AMBIENT LIPSCHITZ GEOMETRY OF NORMALLY EMBEDDED SURFACES

Davi Lopes Medeiros
(Universidade Federal do Ceará)
Joint work with Lev Birbrair

Algebraic Geometry, Lipschitz Geometry and Singularities

Pipa, December 11-15, 2023

Lipschitz Classification Problem

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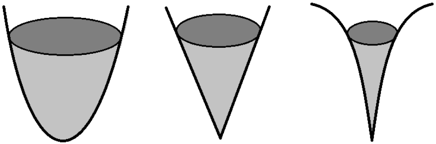

Lipschitz Metrics

Given $C \geq 1$, metric spaces (X, d_1) e (Y, d_2), a map $\varphi : X \to Y$ is **bi-Lipschitz** (or C-bi-Lipschitz) if φ is bijective and if, for all $p, q \in X$:

$$\frac{1}{C} \cdot d_1(p, q) \leq d_2(\varphi(p), \varphi(q)) \leq C \cdot d_1(p, q)$$

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1. **Outer Metric**: $d(p,q) = \|p - q\|, \forall\, p, q \in X$ (euclidian distance)
2. **Inner Metric**: $d_X(p,q) = \inf\{l(p,q)\}, \forall\, p, q \in X$, where the infimum is considered over all rectifiable paths connecting p to q.

Bi-Lipschitz Equivalences

Bi-Lipschitz Equivalence: We say that $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ are:

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2. **Inner Bi-Lipschitz Equivalent**: There is $\varphi : X \to Y$ bi-Lipschitz on the inner metric;
3. **Ambient Bi-Lipschitz Equivalent**: There is $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ outer bi-Lipschitz, such that $\varphi(X) = Y$ (clearly $m = n$).

The Inner Bi-Lipschitz Classification

The inner bi-Lipschitz classification problem is well understood. In fact, we have a complete local classification of closed semialgebraic surfaces (BIRBRAIR, 1999) and a complete global classification for semiagebraic surfaces with isolated inner Lipschitz singularities (FERNANDES, SAMPAIO, 2022).

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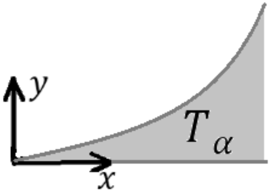
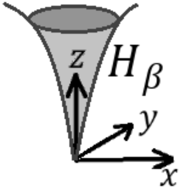
In order to understand the main objects of this (and further) classification, one needs to know the concepts of Hölder triangles and horns.

The Inner Bi-Lipschitz Classification



$\boldsymbol{T_\alpha := \alpha -}$ **Hölder Triangle**

$$\{(x,y)\in\mathbb{R}^2 \mid 0 \le x \le 1; 0 \le y \le x^\alpha\}$$



$\boldsymbol{H_\beta := \beta -}$ **Horn**

$$\{(x,y,t)\in\mathbb{R}^3 \mid x^2 + y^2 \le t^{2\beta}\}$$

The Outer Bi-Lipschitz Classification

The outer bi-Lipschitz classification is a harder problem. One reason is due to topological obstructions (knots). Some interesting results about when we can unknot "Lipschitzly" surfaces germs was given in [BIRBRAIR, MENDES, NUNO-BALLESTEROS, 2017] for specific surfaces germs in $(\mathbb{R}^4, 0)$, and in [BIRBRAIR, FERNANDES, JELONEK, 2020] for sets embedded in a sufficiently high dimension.

The Outer Bi-Lipschitz Classification

Another problem on the outer bi-Lipschitz classification problem is defining a canonical structure for real surfaces in general. Although this problem was solved for complex algebraic plane curves (TARGINO, 2020), for the real case we only have some partial progress.

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The most recent advances in this way, is the decompositon of a surface germ into snakes [GABRIELOV, SOUZA, 2022] and the outer Lipschitz classification of normal pairs of Hölder triangles [BIRBRAIR, GABRIELOV, 2023].

