Mond conjecture for mappings on ICIS

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Still open for $n \geq 3$.

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- $(X,0)$ isolated complete intersection singularity (ICIS) of dimension ≥ 1, then

$$\mu(X,0) \geq \tau(X,0),$$

with equality iff $(X,0)$ is weighted homogeneous. Non-trivial:
  - w.h. implies $=$, Greuel (1980),
  - $\geq$, Looijenga & Steenbrik (1985),
  - $=$ implies w.h., Vosegaard (2002).

- Suppose $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ has isolated instability and (n, p) are nice dimensions, with $n \geq p$. Then,

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It was proved by Damon & Mond (Invent. Math. 1991).

Conjecture (Generalised Mond conjecture)

Suppose $f\colon (X,S) \to (\mathbb{C}^{n+1},0)$ has isolated instability where (X,S) is an n-dimensional ICIS and $(n,n+1)$ are nice dimensions. Then,

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In the same paper: if $n \geq p$ and (n,p) are nice dimensions, then

$$\operatorname{codim}_{\mathscr{A}_e}(X,f) \leq \mu_\Delta(X,f),$$

with equality if (X,f) is weighted homogeneous (generalised Damon & Mond theorem).

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Our proof is based on the construction of a generalised Jacobian module, which controls the image Milnor number.

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- It is also a generalisation of the classical $\mu \geq \tau$-inequality for IHS. In fact, if $(X, 0)$ is an IHS and $i\colon (X, 0) \hookrightarrow (\mathbb{C}^{n+1}, 0)$ is the inclusion, then

$$\operatorname{codim}_{\mathscr{A}_e}(X, i) = \tau(X, 0), \quad \mu_I(X, i) = \mu(X, 0).$$

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  If f has corank one, the double point space $D^2(f) \subset \mathbb{C}^n \times \mathbb{C}^n$ is an ICIS of dimension $n-1$ and consider the projection $\pi_1\colon D^2(f) \to \mathbb{C}^n$. Then $\mu_I(D^2(f),\pi_1)$ is strongly related to $\mu_I(f)$, Giménez-Conejero & JJNB (Adv. Math. 2023).

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$\mathscr{A}$-equivalence:

$$\begin{array}{ccc} (X,S) & \xrightarrow{f} & (\mathbb{C}^p,0) \\ \downarrow \phi & & \downarrow \psi \\ (X',S') & \xrightarrow{f'} & (\mathbb{C}^p,0) \end{array}$$

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Unfoldings (deformations): we deform both the variety X and the map f simultaneously.

An unfolding is a pair $(\mathcal{X}, F)$, where $F\colon (\mathcal{X}, S) \to (\mathbb{C}^p \times \mathbb{C}^r, 0)$ is a map germ together with a flat projection $\pi\colon (\mathcal{X}, S) \to (\mathbb{C}^r, 0)$ such that the following diagram commutes

$$\begin{array}{ccc} (\mathcal{X}, S) & \xrightarrow{\quad F \quad} & (\mathbb{C}^p \times \mathbb{C}^r, 0) \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi_2} & \\ & (\mathbb{C}^r, 0) & \end{array}$$

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For each parameter $u \in \mathbb{C}^r$, we have a mapping $f_u\colon X_u \to V_u \subseteq \mathbb{C}^p$, where $X_u = \pi^{-1}(u)$ and f_u is the restriction of $\pi_1 \circ F$. This is called a perturbation, denoted by (X_u, f_u).

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We can see the unfolding $(\mathcal{X}, F)$ as a family $\{(X_u, f_u)\}_{u \in \mathbb{C}^r}$ which deforms (X, f).

$\mathscr{A}$-equivalence of unfoldings: There exist Φ and Ψ unfoldings of the identity on (X, S) and $(\mathbb{C}^p, 0)$ resp, such that for each parameter $u \in \mathbb{C}^r$ we have $\mathscr{A}$-equivalence of mappings:

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A stabilisation is a 1-parameter unfolding such that (X_t, f_t) has only stable singularities, if $t \in \mathbb{C}$, $t \neq 0$ (stable perturbation).

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Change of parameter space: Given $\varphi\colon (\mathbb{C}^s, 0) \to (\mathbb{C}^r, 0)$ we construct a new unfolding whose perturbation is $(X_{\varphi(v)}, f_{\varphi(v)})$ for each $v \in \mathbb{C}^s$.

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An unfolding is versal if any other unfolding is obtained by change of parameter space and $\mathscr{A}$-equivalence.

