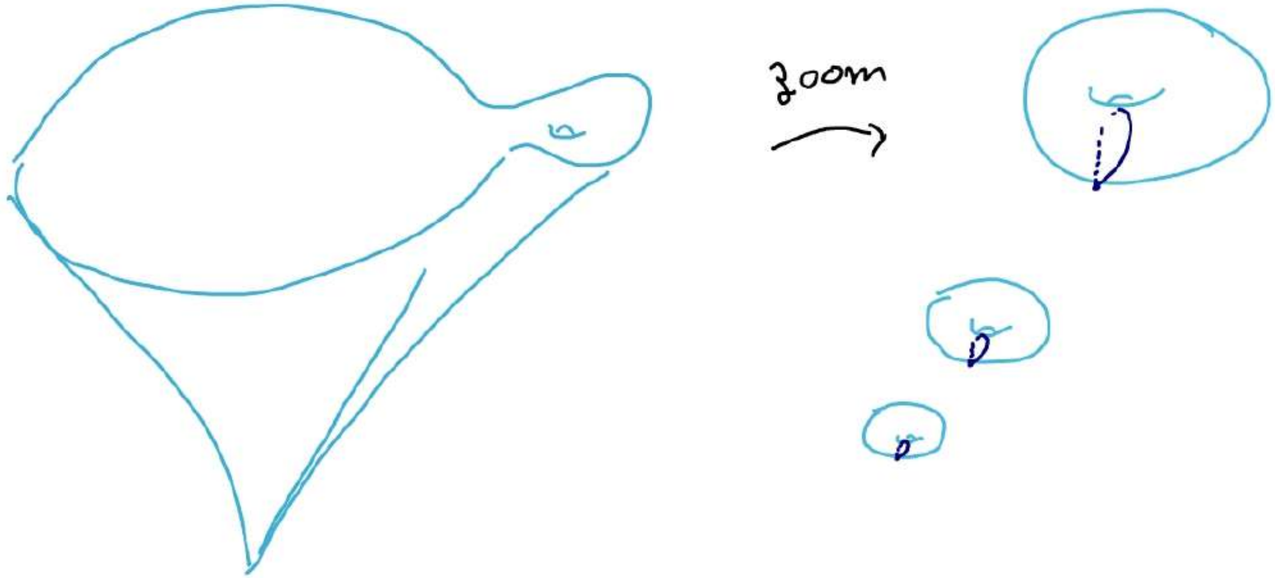

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They given an algorithm to verify the (non) IMC property via the resolution graph of the singularity.

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<u>Thm</u> [BNP. 14] Let $(X,0) \subseteq (\mathbb{C}^N,0)$ be a normal complex surface germ. Then, $(X,0)$ is IMC iff $(X,0)$ has no fast loops.

Hence, for normal surface germs, the only obstruction for the metric conical property is the existence of so-called *fast loop*.

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$$T_{(Z,0)} = \left\{ v \in \mathbb{R}^N ;\ \forall \epsilon > 0,\ \exists z \in Z \setminus \{0\} ;\ \left\| \frac{v}{\|v\|} - \frac{z}{\|z\|} \right\| < \epsilon \right\}.$$

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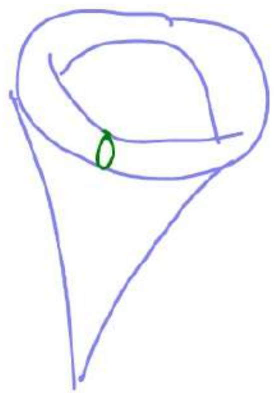
Examples: $X \subseteq \mathbb{R}^5$, $\begin{cases} x_1^2 + x_2^2 = t^2 \\ x_3^2 + x_4^2 = t^3 \end{cases}$, $t \geq 0$



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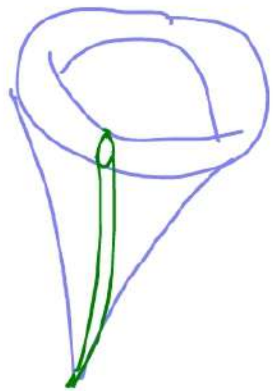

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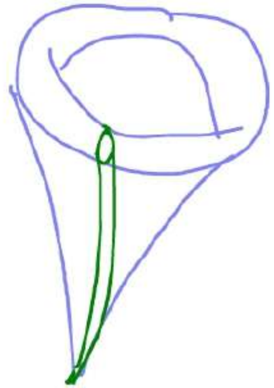

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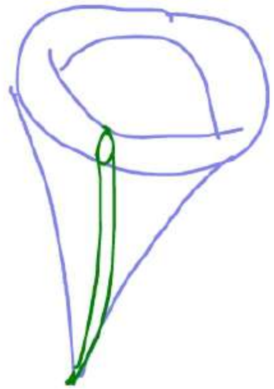

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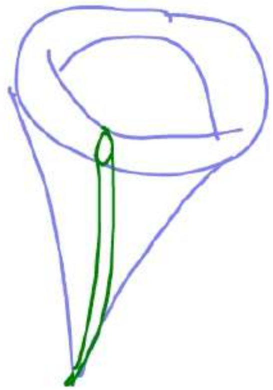