The Outer Bi-Lipschitz Classification

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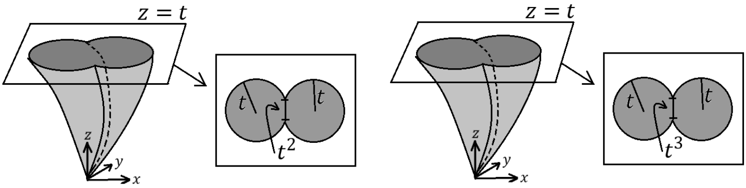

The Outer Bi-Lipschitz Classification

Several invariants were developed to answer when two surfaces are not outer bi-Lipschitz equivalent. Some good examples using homology theory are:

- The **metric homology** (BIRBRAIR, BRASSELET, 2000)
- The **vanishing homology** (VALLETE, 2010)
- The **moderated discontinuous homology** (BOBADILLA, PE PEREIRA, HEINZE, SAMPAIO, 2019)

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One of the first relevant results is in [SAMPAIO, 2016], and it states that if $(X, 0), (Y, 0) \subset (\mathbb{R}^n, 0)$ are subanaytic sets that are ambient bi-Lipschitz equivalent, then their tangent cones $(C(X), 0), (C(Y), 0)$ are also ambient bi-Lipschitz equivalent.

The Ambient Bi-Lipschitz Classification

The most recent contribution in Ambient Lipschitz Geometry is the creation of so-called Metric Knot Theory. There are many examples of pairs of germs of semialgebraic surface germs in $\mathbb{R}^3$, with the same ambient topology, same inner and outer metric structure, but which are not ambient Lipschitz equivalent

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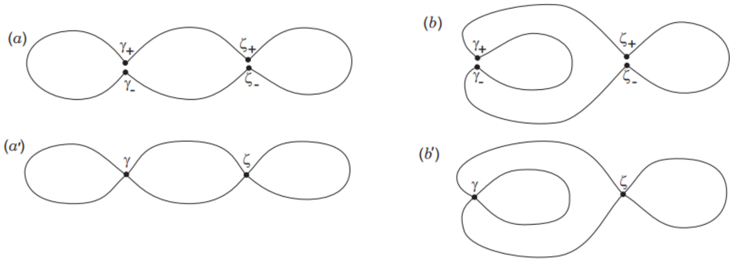

The Ambient Bi-Lipschitz Classification

A surprisingly theorem proved in [BIRBRAIR, GABRIELOV, BRANDENBURSKY, 2020] states that for any semialgebraic surface germ $(X,0) \subset (\mathbb{R}^4,0)$ there are infinite surface germs $(X_i,0) \subset (\mathbb{R}^4,0)$ such that $(X_i,0)$ are topologically ambient equivalent to $(X,0)$, outer bi-Lipschitz equivalent to $(X,0)$, but are not bi-Lipschitz ambient equivalent to each other.

The Ambient Bi-Lipschitz Classification

The main idea of the proof is to "twist" the surface link, therefore breaking the "proximity" between the inner and outer metric.

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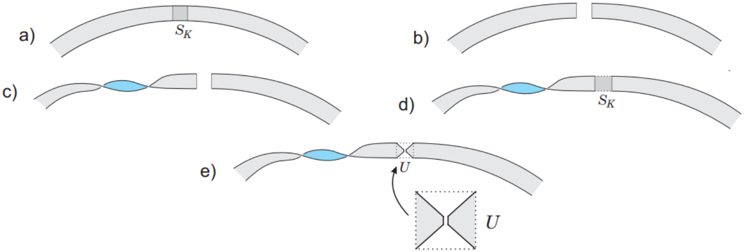

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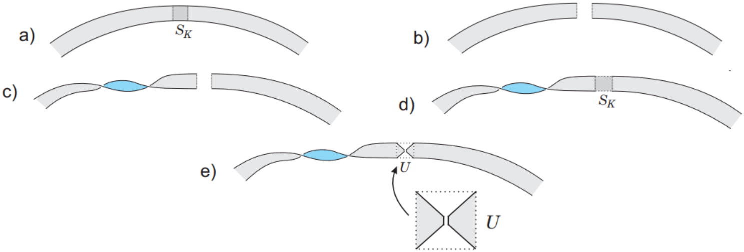

But... if this proximity cannot be broken, what can we say?

