Supercycles and stable supermaps

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ALGEBRAIC GEOMETRY, LIPSCHITZ GEOMETRY AND SINGULARITIES

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Joint work with D. Hernández Ruipérez and Yu. I. Manin

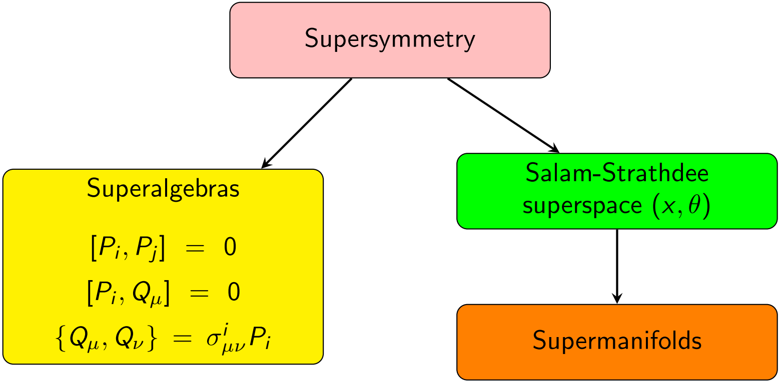
First appearences of supersymmetry:

- H. Miyazawa (1966)
- J. L. Gervais and B. Sakita (1971), Yu. A. Golfand and E. P. Likhtman (1971), D. V. Volkov and V. P. Akulov (1972)
- J. Wess and B. Zumino (1974)

WZ model: hypermultiplet (ϕ, χ, F) with (infinitesimal) supersymmetry

$$\begin{aligned} \delta\phi &= \bar{\varepsilon}\chi \\ \delta\chi &= i\varepsilon\sigma^{\mu}\partial_{\mu}\phi \\ \delta F &= i\bar{\varepsilon}\sigma^{\mu}\partial_{\mu}\chi \end{aligned}$$

$$\mathcal{L} = \partial_{\mu}\bar{\phi}\,\partial^{\mu}\phi + i\bar{\chi}\sigma^{\mu}\partial_{\mu}\chi + \bar{F}F$$



Superspace formalism:

$$\Phi(x, \theta) = \phi(y) + \theta\chi(y) + \theta\bar{\theta}F(y)$$

$$y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$$

Supergeometry

Supergeometry was created to deal with problems coming from supersymmetry using the powerful methods of differential/algebraic geometry

A geometric framework where anti-commutative (fermionic) variables appear $\Longrightarrow$ supermanifolds, supervarieties, superschemes, superstacks

Different first approaches (Berezin-Leĭtes, De Witt-Rogers)

The Berezin-Leĭtes model prevailed (sheaf-theoretic, more elegant, works better). The definition can be adapted so to generalize differentiable manifolds, complex analytic manifolds/spaces, algebraic varieties or schemes, algebraic spaces, stacks ...

Algebraic supergeometry

- First applications of algebraic supergeometry (SUSY curves and their moduli, among others)
- Further developments require an effort to extend fundamental algebraic geometry to the super setting
- Contributions by many people

Super version of classical Grothendieck's *Fondements de la géométrie algébrique:*

Cohomology of coherent sheaves, finiteness theorems, semicontinuity, supervector and superprojective bundles, Hilbert and Picard superschemes, relative duality, supercycles and super Chow groups, stable supercurves, stable supermaps, etc. etc.

• U.B., D. Hernández Ruipérez. *The supermoduli of SUSY curves with Ramond punctures.* Rev. Real Acad. Ciencias Exactas Fis. Nat. Serie A Mat. **115** (2021) Art. no. 114, 33 pp.

• U.B., D. Hernández Ruipérez, A. Polishchuck. *Notes on Fundamental Algebraic Supergeometry. Hilbert and Picard superschemes*, Adv. Math. **415** (2023) 108890, 115 pp.

• U.B., D. Hernández Ruipérez, Yu. I. Manin. *Supercycles, stable supermaps and SUSY Nori Motives*, `arXiv:2203.15855` (35 pp.)

