Kurdyka-Łojasiewicz functions and mapping cylinder neighborhoods

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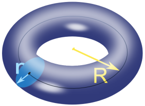
Joint work with Fernando Galaz-Garcia (Univ. of Durham).

Some geometric motivation

- $Z \subset \mathbb{R}^n$ oriented compact submanifold $\dim(M) = d$
- Z has a normal bundle νZ
- $r > 0$ small, $U_r(Z) := \exp(B_r(\nu Z))$ is a tube around Z



Theorem (Weyl tube formula -1939)

$$\mathrm{vol}(U_r(Z)) = \sum_{k\geq 0} C_{k,n,d} \cdot \mu_{2k}(Z) r^{n-d+2k}$$

- $\mu_k(Z) = \int_M R_k(Z)\, d\,\mathrm{vol}_g$
- $R_k(Z) = \mathrm{tr}(R_g^k) \qquad R_g : \Lambda^2 TZ \to \Lambda^2 TZ$
- $k = 1 \Rightarrow R_2 = s_g$,
- $k = d$, d even $\Rightarrow R_k(Z) \cdot \mathrm{vol}_g = \mathrm{Pf}(Z)$.

Some geometric motivation

- Z compact, convex domain: tube formula " = " old Steiner formula.
- Federer (1959) unifies tube formulas via sets of positive reach.

  $X \subset \mathbb{R}^n$ is of positive reach if $\exists \epsilon > 0$ such that

  $$\forall p \in X_\epsilon := \{x \in \mathbb{R}^n \mid d(x, X) < \epsilon\}$$

  $\exists$ a unique projection of p to X.

Some geometric motivation

- Zähle (1986): for manifolds $\mu_k(Z) = \int_{S_1(\nu Z)} \tau_k$
- τ_k are universal forms of degree $n-1$ on $\mathbb{R}^n \times S^{n-1} \supset S_1(\nu Z)$
- For sets of positive reach, Zähle replaced $S_1(\nu Z)$ by a Lipschitz submanifold $N(Z)$ of $\mathbb{R}^n \times S^{n-1}$
- The normal cycle $N(Z)$ defines a closed current in $\mathbb{R}^n \times S^{n-1}$
- Upshot: $\mu_k(Z) = \langle N(Z), \tau_k \rangle$

Some geometric motivation

- For what kind of sets $Z \subset \mathbb{R}^n$ can one define a normal cycle $N(Z)$?

Theorem (Fu, Kashiwara-Schapira 1994)

If Z is a compact subanalytic set then there exists a unique current $N(Z)$ in $\mathbb{R}^n \times S^{n-1}$ that satisfies certain axioms desirable of a normal cycle.

- Other proofs were given by Bernig (2007) and Nicolaescu (2011).
- "Proof philosophy": $\exists$ enough *regular* parallel sets Z_r to Z
- Z_r has normal cycle $N(Z_r)$
- $\exists$ some uniform bound in the mass norm for $N(Z_r)$
- use Compactness Theorem to conclude the convergence of a subsequence.
- prove the limit has the uniqueness properties of normal cycle.

Some geometric motivation

- Nicolaescu uses a subanalytic function $f \geq 0$ such that $f^{-1}(0) = Z$ ...and the regular level sets of f and a similar process.

- What distance functions and subanalytic functions have in common?

Definition (Kurdyka-Łojasiewicz functions)

Let (M, Z) be a relative manifold (i.e. $M \setminus Z$ is a C^2 manifold) and $f : (M, Z) \to ([0, \infty), \{0\})$ a relative C^1 function. Then $p \in \partial Z$ is KŁ-non-degenerate if $\exists$ (ρ, U, ψ) where

- $p \in U \subset M$ is a neighborhood;
- $\rho > 0$;
- $\psi : [0, \rho) \to [0, \infty)$ is C^0 and C^1 on $(0, \rho)$, $\psi' > 0$ and

$$|\nabla_x(\psi \circ f)| \geq 1, \qquad \forall x \in f^{-1}(0, \rho) \cap U$$

- f is KŁ if every point $p \in \partial Z$ is KŁ-non-degenerate.
- f is KŁ if $|f|$ is KŁ.

Examples

- Any C^1 function around a regular point on the zero locus.
- The distance function r to a submanifold $\Rightarrow |\nabla r| = 1$

Theorem (Łojasiewicz -1965)

Let M be a real analytic manifold and $f : M \to \mathbb{R}$ a real analytic function and $p \in f^{-1}(0)$. Then $\exists C > 0$, $\exists U \ni p$ and $0 < \theta < 1$ such that

$$|\nabla_x f| \geq C|f(x)|^{\theta}, \qquad \forall x \in U$$

Obs: here $\psi(t) = t^{1-\theta}$.