Normally Embedded Sets

We say that a set $X \subseteq \mathbb{R}^n$ is **Lipschitz Normally Embedded (LNE)** if the outer metric and the inner metric are equivalent. Analogous definitions can be done to germs of sets in a given point.

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Arcs: An arc in $\mathbb{R}^n$ is a germ at the origin of a semialgebraic map $\gamma : [0, t_0) \to \mathbb{R}^n$, for some $t_0 > 0$ sufficiently small, such that $\gamma(0) = 0$. The set of all arcs $\gamma \subset X$ is called **Valette link of** X, and is denoted by $V(X)$ (VALLETE, 2007)

Normally Embedded Sets

Order of Contact: Given two distinct arcs $\gamma_1, \gamma_2 \in V(X)$, we define the order of contact (or the tangency order) on the following metrics:

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1. **Outer**: $tord(\gamma_1, \gamma_2) = \alpha$, where $\|\gamma_1(t), \gamma_2(t)\| = ct^{\alpha} + o(t^{\alpha})(c > 0)$

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Such orders are rational numbers satisfying

$$1 \leq tord_X(\gamma_1, \gamma_2) \leq tord(\gamma_1, \gamma_2)$$

In addition, $(X, 0)$ is LNE if, and only if, for all $\gamma_1, \gamma_2 \in V(X)$, we have $tord(\gamma_1, \gamma_2) = tord_X(\gamma_1, \gamma_2)$ (BIRBRAIR; MENDES, 2018).

Ambient Lipschitz Equivalence Conjecture

Conjecture: Let $(X, 0)$ and $(Y, 0)$ be two semialgebraic 2-dimensional surface germs on $(\mathbb{R}^n, 0)$ that are outer bi-Lipschitz equivalent and topologically ambient equivalent. If $(X, 0)$ and $(Y, 0)$ are LNE, then $(X, 0)$ e $(Y, 0)$ are ambient bi-Lipschitz equivalent.

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In this talk, I'll prove this conjecture for $n = 3$.

Synchronized Triangles

In what follows, for any $a > 0$, let
$C_a^{n+1} = \{(x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1} \mid t \geq 0;\ x_1^2 + \cdots + x_n^2 \leq (at)^2\}$.

Definition

*Let $(\gamma_0, 0), (\gamma_1, 0) \subseteq (C_a^3, 0)$ distinct curve germs and $(T, 0) \subseteq (C_a^3, 0)$ a triangle germ (boundary arcs $(\gamma_0, 0), (\gamma_1, 0)$). We say that $(T, 0)$ is a **Synchronized Triangle Germ** if for every small t, we have $x_0(t) < x_1(t)$ and:*

- $\gamma_0 = \gamma_0(t) = (x_0(t), y_0(t), t); \gamma_1 = \gamma_1(t) = (x_1(t), y_1(t), t)$
- *$\pi_z(T \cap \{z = t\})$ is graph of a semialgebraic function $f_t : [x_0(t), x_1(t)] \to \mathbb{R}$ with $f_t(x_i(t)) = y_i(t)$ $(i = 1, 2)$.*

*The family of functions $\{f_t\}_{0<t<\varepsilon}$ is called **Generator of the Synchronized Germ** $(T, 0)$.*

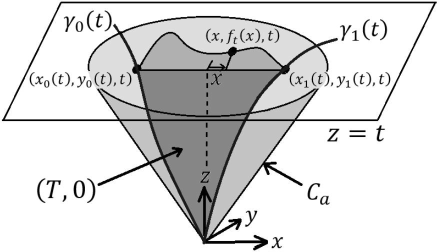
Synchronized Triangles

Arc Coordinate System of (T,0):