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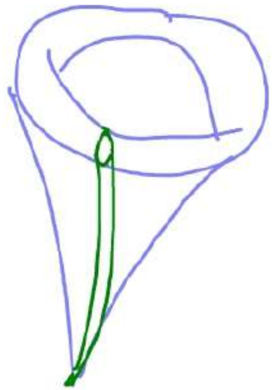

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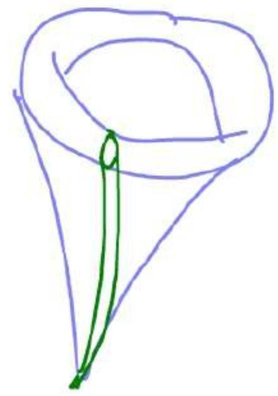

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Examples: • $(X,0) = \{(x,y,z) \in \mathbb{C}^3;\ x^2+y^2=z^3\}$ has fast loop $(X_{\mathbb{R}},0) = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2=z^3\}$.

• $(\mathbb{C}^4,0) \supseteq (X,0) = \{(x,y,z,w);\quad x = w^2+z^2-y^3\}$ is a IMC germ (smooth submanifold) containing a fast loop on $X \cap \{x=0\}$.

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- $(\mathbb{C}^4,0) \supseteq (X,0) = \{(x,y,z,w);\ x^3 = w^2+y^2-z^2\}$ is a non-IHC germ with no fast loops.

[Birbrair - Fernandes - Grandjean - O'Shea (Newman appendix) 2009]

For dim $n > 2$, among A_K - types, with equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^{K+1} = 0$$

The only IMC case is A_1.

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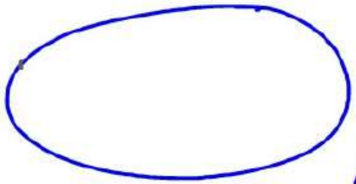
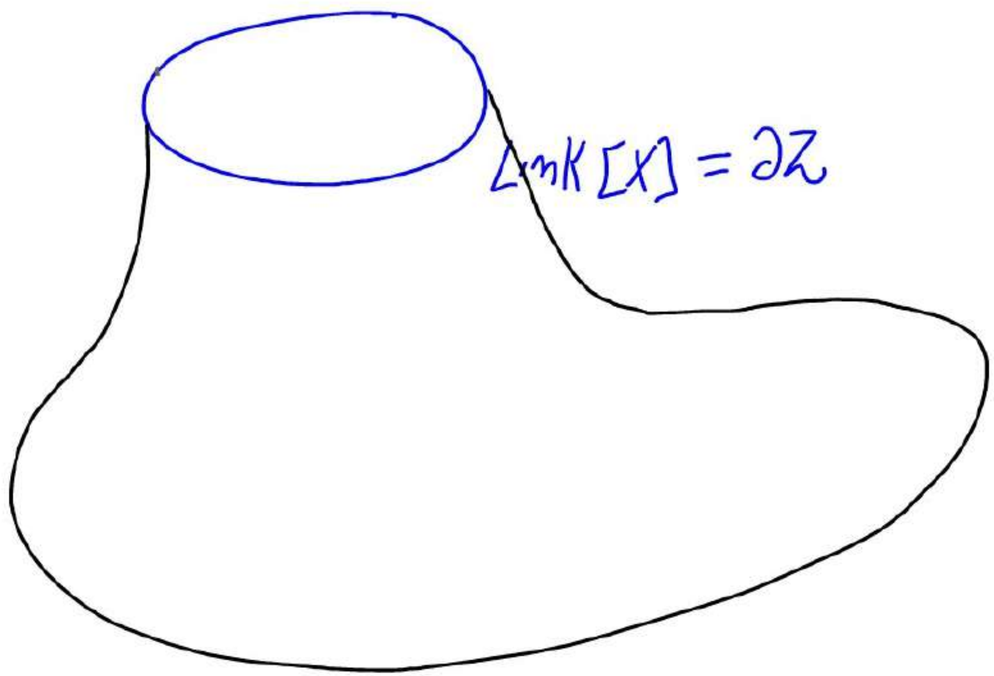