- Kurdyka (1998) extended Łojasiewicz inequality to functions belonging to any o-minimal structure.
- Half-isoperimetric (transnormal) functions, i.e. $|\nabla f| = b(f)$.
- Morse-Bott $f : M \to \mathbb{R}$ near $Z := f^{-1}(0)$ a critical level.

Fundamental property

Proposition

If $x \in U \setminus Z$, then $\text{length}(\alpha_x) \leq \psi(f(x))$*, where* $\alpha_x : [0, \omega_x^+)$ *is the integral curve of* $-\nabla f$ *with* $\alpha_x(0) = x$ *on its maximal interval of definition.*

Proof.

- for $[a, b] \subset [0, \omega_x^+)$ take a partition $a = t_0 < t_1 < \ldots < t_n = b$
- $g := \psi \circ f \Rightarrow g \circ \alpha_x$ is decreasing.
- $g(\alpha_x(t_i)) - g(\alpha_x(t_{i+1})) = |g(\alpha_x(t_i)) - g(\alpha_x(t_{i+1}))| =$
  $= (t_{i+1} - t_i)|\alpha_x'(\theta_i)| \cdot |\nabla_{\alpha_x(\theta_i)} g| \geq (t_{i+1} - t_i)|\alpha_x'(\theta_i)|$
- $\Rightarrow g(\alpha_x(a)) - g(\alpha_x(b)) \geq \sum_{i=1}^{n}(t_{i+1} - t_i)|\alpha_x'(\theta_i)|$
- take the limit over partitions and let $a \to 0$, $b \to \omega_x^+$.

□

Consequence

Corollary

There exista a neighborhood $V \supset Z$ and a continuous retract $R : V \to Z$.

- This was known to Kurdyka, i.e. R is a strong deformation retract.
- What does this say about the zero locus Z?

Proposition

If M is an reasonable space like a topological manifold then the following assertions are equivalent for closed subset Z

- *Z is an NDR (neighborhood deformation retract);*
- *Z is an ANR (absolute neighborhood retract).*

- Any topological manifold is an ANR.
- Hence any embedding of $\mathbb{S}^2$ into $\mathbb{S}^3$ is an NDR.

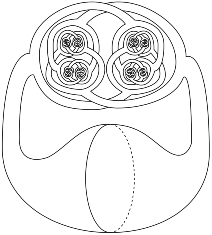


Figure 2.11. The Alexander horned sphere

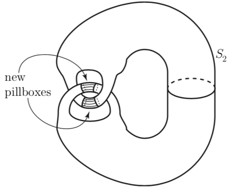


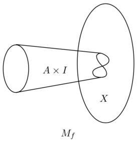
Figure 2.10. The second stage in the construction

Helaman Ferguson - UC Davis

Can this be $Z = f^{-1}(0)$ with f a KŁ function?

Mapping cylinder neighborhoods (MCN)

- $f : A \to X$ continuous $\Rightarrow M_f := I \times A \sqcup X/(1, a) \sim f(a)$



The mapping cylinder of $\mathbb{S}^{n-1} \to$ pt is homeomorphic to $\overline{B}^n$

- Let $Z \subset M$ closed. For simplicity say $\mathring{Z} = \emptyset$.
- Z has a MCN U if $\exists\ \pi : \partial U \to Z$ and a homeomorphism

$$\varphi : \overline{U} \to M_\pi$$

with $\varphi\big|_{\partial U \cup Z} = \text{id}$.

Example

If $Z \subset M$ is a (smooth) submanifold (or set of positive reach) and $U_r := \{x \mid d(x, Z) \leq r\}$ then $\exists \pi : \partial U_r \to Z$ such that $\overline{U_r} \simeq M_\pi$.

Theorem (C. - Galaz)

If $Z \subset M$ is the zero locus of a Kurdyka-Łojasiewicz function then Z has a regular mapping cylinder neighborhood.

Corollary

No wild embedding of a 2-manifold into a 3-manifold M can be the zero locus of KŁ function.

Wild embeddings of $\mathbb{S}^2$

Definition

A topological embedding of a 2-manifold into a 3-manifold is tame if there exists a triangulation of the 3 manifold such that the 2-manifold is a subcomplex. Otherwise it is called wild.

Theorem (Antoine-Alexander 1924)

Wild embeddings of $\mathbb{S}^2$ into $\mathbb{S}^3$ exist.

Theorem (Hajłasz-Zhou 2016)

For every Cantor set $C \subset \mathbb{R}^{n+1}$ there exists $f : S^n \to \mathbb{R}^{n+1}$ a topological embedding with $f_i \in W^{1,n}$, $\forall i$ and $C \subset f(S^n)$.