Arcs $\gamma_u \subset V(T)$ $(0 \leq u \leq 1)$, where:

$$\theta_u(t) = u \cdot x_1(t) + (1-u) \cdot x_0(t) \in [x_0(t), x_1(t)]$$

$$\gamma_u(t) = (\theta_u(t), f_t(\theta_u(t)), t) \ ; \ t > 0$$



Curvilinear Rectangles

Definition

Let $(T_1, 0), (T_2, 0) \subset (C_a^3, 0)$ be synchronized triangle germs. We say that $(T_1, 0), (T_2, 0)$ ***are aligned on boundary arcs*** *if there are distinct curve germs $(\gamma_0, 0), (\gamma_1, 0), (\rho_0, 0), (\rho_1, 0) \subseteq (C_a^3, 0)$ such that, for $i = 0, 1$, $(\gamma_i, 0)$ and $(\rho_i, 0)$ are the boundary arcs of $(T_1, 0), (T_2, 0)$, respectively, and:*

$$\gamma_i = \gamma_i(t) = (x_i(t), y_i(t), t) \ ; \ \rho_i = \rho_i(t) = (x_i(t), w_i(t), t)$$

Let $\{f_t\}, \{g_t\}$ are the families of generating functions of $(T_1, 0), (T_2, 0)$, respectively (WLOG $g_t \leq f_t$) . We define the ***curvilinear rectangle delimited by*** *$(T_1, 0), (T_2, 0)$ as the germ of:*

$$R = \{(x, y, t) \in C_a^3 \ | \ x_0(t) \leq x \leq x_1(t) \ ; \ g_t(x) \leq y \leq f_t(x)\}$$

If $(\gamma_i, 0) = (\rho_i, 0)$ such a rectangle is called ***region delimited by*** *$(T_1, 0), (T_2, 0)$.*

Curvilinear Rectangles

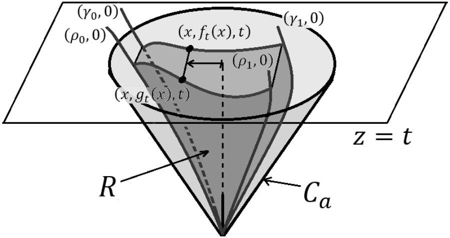
Arc Coordinate System of (R,0):

Arcs $\gamma_{u,v} \subset V(R)$ $(0 \leq u, v \leq 1)$, where:

$$\theta_u(t) = u \cdot x_1(t) + (1-u) \cdot x_0(t) \in [x_0(t), x_1(t)]$$

$$\sigma_{u,v}(t) = v \cdot f_t(\theta_u(t)) + (1-v) \cdot g_t(\theta_u(t)) \in [g_t(\theta_u(t)), f_t(\theta_u(t))]$$

$$\gamma_{u,v}(t) = (\theta_u(t), \sigma_{u,v}(t), t) \; ; \; t > 0$$



Ambient Bi-Lipschitz Isotopy

Definition

*Let $X, X_0, X_1 \subseteq \mathbb{R}^n$ sets such that $X_1, X_2 \subseteq X$. We say that X_1, X_2 are **Ambient Bi-Lipschitz Isotopic in** X if there is a continuous map $\varphi : X \times [0,1] \to X$ such that, if we denote $\varphi_\tau(p) = \varphi(p,\tau)$, then:*

1. *$\varphi_\tau : X \to X$ is a bi-Lipschitz map (with respect to the induced metric of $\mathbb{R}^n$), for all $0 \leq \tau \leq 1$.*
2. *$\varphi_0 = id_X$.*
3. *$\varphi_1(X_1) = X_2$.*

Ambient Bi-Lipschitz Isotopy

Definition

The map φ is called ***Ambient Bi-Lipschitz Isotopy in*** X***, taking*** X_1 ***into*** X_2*. We also say that the isotopy φ is* ***Invariant on the Boundary of*** X *if* $\varphi_\tau|_{\partial X} = id_{\partial X}$*, for all* $0 \leq \tau \leq 1$*.*

