Instituto Federal de Educação Ciência e Tecnologia do Ceará

On the Moser's Bernstein Theorem

Algebraic Geometry, Lipschitz Geometry and Singularities in Pipa

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11/12/2023

Summary

Introduction and motivation
- Bernstein Theorem

Definition and basic results
- Cone tangente
- Spherical blow-up

Kurdyka-Raby's formula at infinity

Parametric versions of the Bernstein Theorem
- Caracterization minimal and definable sets
- Theorem type Moser outside compact set

Bernstein's problem

Goal

The objective of this work is to study the geometric behavior of the solutions of the equation:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{\|\nabla u\|^2+1}}\right)=0.$$

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Bernstein's Problem

If the graph of a function on $\mathbb{R}^n$ is a minimal hypersurface in $\mathbb{R}^{n+1}$, does this imply that the function is linear?

History

1. Bernstein (1915 – 1917): proved Bernstein's theorem that a graph of a real function on $\mathbb{R}^2$ that is also a minimal surface in $\mathbb{R}^3$ must be a plane ($\dim M = 2$);
2. Moser (1961): proved Bernstein's theorem that a graph of a real Lipschitz function on $\mathbb{R}^n$ that is also a minimal surface in $\mathbb{R}^{n+1}$ must be a hyperplane (any dimension $\dim M$)
3. Fleming (1962) gave a new proof of Bernstein's theorem by deducing it from the fact that there is no non-planar area-minimizing cone in $\mathbb{R}^3$;
4. De Giorgi (1965) showed that if there is no non-planar area-minimizing cone in $\mathbb{R}^{n-1}$ then the analogue of Bernstein's theorem is true for graphs in $\mathbb{R}^n$, which in particular implies that it is true in $\mathbb{R}^4$ ($\dim M = 3$).
5. Almgren (1966) showed there are no non-planar minimizing cones in $\mathbb{R}^4$, thus extending Bernstein's theorem to $\mathbb{R}^5$($\dim M = 4$).

History-Higher codimension

1. Simons (1968) showed there are no non-planar minimizing cones in $\mathbb{R}^7$, thus extending Bernstein's theorem to $\mathbb{R}^8$. He also showed that the surface defined by

$$\left\{x \in \mathbb{R}^8 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\right\},$$

   is a locally stable cone in $\mathbb{R}^8$, and asked if it is globally area-minimizing ($\dim M \leq 7$);
2. Bombieri, De Giorgi and Giusti (1969) showed that Simons' cone is indeed globally minimizing, and that in $\mathbb{R}^n$ for $n \geq 9$ there are graphs that are minimal, but not hyperplanes. Combined with the result of Simons, this shows that the analogue of Bernstein's theorem is true in $\mathbb{R}^n$ for $n \leq 8$, and false in higher dimensions.

History-Higher dimension and codimension

1. Lawson (1977) present the graph of the Lipschitz mapping $f\colon \mathbb{R}^4 \to \mathbb{R}^3$ given by $f(0) = 0$ and

$$f(x) = \frac{\sqrt{5}}{2}\|x\|\eta\left(\frac{x}{\|x\|}\right), \quad \forall x \neq 0,$$

   is a minimal cone, where $\eta\colon \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf mapping given by

$$\eta(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1\bar{z}_2).$$

2. Fernandes and Sampaio (2020) showed that if $X \subset \mathbb{C}^n$ be a pure dimensional complex algebraic subset. If X is Lipschitz regular at infinity, then X is an affine linear subspace of $\mathbb{C}^n$.

History-Higher dimension and codimension

1. Sampaio (2022) showed that $X \subset \mathbb{C}^n$ be a pure d-dimensional entire complex analytic set. If X is Lipschitz regular at infinity, then X is an affine linear subspace of $\mathbb{C}^n$.
2. Sampaio and Silva (2023) showed that if $X \subset \mathbb{C}^n$ be a pure dimensional complex algebraic set. If X is blow-spherical regular at infinity, then X is an affine linear subspace of $\mathbb{C}^n$.