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Basic objects

A superring $\mathbb{A}$ is a $\mathbb{Z}_2$-graded supercommutative ring

$$ab = (-1)^{ij} ba \quad \text{if} \quad a \in \mathbb{A}_i, \ b \in \mathbb{A}_j$$

such that (equivalenty) the ideal J generated by the odd elements

1. is finitely generated; or
2. $J^n = 0$ for some $n > 0$ and J/J^2 is a finitely generated module over $A = \mathbb{A}/J$

$A = \mathbb{A}/J$ bosonic reduction

$X = \operatorname{Spec} \mathbb{A} = \{\mathbb{Z}_2\text{-homogeneous prime ideals of } \mathbb{A}\} = \operatorname{Spec} A$

Zariski topology on X. Basis given by the usual open sets $D(f)$ for $f \in \mathbb{A}$ non-nilpotent

Definition

The superspectrum of a superring $\mathbb{A}$ *is a pair* $\operatorname{Spec}\mathbb{A} = (X, \mathcal{O})$ *where* $\mathcal{O}$ *is the sheaf of superrings defined by* $\mathcal{O}(D(f)) = \mathbb{A}_f$ *on the basic open subsets* $D(f)$

Example

Affine superspace of dimension $m|n$ over k:

$$\mathbb{A}^{m|n} = \operatorname{Spec} k[x_1, \ldots, x_m, \theta_1, \ldots, \theta_n], \quad x_i\text{'s even, } \theta_J\text{'s odd.}$$

Locally ringed superspaces and superschemes defined as usual

Superscheme morphisms

All schemes are noetherian and locally of finite type over an algebraically closed field k

Morphism of superschemes: $f\colon \mathcal{X} = (X, \mathcal{O}_\mathcal{X}) \to \mathcal{Z} = (Z, \mathcal{O}_\mathcal{Z})$ is a morphism of locally ringed superspaces, given by a continuous map $f\colon X \to Z$ and an even local morphism of superring sheaves $f_\sharp\colon \mathcal{O}_\mathcal{Z} \to f_*\mathcal{O}_\mathcal{X}$.

The induced morphism $f\colon X \to Z$ is a morphism of schemes

The projection $\mathcal{O}_\mathcal{X} \to \mathcal{O}_X = \mathcal{O}_\mathcal{X}/\mathcal{J} \to 0$ induces a closed embedding of superschemes

$$i\colon X \hookrightarrow \mathcal{X}$$

Projected and split superschemes

$\mathcal{X} = (X, \mathcal{O}_\mathcal{X})$ superscheme. $\mathcal{E} = \mathcal{J}/\mathcal{J}^2$ is an $\mathcal{O}_X$-module

$\mathcal{X}$ is projected if there is a retraction $r\colon \mathcal{X} \to X$, $r \circ i = \mathrm{Id}$

$\mathcal{X}$ is split if $\mathcal{E}$ is locally free and $\mathcal{O}_\mathcal{X} \simeq \bigwedge_{\mathcal{O}_X} \mathcal{E}$ (globally) compatibly with the projection to $\mathcal{O}_X$

Split $\implies$ projected (retraction given by the inclusion $\mathcal{O}_X \hookrightarrow \bigwedge_{\mathcal{O}_X} \mathcal{E}$)

When $\mathcal{X}$ is locally split, one sets $\dim \mathcal{X} = m|n$, where $m = \dim X$ and $n = \operatorname{rk} \mathcal{E}$

Examples

- X scheme, $\mathcal{E}$ l.f. sheaf on X: $\mathcal{X} = S(X, \mathcal{E}) = (X, \bigwedge_{\mathcal{O}_X} \mathcal{E})$ is the split superscheme associated to $\mathcal{E}$
- If $X = \mathbb{P}^m$ and $\mathcal{E} = \mathcal{O}_X(-1)^{\oplus n}$, then

$$\mathbb{P}^{m|n} = (\mathbb{P}^m, \bigwedge_{\mathcal{O}_{\mathbb{P}^m}} \mathcal{E}) \simeq \mathbb{P}\mathrm{roj}\, k[x_0, \dots, x_n, \theta_1, \dots, \theta_n]$$

is the projective superspace of dimension $m|n$ (Manin)

- Supergrassmannian

$$\mathbb{G}\mathrm{r}\,(a|c; k^{m|n}) = (Gr(a; k^m) \times Gr(c; k^n), \mathcal{O}_{\mathbb{G}\mathrm{r}})$$

of $a|c$-dimensional graded subspaces of $k^{m|n}$

Locally split of dimension $a(m-a) + c(n-c) \mid a(n-c) + c(m-a)$

$$\mathbb{G}\mathrm{r}\,(1|0; k^{m|n}) = \mathbb{P}^{m-1|n}$$

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Morphisms & smooth superschemes

Separated, proper, (faithfully) flat morphism, and the notion of fiber of a morphism are defined generalizing the usual notions

Definition

A superscheme $\mathcal{X} = (X, \mathcal{O}_\mathcal{X})$ of dimension $m|n$ is smooth if for every point $x \in X$ (not necessarily closed), the stalk $\Omega_{\mathcal{X},x}$ of the cotangent sheaf at x is a free $\mathcal{O}_{\mathcal{X},x}$-module of rank $m|n$

