Automorphisms of Projective Hypersurfaces

Alvaro Liendo

(joint work with V. González, P. Montero and R. Villaflor)

Instituto de Matemática, Universidad de Talca

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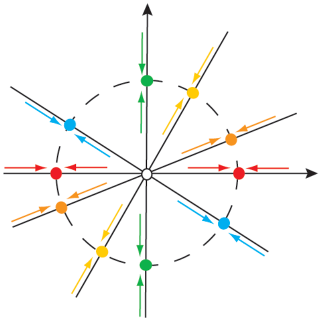
Projective space

Let V be a vector space V of dimension $n+2$

The projective space $\mathbf{P}^{n+1}$ is the quotient of $V \setminus \{0\}$ by the equivalence relation $\sim$ given by

$$x \sim y \quad \text{if and only if} \quad x = \lambda y \quad \text{with } \lambda \in \mathbf{C}$$



Projective hypersurfaces

A homogeneous form F of degree d is an element in the symmetric product $S^d(V^*)$

If we pick a basis $\{x_0, x_1, x_2, \cdots, x_n, x_{n+1}\}$ of V^*, we have the symmetric algebra

$$S(V^*) \simeq \mathbf{C}[x_0, x_1, \cdots, x_n, x_{n+1}]$$

Hence, a homogeneous form F of degree d is a polynomial of degree d.

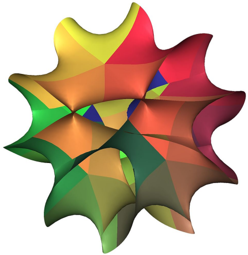
A projective hypersurface X of dimension n and degree d is the zero set in $\mathbf{P}^{n+1}$ of a homogeneous form F of degree d.

$$X = V(F) = \{x \in \mathbf{P}^{n+1} \mid F(x) = 0\}$$

Fermat Hypersurface

The Fermat hypersurface of dimension n and degree d is given by the form

$$F = x_0^d + x_1^d + \ldots + x_{n+1}^d$$

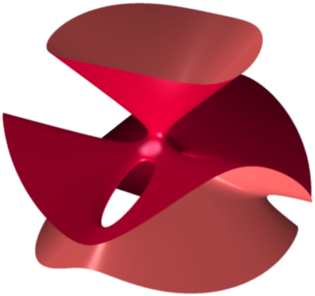


Two-dimensional cross-section of the Fermat quintic threefold

Klein Hypersurface

The Fermat hypersurface of dimension n and degree d is given by the form

$$K = x_0^{d-1}x_1 + x_1^{d-1}x_2 + \ldots + x_{n+1}^{d-1}x_0$$

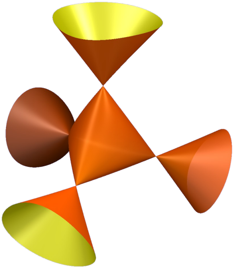


Clebsch cubic surface

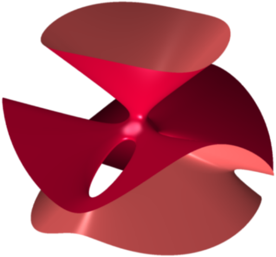
Smooth varieties

A complex variety is smooth if it is locally analytically isomorphic to the Euclidean space

The points where X is not locally analytically isomorphic to the Euclidean space are called the singular points of X.



Cayley nodal cubic surface



Clebsch cubic surface

Smooth hypersurfaces

Jacobian Criterion

A point x in a projective hypersurface $X = V(F)$ of dimension n and degree d in $\mathbf{P}^{n+1}$ is singular if and only if

$$\frac{dF}{dx_0}(x) = \frac{dF}{dx_1}(x) = \cdots = \frac{dF}{dx_{n+1}}(x) = 0$$

Let $n \geq 1$ and $d \geq 3$ with $(n, d) \neq (1, 3), (2, 4)$

Every automorphism of a smooth hypersurface

$$X = \{F = 0\} \subset \mathbf{P}^{n+1}$$

is induced by an automorphism of $\mathbf{P}^{n+1}$ and $\mathrm{Aut}(X)$ is finite. (Matsumura-Monsky '64)

Since

$$\mathrm{Aut}(\mathbf{P}^{n+1}) = \mathrm{PGL}(n, \mathbf{C}),$$

every automorphism φ of X is represented by a matrix $M \in \mathrm{GL}(n, \mathbf{C})$.

