Examples of non-rigid modular vector bundles on hyperkähler manifolds

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Definition

X (compact Kähler manifold) is a *hyperkähler* (HK) manifold if

- $\pi_1(X) = \{0\}$;
- $H^0(X, \Omega^2_X) = \mathbb{C}\sigma$, for σ non degenerate.

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Key result (*Beauville - Bogomolov* decomposition theorem): If X is a compact Kähler manifold with $c_1(X) = 0$, then (up to a étale cover), X decomposes as a product of HK, complex tori and Calabi-Yau manifolds.

Examples of compact HK

There aren't many examples available of (projective) HK in dimension higher than two:

1. (in dimension $2n$, by Beauville):Hilbert schemes of n points on a K3 surfaces $K3^{[n]}$, or generalized Kummer variety $K_n(A)$ constructed from an abelian surface and their deformations;
2. in dimension 6 and 10: two sporadic examples constructed by O'Grady.

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Standard technique to construct these examples: (desingularization of) moduli spaces of (stable) sheaves on K3/Abelian surfaces.
Idea: Construct examples of (possibly known) HK by looking at moduli spaces of sheaves (with extra properties) on higher dimensional HK.

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For example, a possible good notion asks for stable sheaves F for which $\mathbb{P}_X(F)$ extends to all deformations of X. (cf. Verbitsky's projectively hyperholomorphicity of vector bundles)

The notion of *modular sheaves* was recently introduced by O'Grady to expand the previous idea. Recall that the discriminant of a sheaf is defined as

$$\Delta(F) = c_2(End(F)) = -2r(F)\mathrm{ch}_2(F) + \mathrm{ch}_1^2(F).$$

Definition

Let X be a HK of dimension $2n$, and let q_X be the Beauville-Bogomolov-Fujiki form. F a torsion-free coherent sheaf on X is *modular* if there exists $d(F) \in \mathbb{Q}$ such that $\forall \alpha \in H^2(X)$,

$$\int_X \Delta(F) \cdot \alpha^{2n-2} = d(F)(2n-3)!!q_X(\alpha)^{n-1}$$

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If X is of $K3^{[2]}$-type (i.e. deformation equivalent to the Hilbert scheme of two points on a K3 surface), the modularity condition becomes simpler:

Remark

Let X be a HK of $K3^{[2]}$-type. Then F is modular if and only if

$$\Delta(F) = \alpha c_2(X)$$

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Let $Y \subset \mathbb{P}^5$ be a smooth cubic 4-fold. We consider $X = F_1(Y) \subset \mathrm{Gr}(2,6)$ its variety of lines. It is well known that $X \sim K3^{[2]}$ (Beauville-Donagi) - and in fact it gives a locally complete example.
On X we have two very natural vector bundles to consider: the restriction of the (rank 4) quotient bundle $\mathcal{E} := \mathcal{Q}|_X$ and also the restriction of the rank 2 tautological bundle $\mathcal{U}|_X$.

Example

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More in general, there is the following (series of) results by O'Grady:

Theorem (O'Grady)

Results of existence and uniqueness of modular vector bundles on $K3^{[n]}$ under certain numerical conditions. Moreover, if $X = F_1(Y)$,
$r(F) = 4,\ c_1(F) = h,\ \Delta(F) = c_2(X)$, *then* $F \cong \mathcal{E}$.

Rigidity and non-rigidity

Problem

$\mathcal{E}$ (and the others O'Grady's bundles) are rigid, i.e. $Ext^1(\mathcal{E},\mathcal{E}) = 0$, so not useful to construct positive-dimensional examples of HK.