$\Rightarrow$ definition of smooth of relative dimension $m|n$ morphism

Supercycles

Give $\mathbb{Z}^2$ a superring structure by writing it as $\mathbb{Z} \oplus \Pi\mathbb{Z}$

$(m + \Pi n)(m' + \Pi n') = (mm' + nn' + \Pi(mn' + m'n))$

Definition

An h-supercycle of $\mathcal{X}$ is a finite sum

$$\alpha = \sum_i (m_i + \Pi n_i)[Y_i]$$

where $m_i + \Pi n_i \in \mathbb{Z}^2$ and the Y_i are closed subvarieties of X of dimension h. The set $Z_h(\mathcal{X})$ of h-supercycles is a free $\mathbb{Z}_2$-graded module over $\mathbb{Z}^2$. The group of supercycles of $\mathcal{X}$ is the bigraded $\mathbb{Z}^2$-module

$$Z_\bullet(\mathcal{X}) = \bigoplus_{h=0}^{m} Z_h(\mathcal{X}) = Z_\bullet(X) \oplus \Pi Z_\bullet(X)$$

which has a natural ring structure using the ordinary intersection product

(Compare with the Manin-Penkov-Voronov definition of K-theory rings $K^S(\mathcal{X}) = K(X) \oplus \Pi K(X)$)

For these supercycles one can define

- functorial flat pullbacks
- proper pushforwards

Definition

• *$\alpha \in Z_h(\mathcal{X})$ is rationally equivalent to zero if there are t sub-supervarieties $\delta_i \colon \mathcal{W}_i \hookrightarrow \mathcal{X}$ of even dimension $h+1$ and pure odd dimension $s = 0$ or 1 and nonzero rational even superfunctions $g_i \in \mathbb{K}(\mathcal{W}_i)^*$ such that*

$$\alpha = \sum_{i=0}^{t} \delta_{i*} \operatorname{div}(g_i)$$

• *The $\mathbb{Z}^2$-module of h-supercycles modulo rational equivalence is*

$$A_h(\mathcal{X}) = Z_h(\mathcal{X})/W_h(\mathcal{X})$$

where $W_h(\mathcal{X}) \subset Z_h(\mathcal{X})$ is the graded $\mathbb{Z}^2$-submodule formed by the h-supercycles rationally equivalent to zero

If $f \colon \mathcal{X} \to \mathcal{Y}$ is proper, there is a pushforward morphism

$$f_* \colon A_\bullet(\mathcal{X}) \to A_\bullet(\mathcal{Y})$$

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Supercurves

Definition

A supercurve is a reduced superscheme $\mathcal{X}$ of pure dimension $1|1$

One has

$$\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X \oplus \mathcal{L}$$

so that $\mathcal{X}$ is projected. We are not assuming that $\mathcal{L}$ is a line bundle hence $\mathcal{X}$ may not be split

Definition

A relative supercurve is a flat morphisms of superschemes $f: \mathcal{X} \to \mathcal{S}$ whose fibres are supercurves

SUSY curves

Definition

A SUSY curve is a relative supercurve $\pi\colon \mathcal{X} \to \mathcal{S}$ *with locally free submodule* $\mathcal{D} \hookrightarrow \Theta_{\mathcal{X}/\mathcal{S}}$ *of rank* $(0,1)$ *such that*

$$\mathcal{D} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{D} \xrightarrow{[,]} \Theta_{\mathcal{X}/\mathcal{S}} \to \Theta_{\mathcal{X}/\mathcal{S}}/\mathcal{D}$$

is an isomorphism

$\mathcal{D}$ is a conformal structure for $\pi\colon \mathcal{X} \to \mathcal{S}$

Definition

A relative effective superdivisor of degree n is a closed sub-superscheme $\mathcal{Z} = (Z, \mathcal{O}_\mathcal{Z}) \hookrightarrow \mathcal{X}$ *whose ideal is a line bundle* $\mathcal{O}_\mathcal{X}(-\mathcal{Z})$ *of rank* $(1,0)$ *and whose structure sheaf* $\mathcal{O}_\mathcal{Z}$ *is a finite flat* $\mathcal{O}_\mathcal{S}$*-module of rank* (n,n)