Link [X]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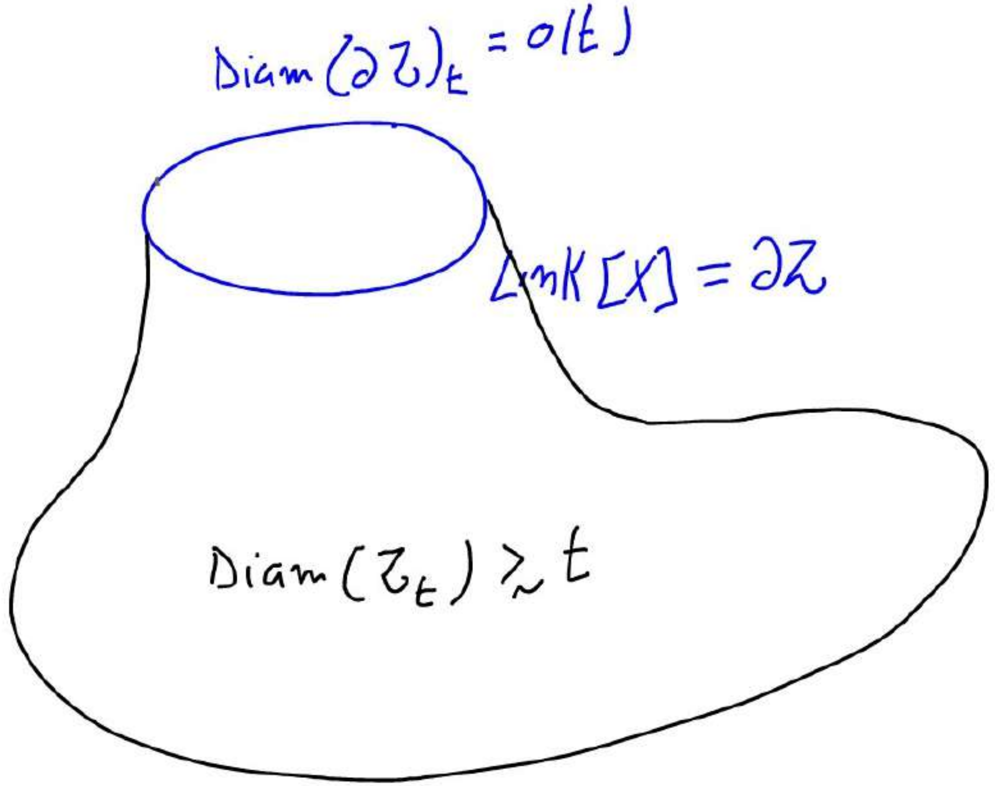
Diam (∂Z)_t = o(t)
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Definition. A germ $\underline{Y}_{,0} \subseteq \mathbb{R}^N_{,0}$ is called thin if $\dim_{\mathbb{R}} \underline{Y} > \dim_{\mathbb{R}} T_{(\underline{Y},0)}$, where $T_{(\underline{Y},0)} = \{ v \in \mathbb{R}^N ; \forall \epsilon > 0, \exists y \in \underline{Y} ; \| \frac{v}{\|v\|} - \frac{y}{\|y\|} \| < \epsilon \}$.

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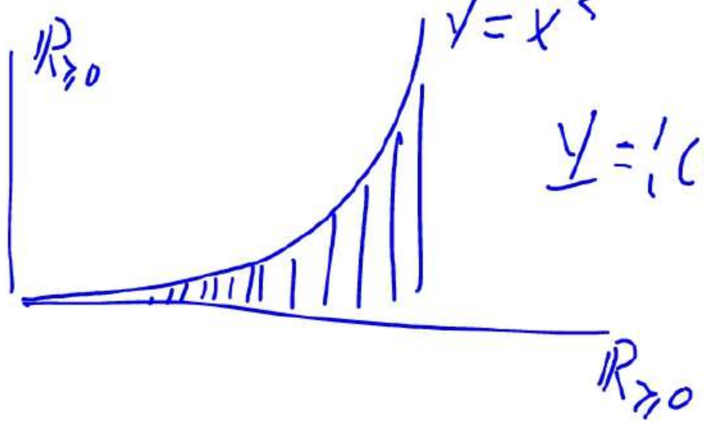

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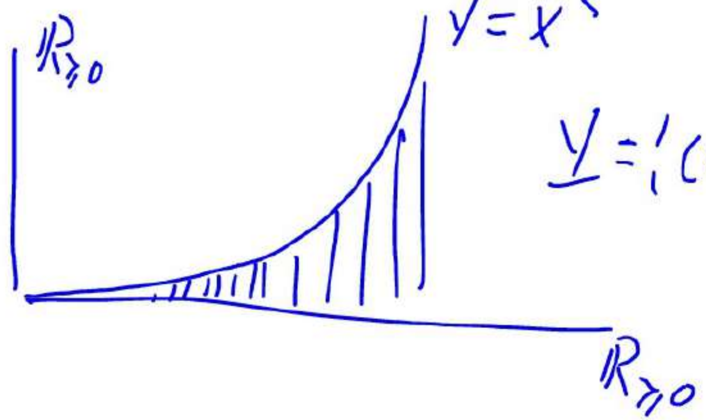

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Definition 2. A horn neighbourhood of $(\underline{Y}, 0) \subseteq \mathbb{R}^m$ is a $\mathcal{U}(\underline{Y}) = \{ x \in (\mathbb{R}^n_{,0}) ; \operatorname{dist}(x, \underline{Y}) < c|x|^{\beta} \}$ for some $\beta > 1$.