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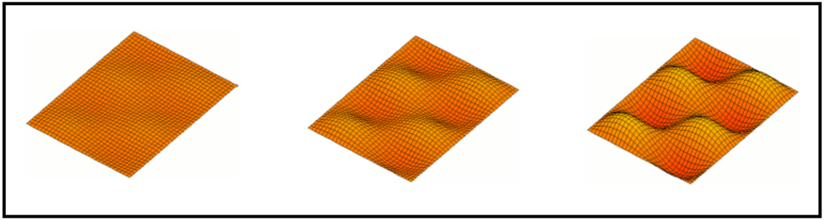


Theorem (Ambient bi-Lipschitz Isotopy in Curvilinear Rectangles)

Let:

$$(T_1, 0), (T_2, 0), (W_1, 0), (W_2, 0) \subset (C_a^3, 0)$$

be germs of synchronized triangles, two by two alligned on the boundary arcs. If for all $t > 0$ *small, there is* $M > 1$ *such that:*

- $(T_1, 0), (T_2, 0), (W_1, 0), (W_2, 0)$ *have M-bounded derivative and that* $\{f_t\}, \{g_t\}, \{a_t\}, \{b_t\}$ *are their respective families of generating functions;*
- $(T_1, 0)$ *has* $(\gamma_0, 0), (\gamma_1, 0)$ *as boundary arcs, where:*

$$\gamma_0 = \gamma_0(t) = (x_0(t), y_0(t), t)\ ;\ \gamma_1 = \gamma_1(t) = (x_1(t), y_1(t), t)$$

and $x_0(t) < x_1(t)$*;*

Ambient Bi-Lipschitz Isotopy

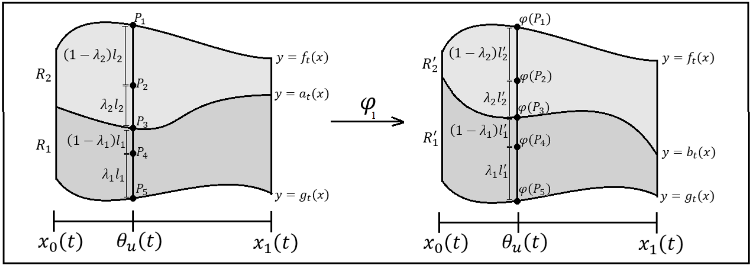
Theorem

- *for all* $x \in (x_0(t), x_1(t))$, *the inequalities are satisfied:*

$$g_t(x) < a_t(x), b_t(x) < f_t(x)$$

$$\frac{1}{M} \leq \frac{a_t(x) - g_t(x)}{f_t(x) - g_t(x)}, \frac{b_t(x) - g_t(x)}{f_t(x) - g_t(x)} \leq 1 - \frac{1}{M}$$

If $(R, 0)$ *is the curvilinear rectangle delimited by* $(T_1, 0), (T_2, 0)$, *then there is a ambient isotopy in* $(R, 0)$, *taking* $(W_1, 0)$ *into* $(W_2, 0)$. *Moreover, if* $(R, 0)$ *is a region, then this ambient isotopy is invariant on the boundary.*

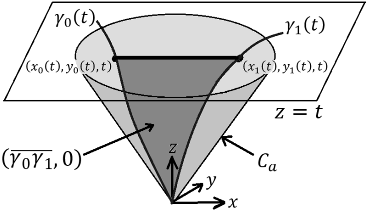


The proof of this theorem is basically a extension to what was done for the $\mathcal{K}$-bi-Lipschitz equivalence criterion (BIRBRAIR, COSTA, FERNANDES, RUAS, 2007) and the $\mathcal{C}^p$-parametrization in o-minimal structures (KOCEL-CYNK, PAWLUCKO, VALLETE, 2019).