Ramond-Ramond SUSY curves

Definition

A (smooth) Ramond-Ramond SUSY curve $\pi\colon \mathcal{X} \to \mathcal{S}$ *along an effective relative superdivisor* $\mathcal{Z} \hookrightarrow \mathcal{X}$ *is a smooth relative supercurve with a locally free submodule* $\mathcal{D} \hookrightarrow \Theta_{\mathcal{X}/\mathcal{S}}$ *of rank* $(0,1)$ *such that the composition*

$$\mathcal{D} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{D} \xrightarrow{[\,,]} \Theta_{\mathcal{X}/\mathcal{S}} \to \frac{\Theta_{\mathcal{X}/\mathcal{S}}}{\mathcal{D}}$$

induces an isomorphism of $\mathcal{O}_\mathcal{X}$*-modules*

$$\mathcal{D} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{D} \simeq \frac{\Theta_{\mathcal{X}/\mathcal{S}}}{\mathcal{D}}(-\mathcal{Z})$$

$\mathcal{D}$ is a RR conformal structure for $\pi\colon \mathcal{X} \to \mathcal{S}$ along $\mathcal{Z}$

The irreducible components $\mathcal{Z}_i$ of the superdivisor $\mathcal{Z}$ are the *Ramond-Ramond* punctures

Proposition

If $f : \mathcal{X} \to \mathcal{S}$ is a proper smooth supercurve and $\mathcal{Z} \hookrightarrow \mathcal{X}$ is an effective relative superdivisor, a superconformal structure $\mathcal{D} \hookrightarrow \Theta_f = \Omega_f^*$ with RR punctures along $\mathcal{Z}$ is equivalent to a sheaf epimorphism

$$\Omega_f \xrightarrow{\bar{\delta}} \mathcal{B}er_f(\mathcal{Z}) \to 0$$

such that the composition

$$\ker \bar{\delta} \hookrightarrow \Omega_f \xrightarrow{d} \Omega_f \wedge \Omega_f \xrightarrow{\bar{\delta} \wedge \bar{\delta}} \mathcal{B}er_f^{\otimes 2}(2\mathcal{Z})$$

yields an isomorphism

$$\ker \bar{\delta} \simeq \mathcal{B}er_f^{\otimes 2}(\mathcal{Z})$$

$\mathcal{D}$ is recovered as the image of $\bar{\delta}^* \colon \mathcal{B}er_f^*(-\mathcal{Z}) \hookrightarrow \Theta_f$

In the singular case we may take this is the definition of SUSY curve with RR punctures

Stable supercurves

Definition

A (pre)stable SUSY curve of arithmetic genus g with punctures is a proper and Cohen-Macaulay supercurve $f\colon \mathcal{X} \to \mathcal{S}$ *with:*

1. *Disjoint closed sub-superschemes* $\mathcal{X}_i \hookrightarrow \mathcal{U}$ $(i = 1, \dots, \mathfrak{n}_{NS})$*, where* $\mathcal{U}$ *is the smooth locus of* f*, such that* $\pi\colon \mathcal{X}_i \to \mathcal{S}$ *is an isomorphism for every* i *(NS punctures)*
2. *Disjoint irreducible Cartier divisors* $\mathcal{Z}_j$ *of relative degree 1* $(j = 1, \dots, \mathfrak{n}_{RR})$*, contained in* $\mathcal{U}$ *(RR punctures)*
3. *An epimorphism* $\bar{\delta}\colon \Omega_f \to \mathcal{B}er_f(\mathcal{Z})$*, where* $\mathcal{Z} = \sum \mathcal{Z}_j$*, satisfying the conditions of the previous Proposition*

These data must fulfil the following condition: if for every bosonic fibre X_s *of* $f\colon \mathcal{X} \to \mathcal{S}$ *we write* $x_{s,i} = X_i \cap X_s$ *and* $z_{s,j} = Z_j \cap X_s$*, then* (X_s, D_s) *with* $D_s = \{x_{s,1} \dots x_{s,\mathfrak{n}_{NS}}, z_{s,1} \dots, z_{s,\mathfrak{n}_{RR}}\}$ *is a (pre)stable* $(\mathfrak{n}_{NS} + \mathfrak{n}_{RR})$*-pointed curve of arithmetic genus g*

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Stable supermaps

Stable supermaps with values in a superscheme $\mathcal{Y}$, which when $\mathcal{Y}$ is a point coincide with stable SUSY curves with punctures. Fix an algebraically closed ground field k and $\beta \in A_1(\mathcal{Y})$