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In order to find more (non rigid) examples, the notion of *atomic sheaves* has been introduced (cf Beckmann, Markman, Taelman), which is more restrictive than the modular one. There is a encouraging result by Bottini:

Theorem (Bottini)

Let X be $X \sim K3^{[2]}$. There exists a stable, atomic (hence modular) vector bundle F with $ext^1(F,F) = 10$ and
$\bigwedge^2 Ext^1(F,F) \cong Ext^2(F,F)$ (smooth deformation functor).
On a specific HK, the smooth locus of an irreducible component of the moduli space containing $[F]$ is birational (and conjecturally isomorphic) to OG10.

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Question

Can we produce "easy" examples of non rigid, modular vector bundles?

We want to modify O'Grady's bundles, using standard tools from representation theory. For example O'Grady's bundle $\mathcal{E} : \mathcal{Q}|_X$ on $X = F_1(Y) \subset \mathrm{Gr}(2,6)$ is the restriction of a homogeneous bundle on $\mathrm{Gr}(2,6)$.

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Standard construction from representation theory: every irreducible, homogeneous vector bundle on $\mathrm{Gr}(k,n)$ is of the form $\Sigma_\alpha \mathcal{U} \otimes \Sigma_\beta \mathcal{Q}$. The cohomology of these objects on $\mathrm{Gr}(k,n)$ is completely explicit via the famous Borel-Weil-Bott theorem.

Main idea

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Definition

For E a vector bundle on X, we denote with $End_0(E)$ the subbundle of the traceless endomorphisms, i.e.

$$E \otimes E^\vee \cong End_0(E) \oplus \mathcal{O}_X$$

We observe that if X is HK, then $H^i(End(E)) \supset H^i(\mathcal{O}_X) \cong \mathbb{C}$ for $i = 2p$.

We state our first theorem:

Theorem (-, 2023)

Let $X \subset \mathrm{Gr}(2,6)$ *be* $X = F_1(Y)$, $\mathcal{E}$ *as above. Set* $\mathcal{F} = \bigwedge^2 \mathcal{E}$. *Assume that* X *is generic in the moduli space. Then* $\mathcal{F}$ *is* μ*-stable, modular and not rigid. We have in fact:*

$$H^p(X, End_0(\mathcal{F})) = \begin{cases} \bigwedge^3 V_6^\vee & p = 1,3 \\ \mathbb{C} & p = 2 \\ 0 & otherwise \end{cases}$$

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$$H^p(X, End_0(\mathcal{F})) = \begin{cases} \bigwedge^3 V_6^\vee & p = 1,3 \\ \mathbb{C} & p = 2 \\ 0 & otherwise \end{cases}$$

As a corollary, we have

$$ext^p(X, \mathcal{F}) = \begin{cases} 20 & p = 1,3 \\ 2 & p = 2 \\ 1 & p = 0,4 \end{cases}$$

and also

- $H^0(X, \mathcal{F}) = \bigwedge^2 V_6$;
- $\mathrm{ch}(\mathcal{F}) = 6 + 3h + \frac{1}{4}(3h^2 - c_2(X)) - \frac{1}{20}hc_2(X) - \frac{c_4(X)}{h^4}$

There are several things to prove before making the statement above a theorem. First of all, the modularity. We proved it using an auxiliary result.

Propostion

Let A be a torsion-free coherent sheaf of rank r on a smooth projective variety. Then

$$\Delta(\bigwedge^{p} A) = \lambda_p \Delta(A),$$

with $\lambda_p = \frac{1}{p-1}\binom{r-1}{p}\binom{r-2}{p-2}$

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For the stability, since $\mathcal{E}$ is stable for X generic, $\mathcal{F} = \bigwedge^2 \mathcal{E}$ is polystable. But $\mathcal{F}$ is also simple, then for X generic $\mathcal{F}$ is stable.