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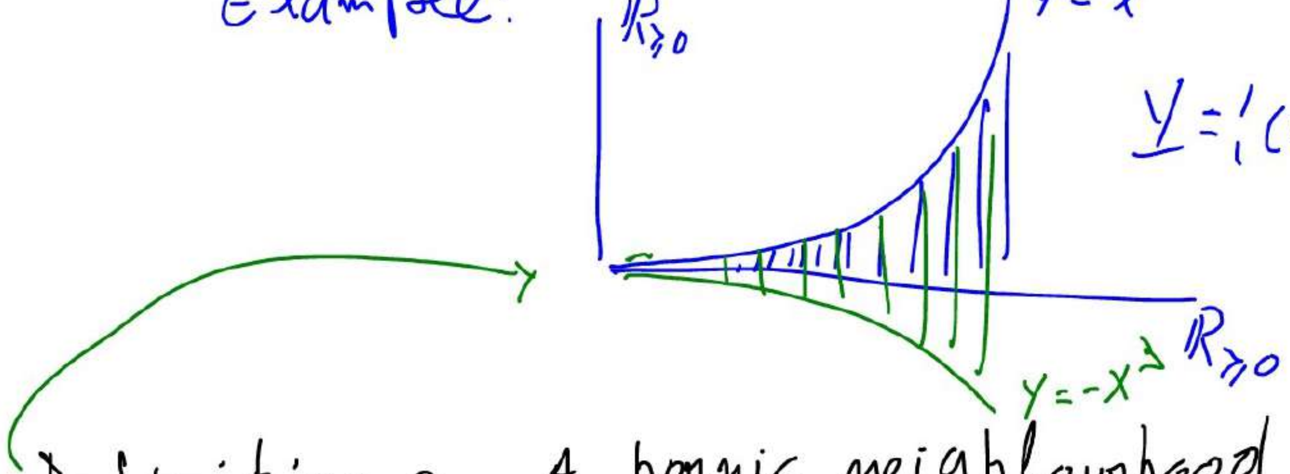

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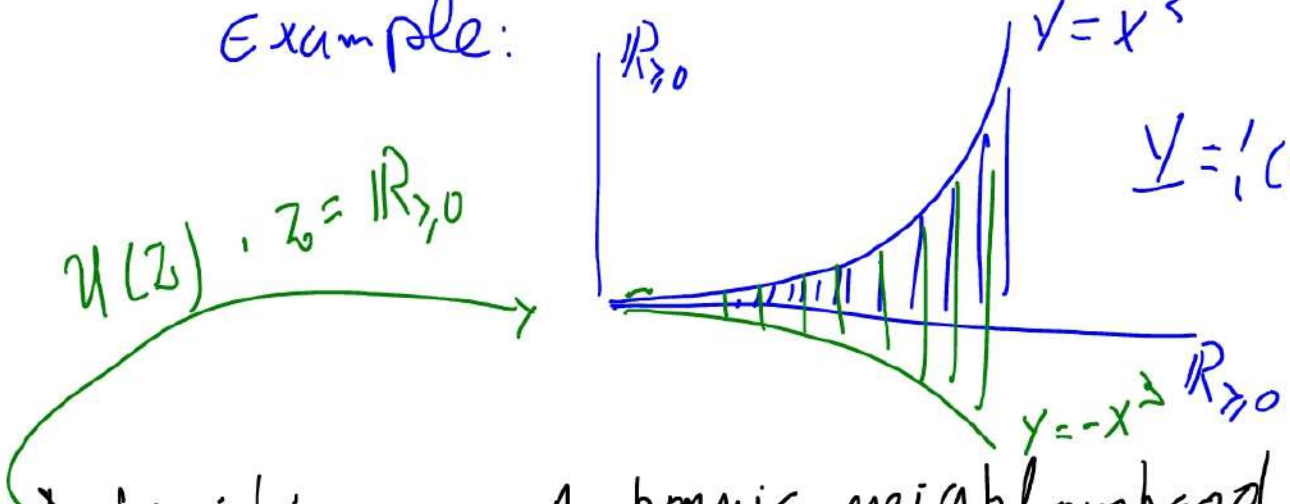
Definition 2. A horn neighbourhood of $(\underline{Y},0) \subseteq \mathbb{R}^n$ is a $\mathcal{U}(\underline{Y}) = \{ x \in (\mathbb{R}^n_{,0}) ; \operatorname{dist}(x, \underline{Y}) < c|x|^{\beta} \}$ for some $\beta > 1$.

Definition 1. A germ $\underline{Y}_{,0} \subseteq \mathbb{R}^N_{,0}$ is called thin if $\dim_{\mathbb{R}} \underline{Y} > \dim_{\mathbb{R}} T_{(\underline{Y},0)}$, where $T_{(\underline{Y},0)} = \{ v \in \mathbb{R}^N ; \forall \epsilon > 0, \exists y \in \underline{Y} ; \| \frac{v}{\|v\|} - \frac{y}{\|y\|} \| < \epsilon \}$.

(the Whitney tangent cone (C_3)).

Example:



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Definition 3 (Fast cycles)

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→ $\underline{Y}' \subset \underline{Y}$ with $\mathrm{Link}[\underline{Y}] \rightsquigarrow \mathrm{Link}[\underline{Y}']$

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Intuitively Speaking:

Given a germ $(X,0)$, the notion of dynamic "fast than linear" of the family of links $\{\mathrm{Link}_t[X]\}_{0<t<<1}$ is strongly related with the existence of "thin pieces" inside of $(X,0)$.