Let φ be an automorphism of $X = V(F)$

Since the order of φ is finite (say p), we can chose a basis that makes it diagonal. Hence,

$$(x_0 : x_1 : \cdots : x_{n+1}) \xmapsto{\varphi} (\xi^{\sigma_0} \cdot x_0 : \xi^{\sigma_1} \cdot x_1 : \cdots : \xi^{\sigma_{n+1}} \cdot x_{n+1}),$$

Hence, the automorphism if characterized by

$$\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{n+1}) \in (\mathbf{Z}/p\mathbf{Z})^{n+2}$$

Let X be a hypersurface of dimension n and degree d, given by the homogeneous form $F \in S^d(V^*)$. If $\deg_i(F) \leq d-2$, for some $i \in \{0, \ldots, n+1\}$, then X is singular.

Proof.

$$F = x_0^{d-2} L_2 + x_0^{d-3} L_3 + \ldots + x_0 L_{d-1} + L_d \,,$$

where L_j is a form of degree j in the variables $\{x_1, \ldots, x_{n+1}\}$.

$$\frac{\partial F}{\partial x_0} = (d-2) x_0^{d-3} L_2 + (d-3) x_0^{d-4} L_3 + \ldots + L_{d-1},$$
$$\frac{\partial F}{\partial x_i} = x_0^{d-2} \frac{dL_2}{dx_i} + x_0^{d-3} \frac{dL_3}{dx_i} + \ldots + \frac{dL_d}{dx_i}, \quad i \in \{1, \ldots, n+1\} \,.$$

Now, the Jacobian criterion shows that the point $(1 : 0 : \ldots : 0)$ is singular. □

Let $\varphi \in \mathrm{Aut}(X)$. We have $\varphi^*(F) = \lambda F$. Assume for a moment that $\varphi^*(F) = F$.

$$\text{if} \quad F = x_0^d + \dots \quad \text{then} \quad \varphi^*(F) = \xi^{d\sigma_0} x_0^d + \dots$$

And so p divides d

$$F = x_0^{d-1} x_1 + \dots \quad \text{then} \quad \varphi^*(F) = \xi^{(d-1)\sigma_0 + \sigma_1} x_0^{d-1} x_1 + \dots$$

And so $\begin{cases} p \text{ divides } d-1 \text{ and } \sigma_1 = 0 \mod p, \text{ or} \\ \sigma_0, \sigma_1 \neq 0 \text{ and } (d-1)\sigma_0 + \sigma_1 = 0 \mod p \end{cases}$

In the second case, we can iterate with x_1 to obtain

$$\sigma_i = (1-d)^i \sigma_0 \mod p$$

Theorem (González, L. 2013)

Let n, d, p be positive integers with $d \geq 3$, $(n, d) \neq (1,3), (2,4)$ and p prime.
Then p is the order of a smooth hypersurface of dimension n and degree d if and only if p divides $d(d-1)$ or

$$\text{there exists } \ell \in \{1, 2, \ldots, n+2\} \text{ such that } (1-d)^{\ell} \equiv 1 \mod p.$$

Corollary

Let n, d be positive integers with $d \geq 3$ and $(n, d) \neq (1,3), (2,4)$. If a prime number p is the order of an automorphism of a smooth hypersurface of dimension n and degree d, then

$$p < (d-1)^{n+1}.$$

Theorem (González, L. 2013)

Let $n \geq 3$, $d \geq 3$. Then, a smooth hypersurface $X \subseteq \mathbf{P}^{n+1}$ of degree d admits an automorphism of prime order $p > (d-1)^n$ if and only if

$$X \simeq \{x_0^{d-1}x_1 + x_1^{d-1}x_2 + \ldots + x_{n+1}^{d-1}x_0 = 0\},$$

$n+2$ is prime and $p = \frac{(d-1)^{n+2}+1}{d}$ is prime.

We say that X is a **Klein hypersurface of Wagstaff type** of dimension n and degree d.

Theorem (González, Montero, L. 2022)

Let n, d, p, r *be positive integers with* $d \geq 3$, $(n, d) \neq (1, 3), (2, 4)$ *and* p *prime.*
Then p^r *is the order of an automorphism of a smooth hypersurface of dimension* n *and degree* d *if and only if*

(i) p *divides* $d - 1$ *and* $r \leq k(n + 1)$, *where* $d - 1 = p^k e$ *with* $\gcd(p, e) = 1$, *or*

(ii) p *divides* d *and and there exists* $\ell \in \{1, 2, \ldots, n + 1\}$ *such that* $(1 - d)^\ell \equiv 1 \mod p^r$, *or*

(iii) p *does not divide* $d(d - 1)$ *and there exists* $\ell \in \{1, 2, \ldots, n + 2\}$ *such that* $(1 - d)^\ell \equiv 1 \mod p^r$.