Principal curvatures

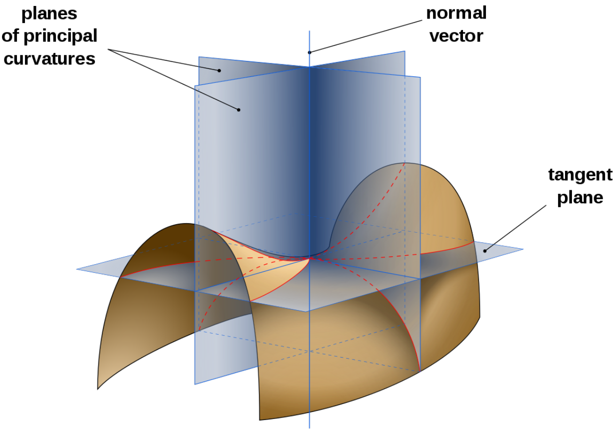


Figure: Principal curvatures

Minimal hypersurface

Mean curvature

We define the mean curvature of $X \subset \mathbb{R}^{n+1}$ by:

$$H = \frac{1}{n}\sum_{j=1}^{n} k_j,$$

where $k_j's$ are principal curvature of X. We say that X is a minimal hypersurface if $H = 0$.

Minimal surface equation

Remind that if $X \subset \mathbb{R}^{n+1}$ is a smooth hypersurface and whenever X is locally expressed as the graph of a smooth function $u\colon B^n_\varepsilon(p) \to \mathbb{R}$, then u is a solution of the following PDE

$$\operatorname{div}\left((1 + |\nabla u|^2)^{-\frac{1}{2}} \nabla u\right) = nH, \tag{1}$$

we say that X is a **minimal hypersurface** and when $H = 0$.

Examples minimal surfaces

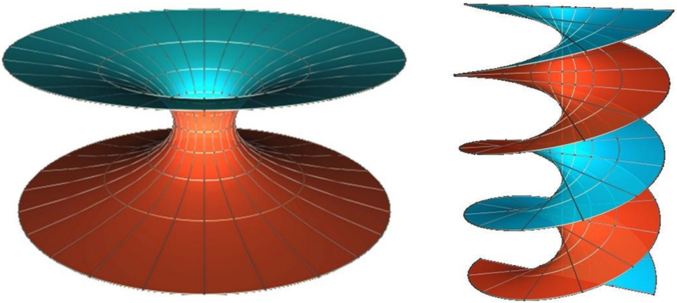


Figure: Catenóide e Helicóide

Tangent cone at infinity

Definition

Let $X \subset \mathbb{R}^{n+1}$ be an unbounded definable set (resp. $p \in \overline{X}$). We say that $v \in \mathbb{R}^{n+1}$ is a tangent vector of X at infinity (resp. p) if there are a sequence of points $\{x_i\} \subset X$ tending to infinity (resp. p) and a sequence of positive real numbers $\{t_i\}$ such that

$$\lim_{i \to \infty} \frac{1}{t_i} x_i = v \quad (\text{resp. } \lim_{i \to \infty} \frac{1}{t_i}(x_i - p) = v).$$

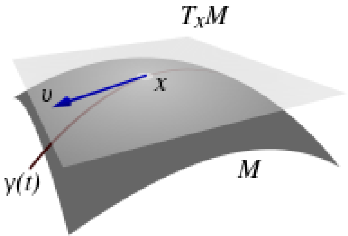
Let $C(X, \infty)$ (resp. $C(X, p)$) denote the set of all tangent vectors of X at infinity (resp. p). We call $C(X, \infty)$ the **tangent cone of X at infinity** (resp. p).

Example tangent cone

Now let's present some examples of tangent cones. Consider M a smooth submanifold in Euclidean space, then its tangent cone at point x is equal to the tangent space at x.

Example

Let M is a $\mathcal{C}^k$ differentiable submanifold in Euclidian space and $x \in M$, then $C(M, x) = T_x M$.

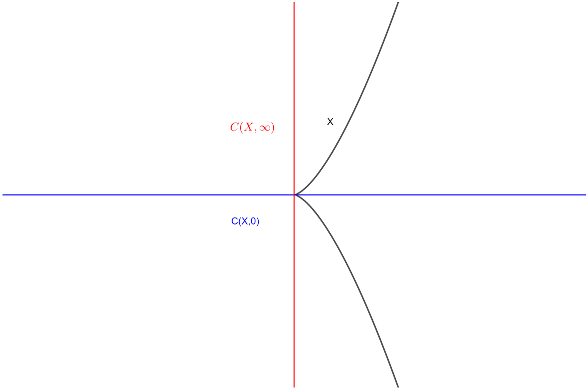


Cusp

Another important example is the complex cusp. Its tangent cone at the origin is the blue line and tangent cone at infinity is tha red line.