Theorem (Nicholson 1969)

A closed embedding of a 2 manifold into a 3-manifold is tame if and only if the embedding has a mapping cylinder neighborhood.

Alternative characterizations of KŁ functions

- interest from convex optimization, non-smooth and variational analysis...
- ...even PDE (non-linear heat and damped wave equation)
- impact in applied mathematics (minimization algorithms, neural networks, complexity theory)

Theorem (Bolte-Daniilidis-Mazet-Ley, 2009)

If $f : X \to \mathbb{R} \cup \{\infty\}$ is a lower semi-continuous function defined on a complete metric space X, then the following are equivalent.

- $|\nabla f|(x) \geq \frac{1}{k}, \qquad \forall x \in f^{-1}(0, r_0)$
- $\rho(f^{-1}[0, r_1], f^{-1}[0, r_2]) \leq k|r_1 - r_2|, \forall r_1, r_2 \in (0, r_0)$

where $|\nabla f|(x) = \limsup_{y \to x} \dfrac{(f(x) - f(y))^+}{d(x, y)}$ *is the strong slope and* ρ *is the Hausdorff distance in a metric space.*

Locally compact ambient

Definition

$f : (M, Z) \to ([0, \infty), \{0\})$ relative C^1 has simple non-degenerate points on $Z = f^{-1}(0)$ if there exists $U \supset Z$ such that $\nabla f \neq 0$ on $U \setminus Z$

Theorem (C. -Galaz)

The following are equivalent for a simple non-degenerate point $p \in Z$

(a) *p is KŁ- nondegenerate*

(b) *p is weakly KŁ non-degenerate*

(c) *$\exists K \ni p$ compact neighborhood such that* $\alpha^K(t) := \dfrac{1}{\inf\limits_{x \in f^{-1}(t) \cap K} |\nabla_x f|}$

is in $L^1(0, \rho)$ for some $\rho > 0$

(d) *$\exists U \ni p$ open neighborhood, $\exists \rho > 0$, and continuous $a : (0, \rho] \to (0, \infty)$ with $a^{-1} \in L^1(0, \rho)$ and*

$$|\nabla_x f| \geq a(f(x)), \qquad \forall x \in U \setminus Z$$

- A point p is weakly KŁ if $\exists \psi : [0, \rho] \to [0, \infty)$ absolutely continuous and non-decreasing such that

$$|\nabla_x(\psi \circ f)| \geq 1, \qquad \forall x \in f^{-1} D_\psi$$

D_ψ the points of differentiability of ψ.

Some questions

- Are simple non-degenerate points also KŁ non-degenerate?
- NOOOO!!

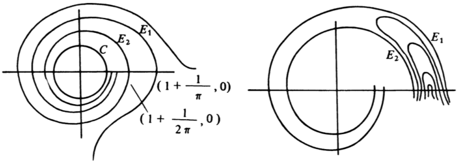


Figure 7 Figure 8

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(r\cos\theta, r\sin\theta) = \begin{cases} e^{1/(r^2-1)}, & \text{if } r < 1; \\ 0, & \text{if } r = 1; \\ e^{-1/(r^2-1)}\sin(1/(r-1)-\theta), & \text{if } r > 1. \end{cases}$$

- Bolte, Pauwels (2020) prove $\exists C^k$ *convex* functions which are not KŁ around the zero=min locus.

Proposition

A simple non-degenerate point $p \in f^{-1}(0)$ falls into one of the three classes:

- *the "good":* $\int_0^\rho \frac{1}{\inf_{x\in f^{-1}(t)\cap K} |\nabla_x f|}\, dt < \infty$*: these are KŁ non-degenerate*
- *the "bad":* $\int_0^\rho \frac{1}{\sup_{x\in f^{-1}(t)\cap K} |\nabla_x f|}\, dt < \infty = \int_0^\rho \frac{1}{\inf_{x\in f^{-1}(t)\cap K} |\nabla_x f|}\, dt$
- *the "ugly":* $\int_0^\rho \frac{1}{\sup_{x\in f^{-1}(t)\cap K} |\nabla_x f|}\, dt = \infty$

The "ugly" satisfy the property that there exists no relative C^1 curve $\gamma : ([0,\epsilon),\{0\}) \to (M,\{p\})$ of finite length.

Example

Take $M = \Gamma_h$ where $h = \begin{cases} x\sin(x^{-1}), & x > 0 \\ 0, & x = 0 \end{cases}$ and $f(x,y) = x$.

Food for thought

Can a topological wild embedding of a 2 manifold into a 3 manifold be the zero locus of a C^1 function with only simple non-degenerate points?