Let $\pi\colon \mathrm{PGL}(V) \to \mathrm{GL}(V)$.

Let $G \subset \mathrm{Aut}(X) \subset \mathrm{PGL}(V)$ with $X = V(F)$.

We say that G is liftable if and only if there exists $\widetilde{G} \in \mathrm{GL}(V)$ such that $\widetilde{G} \xrightarrow{\pi} G$ is an isomorphism and $\varphi^*(F) = F$ for all $\varphi \in \widetilde{G}$.

Theorem (González, Montero, L. 2022)

Let $n \geq 1$ and $d \geq 3$ with $(n, d) \neq (1,3), (2,4)$. Then the automorphism group of every smooth hypersurface of dimension n and degree d in $\mathbb{P}(V)$ is liftable if and only if d and $n+2$ are relatively prime.

We also obtain that non-liftable automorphism always have order dividing d and $n+2$.

This theorem was inspired by papers by Oguiso and his coauthors.

The Klein hypersurface

$$X_K = \{x_0^{d-1}x_1 + x_1^{d-1}x_2 + \ldots + x_{n+1}^{d-1}x_0 = 0\}$$

has two natural automorphisms: if we define

$$m := \frac{(d-1)^{n+2} - (-1)^{n+2}}{d}$$

then X admits an automorphism σ of order m and an automorphism ν of order $n+2$. They are given by

$$(x_0 : x_1 : \cdots : x_{n+1}) \overset{\sigma}{\mapsto} (\zeta_{dm}x_0 : \zeta_{dm}^{1-d}x_1 : \cdots : \zeta_{dm}^{(1-d)^{n+1}}x_{n+1}),$$

$$(x_0 : x_1 : \cdots : x_{n+1}) \overset{\nu}{\mapsto} (x_1 : x_2 : \cdots : x_{n+1} : x_0)$$

Hence, the following group acts faithfully on X

$$\mathcal{K}(n,d) := (\mathbf{Z}/m\mathbf{Z}) \rtimes \mathbf{Z}/(n+2)\mathbf{Z}$$

Theorem (González, Montero, Villaflor, L. 2023)

Let $n \geq 1$ and $d \geq 3$, and $X_d \subseteq \mathbf{P}^{n+1}$ be the Klein hypersurface X. Then

$$\mathrm{Aut}(X) = \begin{cases} X \rtimes \mathbf{Z}/6\mathbf{Z} & \text{if } n = 1, d = 3 \\ \mathrm{PSL}_2(\mathbf{F}_7) & \text{if } n = 1, d = 4 \\ \mathcal{K}(n,d) & \text{if } n = 1, d \geq 5 \\ \mathfrak{S}_5 & \text{if } n = 2, d = 3 \\ ?\ \ (+\infty) & \text{if } n = 2, d = 4 \\ \mathcal{K}(n,d) & \text{if } n = 2, d \geq 5 \\ \mathrm{PSL}_2(\mathbf{F}_{11}) & \text{if } n = 3, d = 3 \\ \mathcal{K}(n,d) & \text{if } n = 3, d \geq 4 \\ \mathcal{K}(n,d) & \text{if } n \geq 4, d \geq 3 \end{cases}$$

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arXiv:2212.13308: On a Torelli Principle for automorphisms of Klein hypersurfaces

Remark that

$$\mathcal{K}(n,d) := (\mathbf{Z}/m\mathbf{Z}) \rtimes \mathbf{Z}/(n+2)\mathbf{Z} = \langle\sigma\rangle \rtimes \langle\nu\rangle$$

σ is diagonal

ν and σ have at most one coefficient different from 0 in each row and each column

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$$\mathcal{K}(n,d) := (\mathbf{Z}/m\mathbf{Z}) \rtimes \mathbf{Z}/(n+2)\mathbf{Z} = \langle\sigma\rangle \rtimes \langle\nu\rangle$$

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A matrix M is a *generalized permutation matrix* if there is at most one coefficient different from 0 in each row and each column.
$M = P_1 D P_2$ with P_1, P_2 permutation matrices and D diagonal.

A matrix M is a *generalized triangular matrix* if $M = P_1 T P_2$ with P_1, P_2 permutation matrices and T triangular.