Kneadable Triangles

Let $a > 0$, $\gamma_1, \gamma_2 \subset C_a^3$ be two arcs satisfying $tord(\gamma_1, \gamma_2) \neq \infty$, with $\gamma_i(t) = (x_i(t), y_i(t), t)$ $(i = 1.2)$, for every $t > 0$ small enough. We define the **Linear Triangle Delimited by** γ_1, γ_2 as the germ at the origin of the set:

$$\overline{\gamma_1\gamma_2} = \{\lambda\gamma_1(t) + (1-\lambda)\gamma_2(t) \mid t > 0 \; ; \; 0 \leq \lambda \leq 1\}$$



Kneadable Triangles

Definition

Let $(T,0) \subset (C_a^3,0)$ be a triangle with main vertex at the origin, γ_1,γ_2 its boundary arcs and $(U,0)$ be a germ of a closed set containing $(T,0)$. We say that $(T,0)$ is ***kneadable in*** $(U,0)$ *if there is an ambient bi-Lipschitz isotopy in U that takes $(T,0)$ into $(\overline{\gamma_1\gamma_2},0)$, invariant on the boundary of $(U,0)$.*

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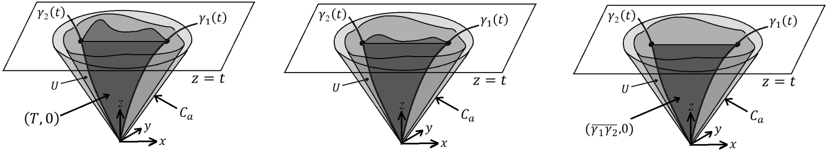
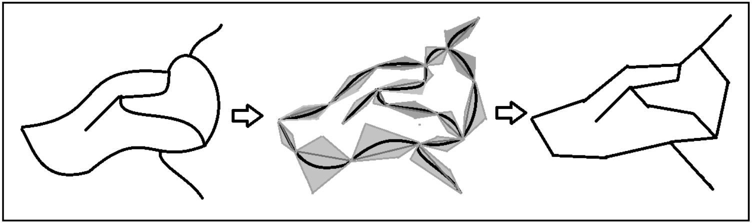


Kneadable Triangles

Proposition

Let $a > 0$ and let $(X, 0) \subset (C_a^3, 0)$ be a pure, closed, semi-algebraic, 2-dimensional LNE surface germ with connected link. Then, $(X, 0)$ is ambient bi-Lipschitz equivalent to a germ of a surface formed by a finite union of linear triangles delimited by arcs.



Kneadable Triangles

Proposition

Let $a > 0$ and let $(X, 0) \subset (C_a^3, 0)$ be a polygonal LNE surface germ.

1. *If $(X, 0)$ is open polygonal, then $(X, 0)$ is ambient bi-Lipschitz equivalent to a α-Hölder triangle, for some $\alpha \in \mathbb{Q}_{\geq 1}$.*
2. *If $(X, 0)$ is closed polygonal, then $(X, 0)$ is ambient bi-Lipschitz equivalent to a β-horn, for some $\beta \in \mathbb{Q}_{\geq 1}$.*

Main Theorem

Theorem

Let $(X,0),(Y,0) \subset (\mathbb{R}^3,0)$ be a normally embedded semi-algebraic surface germs. Then, $(X,0)$ and $(Y,0)$ are ambient bi-Lipschitz equivalent if and only if $(X,0)$ and $(Y,0)$ are inner bi-Lipschitz equivalent.

Sketch of the Proof

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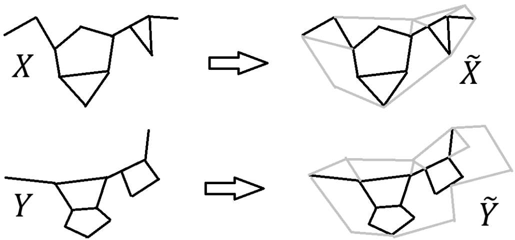
It's also enough to consider X, Y as the union of linear triangles. Now, take $\tilde{X} \supset X$, $\tilde{Y} \supset Y$ LNE surface germs whose link are the union of horns (this is possible by transversality of synchronized triangles).