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(we are assume the germs and maps here inside of subanalytic (e.g. semialgebraic structure)

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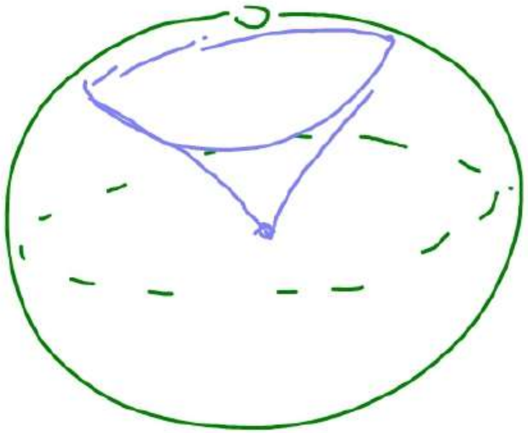
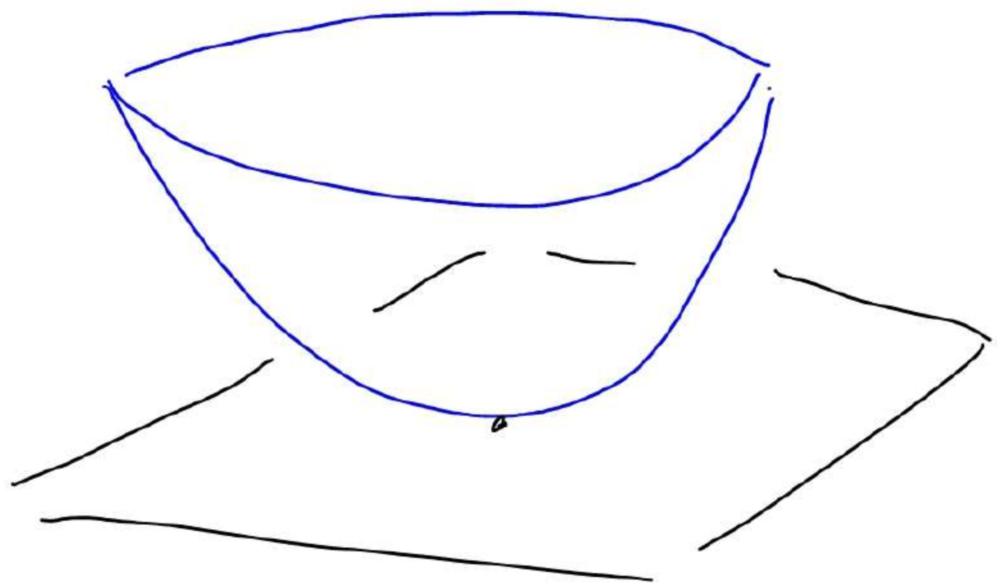
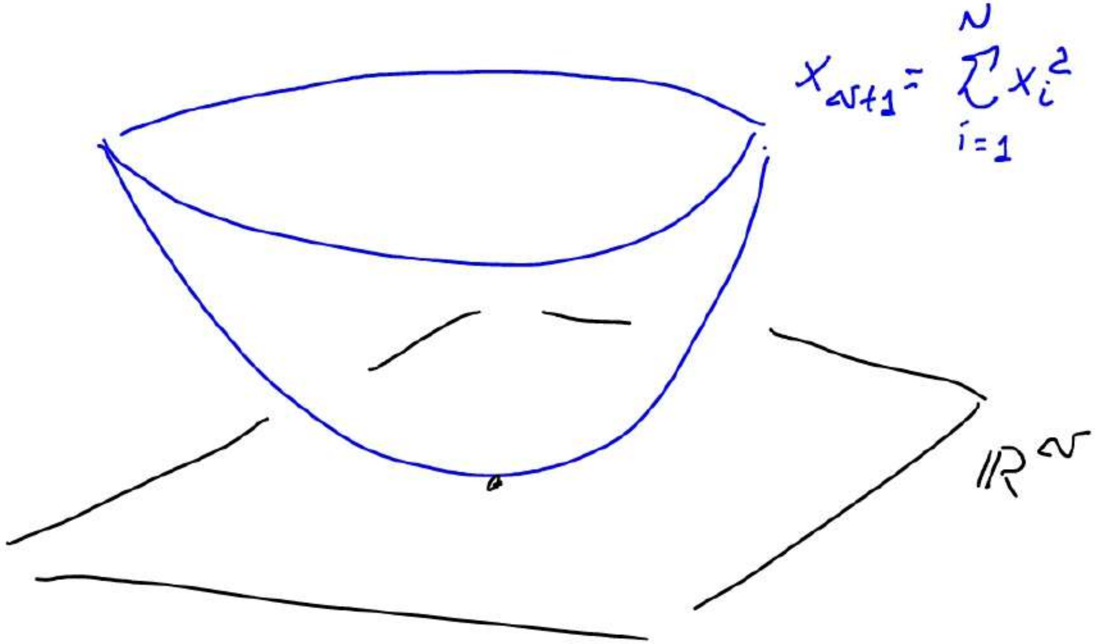
$X = \mathbb{R}^3 - \{(0,0,x_3); x_3 \geq 0\}$, $Y = \{\|x\|^2 = t^3\}$