To compute $\mathrm{Aut}(X)$ in the remaining cases, we improve the **differential method** introduced by Poonen (2005) and Oguiso–Yu (2019).

Let $V = \mathbf{C}^{n+2}$ so that $\mathbf{P}^{n+1} = \mathbf{P}(V)$.

$V^* = \mathrm{span}(x_0, \dots, x_{n+1}) \simeq \mathbf{C}^{n+2}$

$S(V^*) = \mathbf{C}[x_0, \dots, x_{n+1}]$

$K \in S^d(V^*)$ homogeneous forms in $n+2$ variables of degree d.

Let D be the directional derivative operator

$$D\colon V^* \times S(V^*) \to S(V^*), \qquad (x, F) \mapsto \frac{\partial F}{\partial x} = \nabla(F) \cdot x\,.$$

For a fixed homogeneous form $F \in S(V^*)$ define the *specialization of D* given by

$$D_F\colon V^* \to S(V^*), \qquad x \mapsto \frac{\partial F}{\partial x} = \nabla(F) \cdot x\,.$$

We define the *differential rank* of a form $F \in S(V^*)$ as

$$\operatorname{drank}(F) := \operatorname{rank} D_F\,.$$

Theorem (Differential method)

Let X be a hypersurface in $\mathbf{P}^{n+1}$ given by a homogeneous form $F \in S(V^*)$ and let $\varphi \in GL(V)$ be a linear automorphism of X. Then for every $x \in V^*$ we have

$$\operatorname{drank}\left(\frac{\partial F}{\partial x}\right) = \operatorname{drank}\left(\frac{\partial F}{\partial(\varphi^* x)}\right).$$

The differential method is useful when there exists $x \in V^*$ with small

$$\operatorname{drank}\left(\frac{\partial F}{\partial x}\right)$$

Example: Fermat hypersurface

$$F = x_0^d + x_1^d + \ldots + x_{n+1}^d, \quad d \geq 3$$

Let $\varphi = (\varphi_{ij})$ be an automorphism of

$$Y = \{F = 0\} \in \mathbf{P}^{n+1}$$

We have

$$\mathrm{drank}\left(\frac{\partial F}{\partial x_i}\right) = \mathrm{drank}(dx_i^{d-1}) = \mathrm{drank}(x_i^{d-1}) = 1$$

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Hence

$$1 = \operatorname{drank}\left(\frac{\partial F}{\partial \varphi(x_i)}\right) = \operatorname{drank}\left(\nabla F \cdot \varphi(x_i)\right) = \operatorname{drank}\left(\sum_j \varphi_{ij} \cdot x_j^{d-1}\right)$$

Example: Fermat hypersurface

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and so

$$\varphi \text{ is a generalized permutation matrix}$$

Example: Fermat hypersurface

$$F = x_0^d + x_1^d + \ldots + x_{n+1}^d, \quad d \geq 3$$

Theorem

The automorphism group of the Fermat hypersurface $X = V(F)$ of dimension $n \geq 1$ and degree $d \geq 3$, with $(n, d) \neq (1, 3), (2, 4)$ is isomorphic to

$$\mathrm{Aut}(X) = (\mathbf{Z}/d\mathbf{Z})^{n+1} \rtimes S_{n+2}$$

Let $F \in S^d(V^*)$ be a homogeneous form of degree d.

(*i*) We define the distance between two monomials $\prod x_i^{a_i}$ and $\prod x_i^{b_i}$ as the sum $\sum_i |a_i - b_i|$.

(*ii*) We define the *sparsity* of F with respect to β, denoted by $\mathrm{spar}(F)$, as the minimum of the distance between the monomials of F after identifying $S(V^*) \simeq \mathbf{C}[x_0, \ldots, x_{n+1}]$ via the basis β.

(*iii*) We define the variables of F, denoted by $\mathrm{vars}(F)$, as the set of variables appearing in F, i.e., $\mathrm{vars}(F)$ is the smallest subset of β^* such that F is contained in $S^d(W)$ with W the span of $\mathrm{vars}(F)$.