Example

Let $X = \{(x, y) \in \mathbb{C}^2; x^3 = y^2\}$



Spherical blow-up

Another way to present the tangent cone at infinity (resp. p) of a subset $X \subset \mathbb{R}^{n+1}$ is via the spherical blow-up at infinity (resp. p) of $\mathbb{R}^{n+1}$.

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Definition

Let us consider the **spherical blowing-up at infinity** (resp. p) of $\mathbb{R}^{n+1}$, $\rho_\infty : \mathbb{S}^n \times (0, +\infty) \to \mathbb{R}^{n+1}$ (resp. $\rho_p : \mathbb{S}^n \times [0, +\infty) \to \mathbb{R}^{n+1}$), given by $\rho_\infty(x, r) = \frac{1}{r}x$ (resp. $\rho_p(x, r) = rx + p$).
Note that $\rho_\infty : \mathbb{S}^n \times (0, +\infty) \to \mathbb{R}^{n+1} \setminus \{0\}$ (resp. $\rho_p : \mathbb{S}^n \times (0, +\infty) \to \mathbb{R}^{n+1} \setminus \{0\}$) is a homeomorphism with inverse mapping $\rho_\infty^{-1} : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{S}^n \times (0, +\infty)$ (resp. $\rho_p : \mathbb{S}^n \times (0, +\infty) \to \mathbb{R}^{n+1} \setminus \{0\}$) given by $\rho_\infty^{-1}(x) = (\frac{x}{\|x\|}, \frac{1}{\|x\|})$ (resp. $\rho_p^{-1}(x) = (\frac{x-p}{\|x-p\|}, \|x - p\|)$).

Strict transform

Strict transform

Definition

The **strict transform** of the subset X under the spherical blowing-up ρ_∞ is $X'_\infty := \overline{\rho_\infty^{-1}(X \setminus \{0\})}$ (resp. $X'_p := \overline{\rho_p^{-1}(X \setminus \{0\})}$). The subset $X'_\infty \cap (\mathbb{S}^n \times \{0\})$ (resp. $X'_p \cap (\mathbb{S}^n \times \{0\})$) is called the **boundary** of X'_∞ (resp. X'_p) and it is denoted by $\partial X'_\infty$ (resp. $\partial X'_p$).

Remark

If $X \subset \mathbb{R}^{n+1}$ is a semialgebraic set, then $\partial X'_\infty = (C(X,\infty) \cap \mathbb{S}^n) \times \{0\}$ (resp. $\partial X'_p = (C(X,p) \cap \mathbb{S}^n) \times \{0\}$).

Simple point

Simple point

Definition

Let $X \subset \mathbb{R}^{m+1}$ be a d-dimensional subanalytic subset and $p \in \mathbb{R}^{m+1} \cup \{\infty\}$. We say $x \in \partial X'_p$ is **simple point of $\partial X'_p$**, if there is an open subset $U \subset \mathbb{R}^{m+2}$ with $x \in U$ such that:

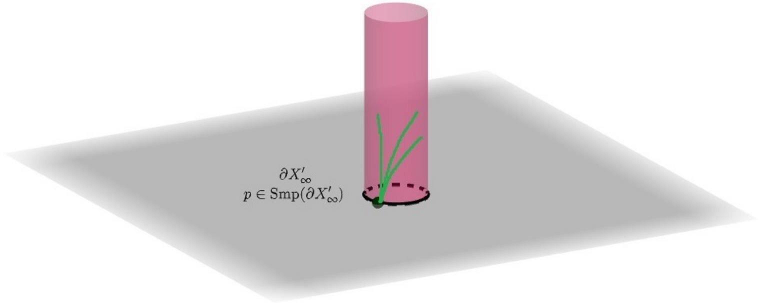
a) the connected components $X_1, \cdots, X_r$ of $(X'_p \cap U) \setminus \partial X'_p$ are topological submanifolds of $\mathbb{R}^{m+2}$ with $\dim X_i = \dim X$, for all $i = 1, \cdots, r$;

b) $(X_i \cup \partial X'_p) \cap U$ are topological manifolds with boundary, for all $i = 1, \cdots, r$.