The retraction to a non-thin is not hornic

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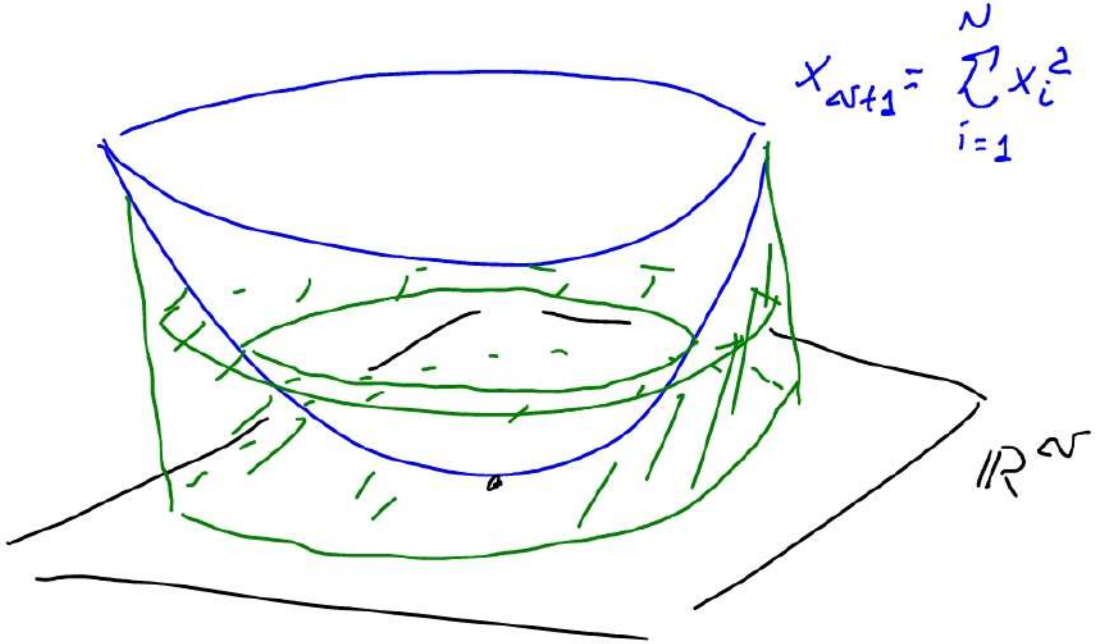
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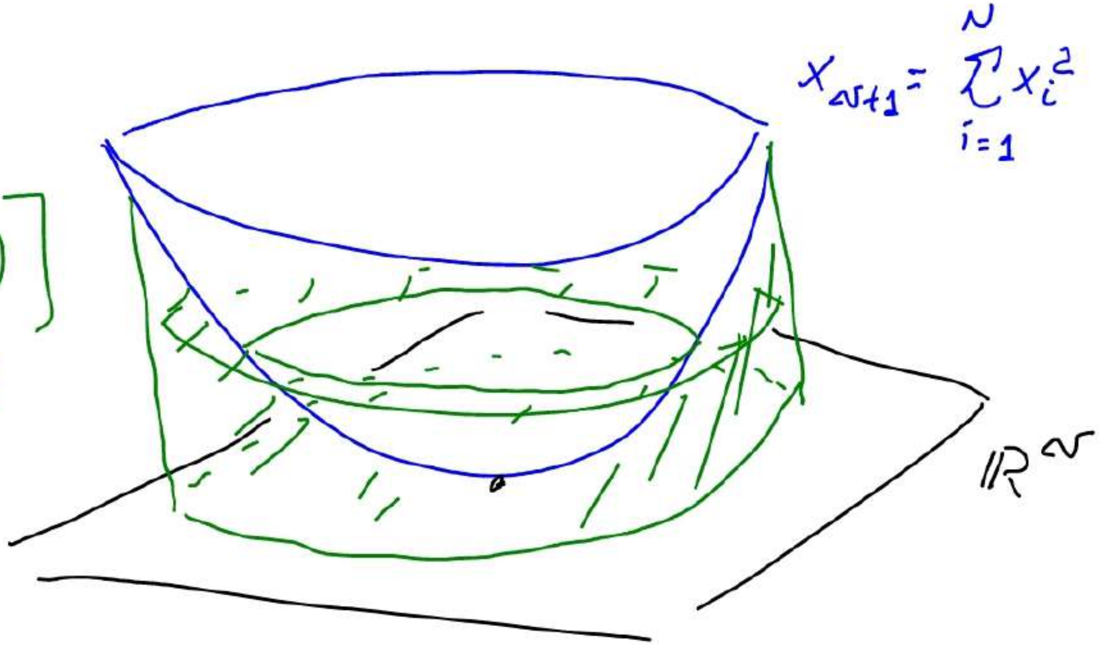
The band $[\mathbb{R}^N, \mathrm{Graph}(x_{N+1})]$ is thin, but not contains fast cycles.