Definition

A stable supermap into $\mathcal{Y}$ of class β with NS and RR punctures is

1. *a prestable SUSY curve $(f : \mathcal{X} \to \mathcal{S}, \{\mathcal{X}_i\}, \{\mathcal{Z}_j\}, \bar{\delta})$ with $\mathfrak{n}_{NS}$ NS punctures and $\mathfrak{n}_R$ RR punctures*
2. *a morphism $\phi \colon \mathcal{X} \to \mathcal{Y}$ such that $\phi_*[\mathcal{X}_s] = \beta$ for every closed $s \in S$*
3. *for every geometric point $s \in S$, if $\tilde{X}'_s$ is a component of the normalization $\pi \colon \tilde{X}_s \to X_s$ of the bosonic fibre X_s which is contracted by $\phi \circ \pi$ to a point, then*
   1. *if $\tilde{X}'_s$ is rational, it contains at least three points from $\pi^{-1}(D_s \cup X'_{s,sing})$, where $D_s = \{x_{s,1} \dots x_{s,\mathfrak{n}_{NS}}, z_{s,1} \dots, z_{s,\mathfrak{n}_R}\}$ with $x_{s,i} = X_i \cap X_s$ and $z_{s,j} = Z_j \cap X_s$*
   2. *if $\tilde{X}'_s$ has genus 1, it contains at least one such point*

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Remarks

- The supercycle class $\phi_*[\mathcal{X}_s]$ is defined even when ϕ is not proper, because the restriction of ϕ to $\mathcal{X}_s$ is automatically proper

- When $\mathcal{Y}$ is a point the second condition is automatically fulfilled and stable supermaps into a point are the same thing as stable SUSY curves

The stack of stable supermaps

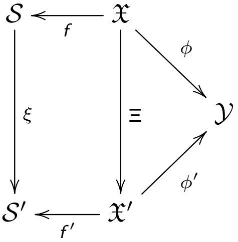
Category fibred in groupoids $\mathsf{S}\mathfrak{M}_{g,\mathfrak{n}_{NS},\mathfrak{n}_R}(\mathcal{Y},\beta) \xrightarrow{p} \mathsf{SSch}$ of stable supermaps of class β and arithmetic genus g into $\mathcal{Y}$, with $\mathfrak{n}_{NS}$ NS punctures and $\mathfrak{n}_R$ RR punctures

- Objects: stable supermaps

$$\mathfrak{X} = ((f\colon \mathcal{X} \to \mathcal{S}, \{\mathcal{X}_i\}, \{\mathcal{Z}_j\}, \bar{\delta}), \phi\colon \mathcal{X} \to \mathcal{Y})$$

  of class β and arithmetic genus g with $\mathfrak{n}_{NS}$ NS punctures and $\mathfrak{n}_R$ RR punctures

- Morphisms: diagrams



  cartesian diagrams compatible with the punctures and the morphisms $\bar{\delta}$, $\bar{\delta}'$

$p(\mathfrak{X}) = \mathcal{S}, \quad p(\mathfrak{X}') = \mathcal{S}', \quad p(\Xi) = \xi$

The definition of morphisms as fibre products has two consequences

- the fibre $S\mathfrak{M}_{g,\mathfrak{n}_{NS},\mathfrak{n}_R}(\mathcal{Y},\beta)(\mathcal{S})$ of $S\mathfrak{M}_{g,\mathfrak{n}_{NS},\mathfrak{n}_R}(\mathcal{Y},\beta)$ over a superscheme $\mathcal{S}$ is a groupoid whose objects are stable supermaps over $\mathcal{S}$
- it equips the CFG $S\mathfrak{M}_{g,\mathfrak{n}_{NS},\mathfrak{n}_R}(\mathcal{Y},\beta)$ with a natural cleavage

Étale descent data for stable supermaps are effective and descent data for morphims to $\mathcal{Y}$ are effective as well so that the descent data for $S\mathfrak{M}_{g,\mathfrak{n}_{NS},\mathfrak{n}_R}(\mathcal{Y},\beta)$ are effective

The isomorphisms between two objects of $S\mathfrak{M}_{g,\mathfrak{n}_{NS},\mathfrak{n}_R}(\mathcal{Y},\beta)$ form a sheaf in the étale topology of superschemes, so that the CFG $S\mathfrak{M}_{g,\mathfrak{n}_{NS},\mathfrak{n}_R}(\mathcal{Y},\beta)$ is a superstack

Short-term developments

- Establish the basic deformation theory of supermaps

$$\mathcal{X} \xrightarrow{\phi} \mathcal{Y}$$

where $\mathcal{Y}$ is fixed and the pair $(\mathcal{X}, \phi)$ is deformed

- Compute the dimension of the stack $\mathrm{S}\mathfrak{M}_{g,\mathfrak{n}_{NS},\mathfrak{n}_R}(\mathcal{Y}, \beta)$ using Manin-Penkov-Voronov's super Grothendieck-Riemann-Roch theorem

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