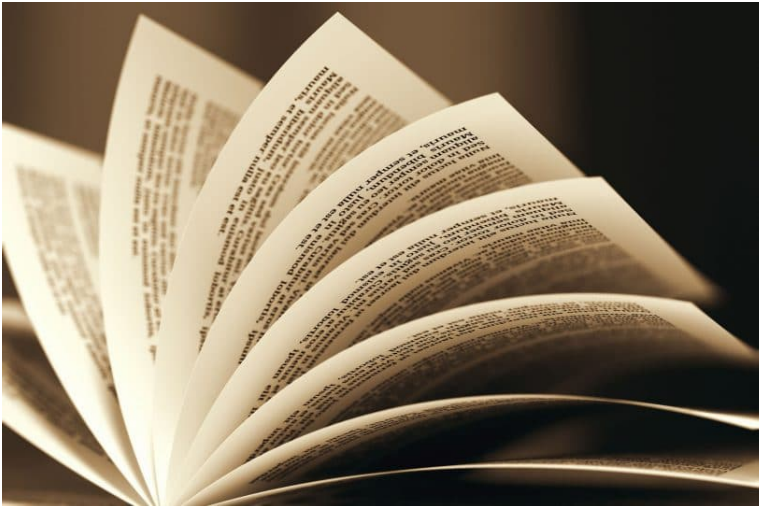
Let $\mathrm{Smp}(\partial X'_p)$ be the set of simple points of $\partial X'_p$ and we define $C_{\mathrm{Smp}}(X, p) = \{t \cdot x;\ t > 0 \text{ and } x \in \mathrm{Smp}(\partial X'_p)\}$. Let $k_{X,p} \colon \mathrm{Smp}(\partial X'_p) \to \mathbb{N}$ be the function such that $k_{X,p}(x)$ is the number of connected components of the germ $(\rho_p^{-1}(X \setminus \{p\}), x)$.

Blow-spherical homeomorphism at p

It is clear the function $k_{X,p}$ is locally constant. In fact, $k_{X,p}$ is constant on each connected component X_j of $\mathrm{Smp}(\partial X'_p)$. Then, we define **the relative multiplicity of X at p (along of X_j)** to be $k_{X,p}(X_j) := k_{X,p}(x)$ with $x \in X_j$. Let $X_1, ..., X_r$ be the connected components of $\mathrm{Smp}(\partial X'_p)$. By reordering indices, if necessary, we assume that $k_{X,p}(X_1) \leq \cdots \leq k_{X,p}(X_r)$. Then we define $k(X,p) = (k_{X,p}(X_1), ..., k_{X,p}(X_r))$.



Relative multiplicity



Blow-spherical sets equivalence

Definition

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be closed sets. Let $p \in \mathbb{R}^n \cup \{\infty\}$, $q \in \mathbb{R}^m \cup \{\infty\}$ and a homeomorphism $\varphi : X \to Y$ such that $q = \lim\limits_{x \to p} \varphi'(x)$, is said to be **blow-spherical homeomorphism at** p, if the homeomorphism

$$\varphi' : X'_x \setminus \partial X'_x \to Y'_y \setminus \partial Y'_y$$

estends to a homeomorphism $\varphi' : X'_x \to Y'_y$.

Blow-spherical sets equivalence

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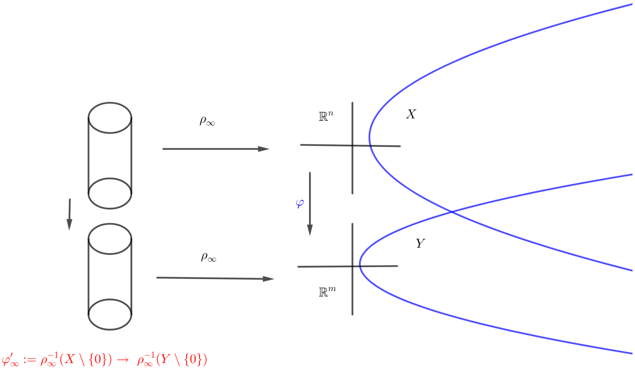
Remark

A subset $X \subset \mathbb{R}^n$ is called blow-spherical regular at infinity if there are compact subsets K and $\widetilde{K}$ in X and $\mathbb{R}^d$ respectively such that $X \setminus K$ is a blow-spherical homeomorphic at ∞ to an $\mathbb{R}^d \setminus \widetilde{K}$.

Blow-spherical sets equivalence



Density at infinity

Definition

Let $X \subset \mathbb{R}^n$ be a set in an o-minimal structure $\mathcal{S}$ of dimensional k. We say that X has a **density at infinity** and we denote it by $\theta(X)$, when the limit exists:

$$\theta^k(X) := \lim_{r \to +\infty} \frac{\mathcal{H}^k(X \cap B_r^n)}{\mu_k r^k},$$

where μ_k is the volume of the k-dimensional Euclidean unit ball, $\mathcal{H}^k(A)$ is the Hausdorff measure of A and $B_r^m(p) \subset \mathbb{R}^m$ is the open Euclidean ball center at p of radius $r > 0$ and by simplicity we denote $B_r^m := B_r^m(0)$.