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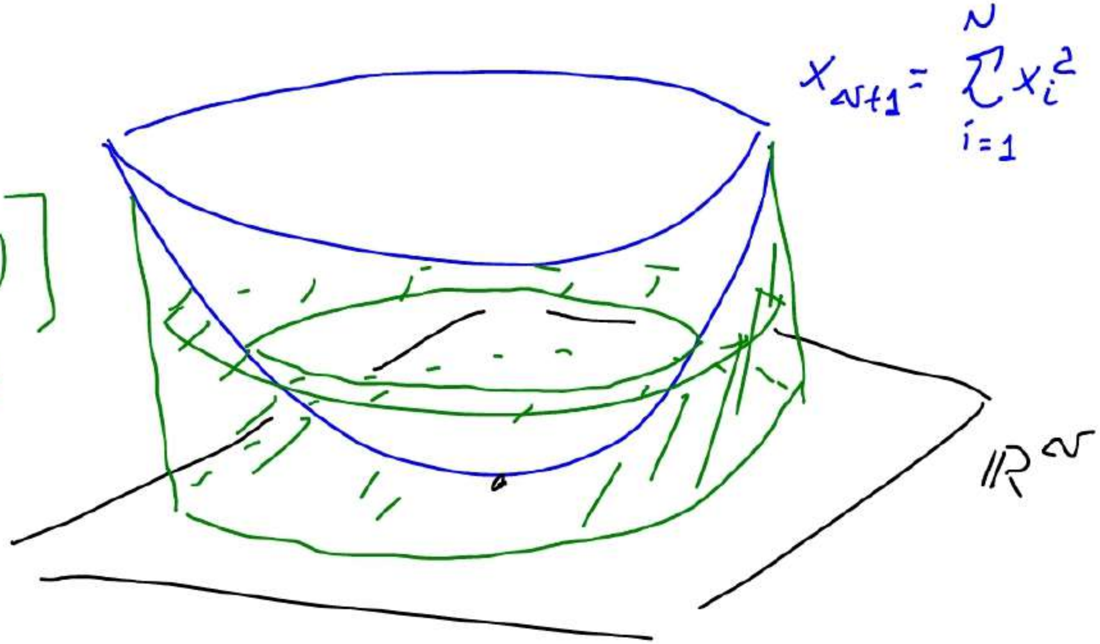

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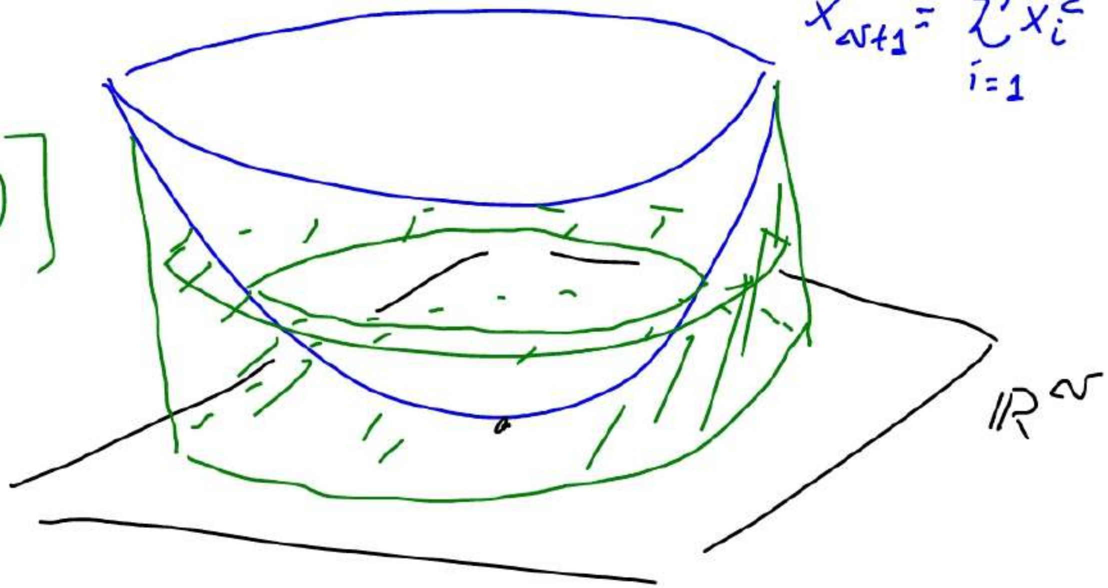

Lemma. An IMC-germ has no fast cycles.

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Observe that $\{\gamma_s\}_{s \in S}$ foliate the ambient K^N. Then, we have an ambient arc foliation compatible with $(X_0, 0) \subseteq (K^N, 0)$.

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Examples.

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Then, X IMC $\Rightarrow p_1 = p_2 = \cdots = p_n$.

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Corollary. Suppose X IMC and $\dim_{\mathbb{C}} \operatorname{Sing}(V(f_\sigma)) < \frac{n+1}{2}$. Then, $w_1 = \cdots = w_n$.