Let $F \in S(V^*)$ be homogeneous of degree $d \geq 2$ respectively. We endow the set $\beta^* = \{x_0, \ldots, x_{n+1}\}$ with the relation $\leq_F$ given by

$$x_i \leq_F x_j \quad \Longleftrightarrow \quad \mathrm{vars}\left(\frac{dF}{dx_i}\right) \subseteq \mathrm{vars}\left(\frac{dF}{dx_j}\right)$$

Let $F \in S(V^*)$ be homogeneous of degree $d \geq 2$ respectively.
We endow the set $\beta^* = \{x_0, \ldots, x_{n+1}\}$ with the relation $\leq_F$ given by

$$x_i \leq_F x_j \quad \iff \quad \text{vars}\left(\frac{dF}{dx_i}\right) \subseteq \text{vars}\left(\frac{dF}{dx_j}\right)$$

The set β^* together with the relation $\leq_F$ may not be a poset.
Indeed, let $\beta^* = \{x_0, x_1, x_2, x_3\}$.

$$F = x_0^{d-1}x_1 + x_1^{d-1}x_0 + x_2^d + x_3^d.$$

Then $(\beta^*, \leq_F)$ is not a poset since $x_0 \leq_{(F,\beta)} x_1$ and $x_1 \leq_{(F,\beta)} x_0$.

$$F = x_0^d + x_1^d + x_2^d + x_3^d.$$

Then $(\beta^*, \leq_F)$ is the trivial poset

Theorem

Let $X = V(F) \subset \mathbb{P}(V)$ *be a smooth hypersurface of dimension* $n \geq 1$ *and degree* $d \geq 3$ *with* $(n, d) \neq (1, 3), (2, 4)$*. If* $\mathrm{spar}(F) > 4$ *and* $(\beta^*, \leq_F)$ *is a poset then* $\mathrm{Aut}(X)$ *is composed of generalized triangular matrices.*

Corollary

Let $X = V(F) \subset \mathbb{P}(V)$ *be a smooth hypersurface of dimension* $n \geq 1$ *and degree* $d \geq 3$ *with* $(n, d) \neq (1, 3), (2, 4)$*. If* $\mathrm{spar}(F) > 4$ *and* $(\beta^*, \leq_F)$ *is the trivial a poset then* $\mathrm{Aut}(X)$ *is composed of generalized permutation matrices.*

$$K = x_0^{d-1}x_1 + x_1^{d-1}x_2 + \ldots + x_{n+1}^{d-1}x_0$$

Corollary

The automorphism group of the Klein hypersurface $X = V(K)$ of dimension $n \geq 2$ and degree $d \geq 4$, with $(n, d) \neq (2, 4)$ is isomorphic to

$$\mathrm{Aut}(X) = (\mathbf{Z}/m\mathbf{Z}) \rtimes \mathbf{Z}/(n+2)\mathbf{Z}$$

where $m = \frac{(d-1)^{n+2} - (-1)^{n+2}}{d}$.

Proof.

$\mathrm{spar}(K) = 2d - 2 > 4$ y $\mathrm{vars}\left(\frac{dK}{dx_i}\right) = \{x_{i-1}, x_i, x_{i+1}\}$

$x_i \leq_K x_j \iff x_i = x_j$ and $(\beta^*, \leq_K)$ is a poset

$\varphi \in \mathrm{Aut}(X)$ is given by a generalized permutation matrix

$$K = x_0^{d-1}x_1 + x_1^{d-1}x_2 + \ldots + x_{n+1}^{d-1}x_0$$

Proof.

The only generalized permutation matrices inducing automorphisms of X have the shape of a multiple of

$$\varphi = \begin{pmatrix} 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \\ * & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Taking all coefficients equal to 1 we obtain a cyclic permutation of the variables. $P < \mathrm{Aut}(X)$ with $|P| = n+2$

Composing any φ with an element of P we obtain a diagonal matrix.

Proof.

Assume that φ is diagonal, and note that, up to changing the representative we can assume that $\varphi_{0,0}^{d-1}\varphi_{1,1} = 1$. Replacing in Klein's equation we get that $\varphi_{i,i}^{d-1}\varphi_{i+1,i+1} = 1$ for all $i = 0, \ldots, n+1$. Hence

$$\varphi_{0,0} = \varphi_{n+1,n+1}^{1-d} = \varphi_{n,n}^{(1-d)^2} = \cdots = \varphi_{1,1}^{(1-d)^{n+1}} = \varphi_{0,0}^{(1-d)^{n+2}}.$$

Thus φ is determined by the value of $\varphi_{0,0}$ which is a *dm*-root of unity.

Hence, we have an epimorphism $\mathbf{Z}/dm\mathbf{Z} \to D$ whose kernel is the d-roots of unity. Therefore D is a cyclic group of order m $\square$

¡Gracias!