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Remark

If X has the density at infinity defined, then it does not depend on the base point p

Kurdyka-Raby's formula at infinity

Let $X \subset \mathbb{R}^n$ be a definable set in an o-minimal structure $\mathcal{S}$ of dimensional k. Fixed $p \in X$, we define the function $\theta^k(X,p,\cdot)\colon (0,+\infty) \to X$ by the following:

$$\theta(X,p,r) = \frac{\mathcal{H}^k(X \cap B_r^n(p))}{\mathcal{H}^k(B_r(p))}.$$

Theorem (Sampaio, —— (2023))

Let $X \subset \mathbb{R}^n$ be a definable set in an o-minimal structure and $d = \dim_H X$. Let $C_1, ..., C_m$ be the connected components of $Smp(\partial X'_\infty)$. Then, for each $p \in \mathbb{R}^n$, we have

$$\theta^d(X) = \lim_{r\to+\infty} \frac{\mathcal{H}^d(X \cap B_r^n(p))}{\mu_d r^d} = \sum_{j=1}^{m} k_{X,\infty}(C_j)\cdot \mathcal{H}^{d-1}(C_j).$$

LNE sets

Let us remind the definition of inner distance. Given a path connected subset $X \subset \mathbb{R}^m$ the *inner distance* on X is defined as follows: given two points $x_1, x_2 \in X$, $d_X(x_1, x_2)$ is the infimum of the lengths of paths on X connecting x_1 to x_2.

Definition (Birbrair and Mostowski: 2000)

A subset $X \subset \mathbb{R}^n$ is called **Lipschitz normally embedded** (or shortly **LNE**) if there exists $\lambda > 0$ such that

$$d_X(x_1, x_2) \leq \lambda \|x_1 - x_2\|$$

for all $x_1, x_2 \in X$.

LNE sets at infinity

Definition (Fernandes and Sampaio: 2020)

A subset $X \subset \mathbb{R}^n$ is **Lipschitz normally embedded at infinity** (or shortly **LNE at infinity**) if there exists a compact subset $K \subset \mathbb{R}^n$ such that $X \setminus K$ is Lipschitz normally embedded.

Caracterization minimal and definable sets

Theorem (Sampaio, ——-, 2023)

Let $X \subset \mathbb{R}^{n+1}$ be a closed and connected set and $d = \dim_H X$. Assume that X is a minimal submanifold or is an area-minimizing set. Then the following statements are equivalent:

(1) *X is an affine linear subspace;*

(2) *X is a definable set that is Lipschitz regular at infinity and $C(X, \infty)$ is a linear subspace;*

(3) *X is a definable set, blow-spherical regular at infinity and $C(X, \infty)$ is a linear subspace;*

(4) *X is an LNE at infinity definable set and $C(X, \infty)$ is a linear subspace;*

(5) $\theta^d(X) = 1$.

Corollary

Let $X \subset \mathbb{R}^{n+1}$ be a complete area-minimizing hypersurface with $2 \leq n \leq 6$. Suppose that X is a definable set and is Lipschitz regular at infinity. Then X is an affine linear subspace.

Theorem type Moser

Theorem (Sampaio, ——- : 2023)

Let $X \subset \mathbb{R}^{n+1}$ be a complete minimal hypersurface with $n \geq 2$. Suppose that there are compact sets $K \subset \mathbb{R}^n$ and $\tilde{K} \subset \mathbb{R}^{n+1}$ such that $X \setminus \tilde{K}$ is the graph of a Lipschitz function $u \colon \mathbb{R}^n \setminus K \to \mathbb{R}$. Then u is the restriction of an affine function and, in particular, X is a hyperplane.