The cohomological computations of the proof are done using a combination of standard tools in algebraic geometry and representation theory. First we decompose in irreducibles

$$\mathcal{F} \otimes \mathcal{F}^\vee \cong \Sigma_{2,2}\mathcal{Q}|_X(-1) \oplus \mathcal{O}_X \oplus \Sigma_{2,1,1}\mathcal{Q}|_X(-1)$$

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In order to compute the cohomology of the factors above, we need to use the Koszul complex. We know that $X \subset \mathrm{Gr}(2,6)$ is $X = V(\sigma)$, with $\sigma \in H^0(\mathrm{Gr}(2,6), \mathrm{Sym}^3\mathcal{U}^\vee)$ a general global section. Therefore $\mathcal{O}_X$ is resolved by

$$0 \to \mathcal{O}_{\mathrm{Gr}(2,6)}(-6) \to \bigwedge^3 \mathrm{Sym}^3\mathcal{U} \to \bigwedge^2 \mathrm{Sym}^3\mathcal{U} \to \mathrm{Sym}^3\mathcal{U} \to \mathcal{O}_X \to 0$$

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One has then to tensor the above complex with the factor involved in the endomorphism bundle, decompose in irreducibles, use the Borel-Weil-Bott to compute the cohomology of every single factor, and finally use all these data together to compute the necessary cohomologies.

A couple of remarks

A first remark is about the obstruction map:

Remark

We can consider the obstruction map given by the (symmetrized) Yoneda pairing:

$$H^1(X, End_0(\mathcal{F})) \times H^1(X, End_0(\mathcal{F})) \to H^2(X, End_0(\mathcal{F})) \cong \mathbb{C}$$

If is zero, then there is a single component of the moduli space of stable sheaves on X containing $\mathcal{F}$, it has dimension 20 and it is smooth at $\mathcal{F}$. If the latter holds, then the component in question has a regular 2-form which is symplectic in a neighborhood of $\mathcal{F}$.

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A second remark is about what happens if we started from the Debarre-Voisin HK:

Remark

On $Z \subset \mathrm{Gr}(6, 10)$, $\mathcal{Q}|_Z$ is also modular, stable and rigid. We can consider $\bigwedge^2 \mathcal{Q}|_Z$, and has the same invariants and ext-table. This should not be a coincidence, since $\mathbb{P}_X(\mathcal{Q}|_X)$ deforms to $\mathbb{P}_Z(\mathcal{Q}|_Z)$, and the same for their $\bigwedge^2$.

What if we iterate the process? In fact, we can consider

$$\mathcal{K} := \bigwedge^2 \mathcal{F} \cong \bigwedge^2 \bigwedge^2 \mathcal{Q}|_X$$

which happens to be still irreducible (in fact, $\mathcal{K} = \Sigma_{2,1,1}\mathcal{Q}|_X$).

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which happens to be still irreducible (in fact, $\mathcal{K} = \Sigma_{2,1,1}\mathcal{Q}|_X$). $\mathcal{K}$ now has $r(\mathcal{K}) = deg(\mathcal{K}) = 15$, and the same argument for modularity and stability holds. More interestingly,

Theorem (-, 2023)

Let $X \subset \mathrm{Gr}(2,6)$ be $X = F_1(Y)$, $\mathcal{K}$ as above. Assume that X is generic in the moduli space. Then $\mathcal{K}$ is μ-stable, modular and not rigid. We have in fact:

$$h^p(X, End_0(\mathcal{K})) = \begin{cases} 20 & p = 1,3 \\ 190 & p = 2 \\ 0 & otherwise \end{cases}$$

and we also have

$$\Sigma_{2,2,1,1} V_6^\vee \oplus \mathbb{C} \cong H^2(X, End_0(\mathcal{K})) \cong \bigwedge^2 H^1(X, End_0(\mathcal{K}))$$

A few observations.

1. The situation of the last theorem is analogous to Bottini's result, with the smoothness of the deformation functor;
2. We could go on and produce more and more non-rigid examples! Also, we could try to understand (combinatorially) which Schur functors give non-rigid examples.
3. A work-in-progress by O'Grady is attempting to insert these examples in a more general context. In particular, these moduli spaces could be birational to $K3^{[10]}$.

Thanks for the attention!