Sketch of the Proof



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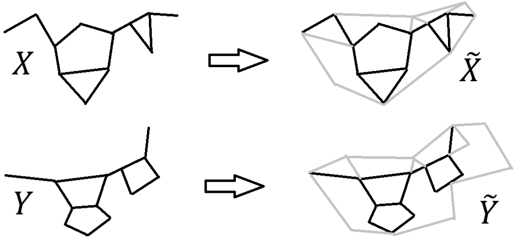
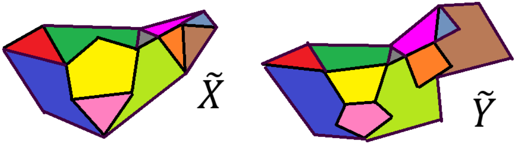


By the Inner bi-Lipschitz Gluing Lemma, we can define sucessively bi-Lipschitz maps to the union of each horn in the Hölder complex, thus constructing a ambient bi-Lipschitz map sending one "maximal horn" to another "maximal horn".

Sketch of the Proof



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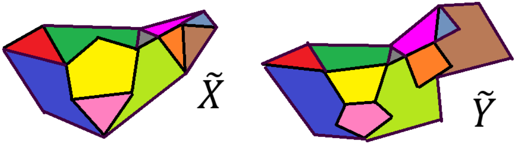

Since those horns are on the same order, they are ambient bi-Lipschitz equivalent, and this ambient map sends X to Y by construction ∎

Remark on Higher Dimension

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Although all the concepts of synchronized triangles, regions and polygonal surfaces can be defined naturally for higher dimensions, reducing edge surfaces via triangulations does not work for higher dimensions. For example, the following set (By Edson Sampaio):

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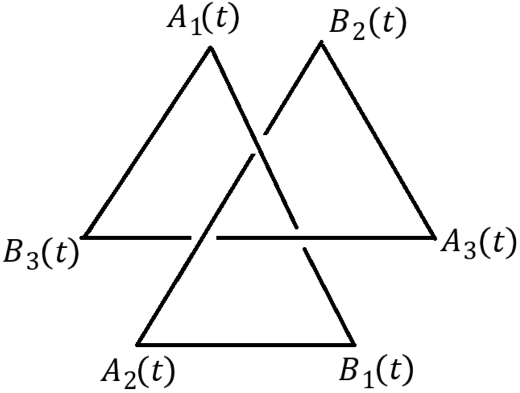
Although all the concepts of synchronized triangles, regions and polygonal surfaces can be defined naturally for higher dimensions, reducing edge surfaces via triangulations does not work for higher dimensions. For example, the following set (By Edson Sampaio):

$$X(t) = \overline{A_1(t)B_1(t)} \cup \overline{B_1(t)A_2(t)} \cup \overline{A_2(t)B_2(t)} \cup \overline{B_2(t)A_3(t)}$$
$$\cup \overline{A_3(t)B_3(t)} \cup \overline{B_3(t)A_1(t)}; X = (\cup_{t>0} X(t)) \cup \{0\}$$

where:

$$A_1(t) = (5t\sqrt{3}, 3t, 3t, t)\ ;\ A_2(t) = (-4t\sqrt{3}, 6t, 3t, t)$$
$$A_3(t) = (-t\sqrt{3}, -9t, 3t, t)\ ;\ B_1(t) = (-4t\sqrt{3}, -6t, -3t, t)$$
$$B_2(t) = (5t\sqrt{3}, -3t, -3t, t)\ ;\ B_3(t) = (-t\sqrt{3}, 9t, -3t, t)$$

X is LNE, and is outer bi-Lipschitz equivalent to the 1-horn embedded in $\mathbb{R}^4$. However, X is not topologically equivalent to the 1-horn, so X is not ambient bi-Lipschitz equivalent to the 1-horn.



Further Research and Open Questions

The proof in $\mathbb{R}^3$ works well on LNE sets by Jordan Theorem on curves. How can we relate topological obstructions with ambient Lipschitz Classification?

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It'll be a pleasure if you can help me, but for now...

Thank You!!!