Rigidity of Brieskorn-Pham affine hypersurfaces

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Affine Brieskorn-Pham hypersurfaces

Definition: Affine Brieskorn-Pham hypersurfaces

A hypersurface $X_{a_0,\ldots,a_n} \subset \mathbb{A}^{n+1} = 0$ defined by an equation of the form

$$X_0^{a_0} + \cdots X_n^{a_n} = 0$$

for an $(n+1)$-tuple of positive integers a_i.

The Rigidity Conjecture

For $n \geq 2$, $X_{a_0,\ldots a_n}$ admits a non-trivial action of the additive group $\mathbb{G}_a$ if and only if:

1. Either $a_i = 1$ for some i,
2. Or at least two distinct a_i's are equal to 2.

Additive group actions and cylinders

Notion of cylinder

1. A *cylinder* in an algebraic variety X is a Zariski open subset U of X isomorphic to $Z \times \mathbb{A}^1$ for some *affine* variety Z.
2. A *polar cylinder* in a polarized variety (X, H) -H an ample Weil $\mathbb{Q}$-divisor on X- is a cylinder U in X such that $D = X \smallsetminus U$ is the support of an effective Weil $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to H.

Affine varieties with $\mathbb{G}_a$-actions have large automorphism groups

1. An affine $\mathbb{G}_a$-variety contains an invariant principal cylinder.
2. Consequence: if $\dim X \geq 2$ then $\mathrm{Aut}(X)$ is an infinite dimensional group: it contains $\mathbb{G}_a^\infty = \mathrm{colim}_n \mathbb{G}_a^n$.

Rigidity for Brieskorn-Pham surfaces

Classical numerology and "easy" exclusion results

For $X = X_{a_0,a_1,a_2} = \{X_0^{a_0} + X_1^{a_1} + X_2^{a_2} = 0\}$, we have the following dichotomy:

1. $1/a_0 + 1/a_1 + 1/a_2 \leq 1 \Rightarrow \bar{\kappa}(X \setminus \{0\}) \geq 0 \Rightarrow X$ does not contain a cylinder.
2. $1/a_0 + 1/a_1 + 1/a_2 > 1$ and then (a_0, a_1, a_2) is one of the *Platonic triplets* $(2,2,m)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$. In this case $\bar{\kappa}(X \setminus \{0\}) = -\infty$: requires an additional algebraic/geometric study!

Theorem (Kaliman-Zaidenberg 2000)

The rigidity conjecture holds for Brieskorn-Pham surfaces.

Well-formed hypersurfaces

Definition

An affine Brieskorn-Pham hypersurface $X = X_{a_0,\ldots,a_n} \subset \mathbb{A}^{n+1}$ is called *well-formed* if the corresponding quasi-smooth weighted hypersurface

$$\hat{X} = \{X_0^{a_0} + \cdots X_n^{a_n} = 0\} \subset \mathbb{P} := \mathbb{P}(w_0, \ldots, w_n), \quad w_i = \mathrm{lcm}_{i=0,\ldots,n}(a_i)/a_i$$

is well-formed:

1. $\gcd(w_0, \ldots, \hat{w}_i, \ldots w_n) = 1, \ \forall i = 0, \ldots, n$
2. $\mathrm{codim}_{\hat{X}}(\hat{X} \cap \mathrm{Sing}(\mathbb{P})) \geq 2.$

Proposition

For $n \geq 2$, $X = X_{a_0,\ldots,a_n} \subset \mathbb{A}^{n+1}$ is well-formed if and only if for every $i = 0, \ldots, n$, a_i divides $\mathrm{lcm}(a_0, \ldots, \hat{a}_i, \ldots, a_n)$.

Reduction of the Conjecture

Chitayat Reduction Theorem

To prove the Rigidity Conjecture in any dimension $n \geq 3$ it suffices to prove it for well-formed Brieskorn-Pham hypersurfaces.