Sketch of the proof

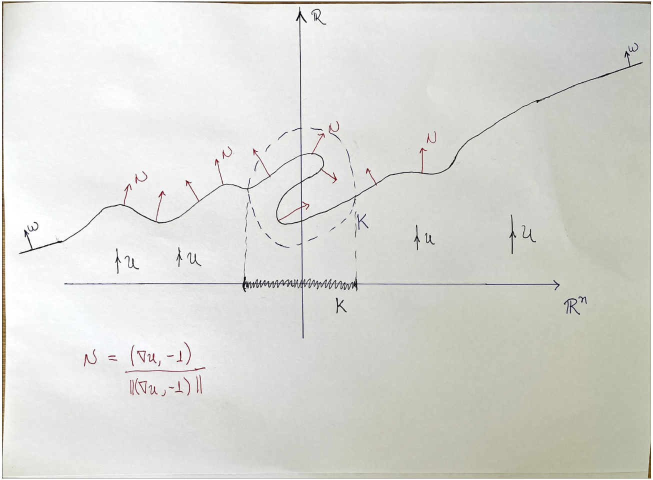
Since u is a C^2 solution of the equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{\|\nabla\|^2+1}}\right)=0,$$

on $\mathbb{R}^n \setminus K$ and u is a Lipschitz function, we have by Simon and Bers Theorem X that that limit exists

$$\lim_{\|x\|\to+\infty}(\nabla u(x),-1)=\omega.$$

Sketch of the proof



Sketch of the proof

Now, we define $F(x, y) = z - u(x)$. Moreover, $\nabla F(x, y) = (-\nabla u(x), 1)$. Thus, $\frac{\nabla F}{\|\nabla F\|} \to \omega$. Now, we choose linear coordinates $(y_1, \cdots, y_{n+1})$ of $\mathbb{R}^{n+1}$ such that P be the hyperplane $\omega^{\perp}$. Thus we have for a larger enough $R > 0$, such that $X \setminus B_R^n \times R$ is the graph of a function $v\colon P \setminus B_R^n \to$. By Bers and Simon Theorem we have

$$\lim_{\|y\| \to +\infty} \nabla v(y) = \widetilde{\omega}.$$

On the other hand, we have $N = \frac{(-\nabla v(y), 1)}{\|(-\nabla v(y), 1)\|}$. Therefore, $\widetilde{\omega} = 0$. Thus, $\|\nabla v(y)\| \leq \epsilon$ for all $y \in P \setminus \overline{B_R^n}$.

Sketch of the proof

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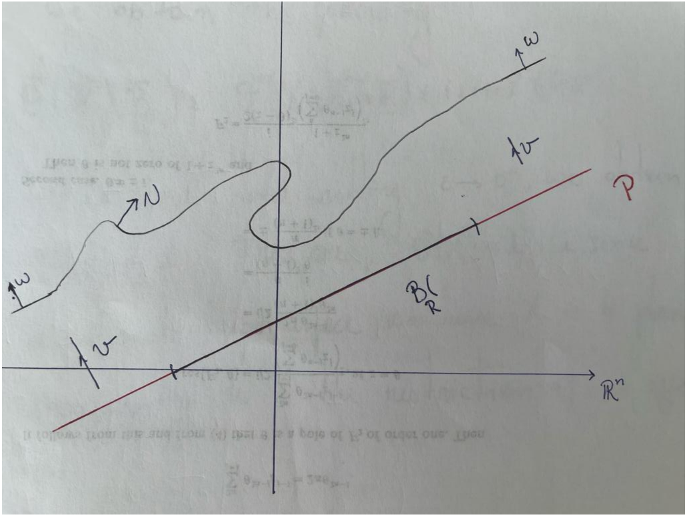

Sketch of the proof

Now, since $P \setminus \overline{B_R^n}$ is a LNE set, we have

$$\|v(x) - v(y)\| \leq \epsilon d_{in}^{P\setminus \overline{B_R^n}}(x, y) \leq \pi\epsilon\|x - y\|.$$

Finally, let $\varphi\colon P \setminus \overline{B_R^n} \to X \setminus \overline{B_R^n} \times \mathbb{R}$ be the mapping given $\varphi(x) = (x, v(x))$. Thus, we have φ is a bi-Lipschitz mapping such that

$$\|x - y\| \leq \|\varphi(x) - \varphi(y)\| \leq (1 + \pi\epsilon)\|x - y\|.$$

Therefore, the density,

$$\theta^n(\mathbb{R}^n \setminus \overline{B_r^n}) \leq \theta^n(X \setminus \overline{B_r^{n+1}}) \leq (1 + \epsilon\pi)^n\theta^n(\mathbb{R}^n).$$

Therefore, $\theta^n(X) = 1$.Consequently, X is a hyperplane.

Non-complete minimal surface

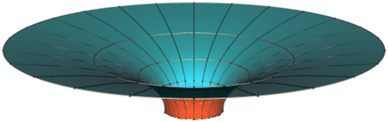

Figure: Catenoid non-complete

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Thank you for your attention