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Then, $X \subseteq \mathbb{C}^{2+c}$ (N.N.D. surface) IHC $\Rightarrow w_1 = w_2$.

About the criterions

<u>Criterion</u> <u>A_1</u> : Let $(X,0) \subseteq K^N,0$, $(K=\mathbb{R},\mathbb{C})$ an analytic germ.

suppose $Y \subset X$ be a subanalytic germ compatible with an action of the form

$$t \circ (x_1, \dots, x_N) = \left(t^{w_1}x_1 + t^{w_1}g_1(\underline{x},t), \dots, t^{w_N}x_N + t^{w_N}g_N(\underline{x},t)\right)$$

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, $0 < w_1 \leq \cdots \leq w_N$. ($g_i(\underline{x},0)=0$, $g_i(\underline{0},t)=0$) continuous functions $\lim_{\underline{x}\to 0} g_i(\underline{x},t) = \lim_{t\to 0} g_i(\underline{x},t)=0$).

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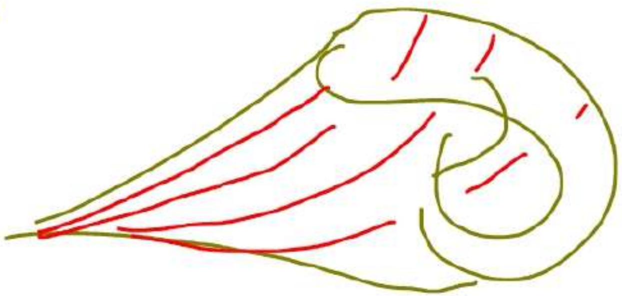

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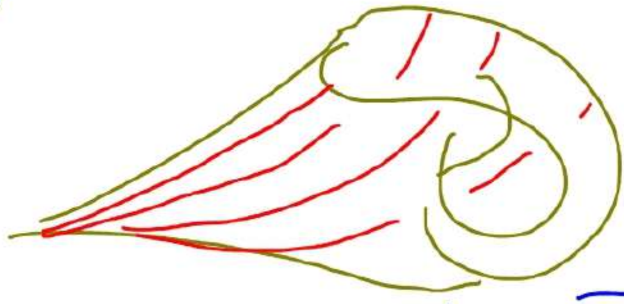

Hence, $\dim T_{(-,0)} \leq r_1 \leq l < \dim \overline{\mathbb{R}_{>0} z}$.

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Theorem (–, Komer (2023)) Let $(X,0) \subset (\mathbb{C}^{n+c}, 0)$ as above, reduced.

1. (Surface case, $n = 2$). If $w_1 < w_2$ and $X \cap \{x_1 = t\}$ contains a smooth (irreducible) non-contractible component, then $(X,0)$ has a fast loop.

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Remark: We partially show how a "essential inner local geometry" and "the topology of the germ" are strongly / directed linked.

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$\Rightarrow$ Moreover, its homotopy type is that of the Milnor fibre.

· Compatibility of deformation of orbit foliations.

Let $\{\gamma_{\underline{a}}(t)\}_{\underline{a}\in\mathfrak{D}}$, $\gamma_{\underline{a}}(t) = t^{\underline{w}}\cdot\underline{a}$. $X_0 = \bigcup_{\underline{a}\in\mathfrak{D}} \gamma_{\underline{a}}$.

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Let $X_\epsilon = V(\underline{f}_{\underline{p}} + \epsilon\cdot \underline{f}_{>\underline{p}}) \subset K^N$, $|\epsilon| \leq 1$ $(\mathrm{ord}_w f_{p_i} < \mathrm{ord}_w f_{>p_i})$

Goal: To deform $\{\gamma_{\underline{a}}\}$ into $\{\gamma_{\underline{a},\epsilon}\}$ compatible with X_ϵ.

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$X_0 = V(x^a + z^c + xy^b)$. weights: $\left(\frac{1}{a}, \frac{1-\frac{1}{a}}{b}, \frac{1}{c}\right)$.

the perturbation $\epsilon \cdot y^{p+1}$ is weigth >1 iff $\frac{b+1}{b}\left(1-\frac{1}{a}\right) > 1$

iff $a > b+1$.

the arc $\gamma_0(t) = (0,t,0) \subset X_0$. Suppose $\gamma_0, \gamma_\epsilon$ tangent at 0. then, $\gamma_\epsilon(t) = (t^\alpha \cdot (\dots), t, t^\beta \cdot (\dots))$, $\alpha, \beta > 1$.

If $c > b+1$, $f_\epsilon(\gamma_\epsilon(t)) = 0$ is a contradiction.

Problem: $X_0 \cap \{x = z = 0\} = \mathbb{C}_y$ is not expected dimension

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Thank you.

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By Birbrair-Fernandes-Neumann 2008, Briançon-Speder Family is not inner Lipschitz trivial.

Hence, $\{\gamma_{s,w}\}$ orbit-foliation of $V(f_0)$ deforms to $\{\gamma_{s,\epsilon,w}\}$ compatible with $V(f_0 + \epsilon z y^6)$ outside of a hornic neighbourhood of $\mathbb{C}_y$.