Consequence for Brieskorn-Pham threefolds

For well-formed affine Brieskorn-Pham threefolds $X = X_{a_0,\dots,a_3}$, we have the following dichotomy:

1. $\sum 1/a_i \leq 1$: $K_{\hat{X}}$ is either ample or trivial.
2. $\sum 1/a_i > 1$ leads to a finite number of cases: $(2,3,3,6)$, $(2,3,6,6)$, $(2,4,4,4)$, $(3,3,3,3)$, $(3,3,4,4)$, $(3,3,5,5)$, $(2,3,4,12)$, $(2,3,5,30)$. In each of these cases, $\hat{X}$ is a del Pezzo surface with (at worse) cyclic quotient singularities.

Reduction to the existence of polar cylinders

Kishimoto-Prokhorov-Zaidenberg correspondance

For well-formed affine Brieskorn-Pham threefolds $X = X_{a_0,\dots,a_3}$, the following are equivalent:

1. X admits a non-trivial $\mathbb{G}_a$-action.
2. $\hat{X}$ admits a $\mathcal{O}_{\hat{X}}(1)$-polar cylinder.

Consequence:

1. If $\sum 1/a_i \leq 1$ then $X_{a_0,\dots,a_3}$ does not admit any nontrivial $\mathbb{G}_a$-action.
2. If $\sum 1/a_i > 1$ then $X = X_{a_0,\dots,a_3}$ admits a nontrivial $\mathbb{G}_a$-action if and only if $\hat{X}$ admits a $-K_{\hat{X}}$-polar cylinder.

Exclusion of anti-canonical polar cylinders in del Pezzo surfaces

Theorem (Cheltsov-Park-Won -2016)

1. A smooth del Pezzo surface of degree d admits an anticanonical polar cylinder if and only if $d \geq 4$.
2. Complete classification of del Pezzo surfaces with at most du Val singularities admiting anticanonical polar cylinders.

Consequences

1. $(3,3,3,3)$: Fermat cubic surface in $\mathbb{P}^3$, no cylinder.
2. $(2,4,4,4)$: Smooth del Pezzo surface of degree 2 in $\mathbb{P}(2,1,1,1)$, no cylinder.
3. $(2,3,6,6)$: Smooth del Pezzo surface of degree 1 in $\mathbb{P}(3,2,1,1)$, no cylinder.
4. $(2,3,3,6)$: del Pezzo surface of degre 3 in $\mathbb{P}(3,2,2,1)$ with three A_1-singularities, no cylinder.

Remaining cases - Take 1

Cylinders and non log-canonicity (Cheltov-Park)

Let S be a del Pezzo surface with at worse log-canonical singularities and let $D \sim_{\mathbb{Q}} -K_S$ be an effective Weil $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $S \setminus \mathrm{Supp}(D)$ is a cylinder. Then the log pair (S, D) is not log-canonical.

Consequence:

A del Pezzo surface S with quotient singularities and α-invariant larger than or equal to 1 does not contain any $-K_S$-polar cylinder.

1. $(3,3,4,4)$: $\alpha = 1$ (Cheltsov-Park-Shramov 2010), no cylinder.
2. $(3,3,5,5)$: $\alpha = 2$ (Cheltsov-Park-Shramov 2010), no cylinder.

Remaining cases - Take 2

The last two cases

1. $(2, 3, 5, 30)$: a del Pezzo surface of degree $2/15$ in $\mathbb{P}(15, 10, 6, 1)$ with three cyclic quotient singularities of type $1/5(1, 1)$, $1/3(1, 1)$ and $1/2(1, 1)$ supported on a line.
2. $(2, 3, 4, 12)$: a del Pezzo surface of degree $2/3$ in $\mathbb{P}(6, 4, 3, 1)$ with three cyclic quotient singularities of type $1/2(1, 1)$, $1/3(1, 1)$ and $1/3(1, 1)$ supported on a line.

Proposition and Theorem

The two surfaces above do not contain any anticanonical polar cylinder. Consequently, the Rigidity Conjecture is solved in dimension 3: *an affine Brieskorn-Pham threefold $X_{a_0,\ldots,a_3}$ admits a nontrivial $\mathbb{G}_a$-action if and only if either one of the a_i's equals 1 or at least two the a_i's are equal to 2.*