The $\mathscr{A}_e$-codimension of (X, f) is defined as

$$\operatorname{codim}_{\mathscr{A}_e}(X, f) = \dim_{\mathbb{C}} \frac{\theta(f)}{tf(\theta_{X,S}) + \omega f(\theta_p)} + \sum_{x \in S} \tau(X, x),$$

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Example

$(X, 0)$ is an ICIS in $(\mathbb{C}^{n+k}, 0)$, inclusion $i \colon (X, 0) \to (\mathbb{C}^{n+k}, 0)$. By construction, $\omega i(\theta_{n+k}) = \theta(i)$, hence

$$\operatorname{codim}_{\mathscr{A}_e}(X, i) = \tau(X, 0).$$

Theorem (Versality theorem, Mond & Montaldi 1996)

(X, f) *admits a versal unfolding iff it is* $\mathscr{A}$*-finite. In such case,* $\operatorname{codim}_{\mathscr{A}_e}(X, f)$ *is the minimal number of parameters in a versal unfolding.*

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Theorem (Generalised Mather-Gaffney criterion)

(X, f) has isolated instability iff it is $\mathscr{A}$-finite.

From now on, we consider only pairs (X, f), where $f\colon (X, S) \to (\mathbb{C}^{n+1}, 0)$ is $\mathscr{A}$-finite and either $(n, n+1)$ are nice dimensions or f has corank one.

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Theorem (Mond, 1991 adapted by Giménez-Conejero & JJNB, 2023)

Take a stable perturbation (X_t, f_t). Then $f_t(X_t) \cap B_\epsilon$ has the homotopy type of a bouquet of n-spheres, $\forall 0 < \eta \ll \epsilon \ll 1$ and $0 < |t| < \eta$. The number of such spheres $\beta_n(f_t(X_t) \cap B_\epsilon)$ is independent of the choice of η, ϵ, t and the stabilisation.

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$\mu_I(X, f) = \beta_n(f_t(X_t) \cap B_\epsilon)$ is called the image Milnor number.

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Theorem (Mond, 1991 adapted by Giménez-Conejero & JJNB, 2023)

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$\mu_I(X, f) = \beta_n(f_t(X_t) \cap B_\epsilon)$ is called the image Milnor number.

Example

Let $(X, 0)$ be an IHS and consider the inclusion $i\colon (X, 0) \to (\mathbb{C}^{n+1}, 0)$. We take a reduced equation $g \in \mathscr{O}_{n+1}$. For each $t \neq 0$, $X_t = g^{-1}(t)$ is smooth and $i\colon X_t \to \mathbb{C}^{n+1}$ the inclusion is stable. We have

$$\mu_I(X, i) = \beta_n(X_t \cap B_\epsilon) = \mu(X, 0).$$

The generalised Jacobian module

The Jacobian module was introduced in a paper by Bobadilla, JJNB & Peñafort (2019) for map germs $f: (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$. Here we present a generalised version for mappings on ICIS.

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Assume $f\colon (X, S) \to (\mathbb{C}^{n+1}, 0)$ is finite, where (X, S) is ICIS of dimension n.

We choose $h\colon (\mathbb{C}^{n+k}, S) \to (\mathbb{C}^k, 0)$ reduced equation of (X, S) and $\tilde{f}\colon (\mathbb{C}^{n+k}, S) \to (\mathbb{C}^p, 0)$ an analytic extension of f. We define

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Then $(\mathbb{C}^{n+k}, \hat{f})$ is an unfolding of (X, f) with smooth base $\mathbb{C}^{n+k}$ and with parameter space $\mathbb{C}^k$.

If $(X, 0)$ has embedding dimension $n + k$, then $(\mathbb{C}^{n+k}, \hat{f})$ is a minimal unfolding with smooth base.

Since f is finite, $\hat{f}$ is also finite and its image is a hypersurface in $(\mathbb{C}^{n+1} \times \mathbb{C}^k, 0)$. Let $\hat{g} \in \mathscr{O}_{n+1+k}$ be a reduced equation of its image.

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The generalised Jacobian module is

$$M(g)=\frac{(\hat{f}^*)^{-1}(J(\hat{g})\cdot\mathscr{O}_{n+k})}{J_y(\hat{g})}\otimes\frac{\mathscr{O}_k}{\mathfrak{m}_k},$$

where $J_y(\hat{g})=$ the relative Jacobian ideal, generated in $\mathscr{O}_{n+1+k}$ by $\partial\hat{g}/\partial y_i$, $i=1,\dots,n+1$.

Example

Let $(X,0)$ be an IHS and consider the inclusion $i\colon (X,0) \to (\mathbb{C}^{n+1},0)$.
We have $\hat{g}(y,t) = g(y) - t$, where $g \in \mathscr{O}_{n+1}$ is the equation of $(X,0)$.
Thus, $J(\hat{g}) = \mathscr{O}_{n+2}$ and

$$M(g) = \frac{\mathscr{O}_{n+2}}{J_y(\hat{g})} \otimes \frac{\mathscr{O}_1}{\mathfrak{m}_1} \cong \frac{\mathscr{O}_{n+1}}{J(g)},$$

the Jacobian algebra.

Theorem

Suppose (X, f) has isolated instability and $(n, n+1)$ nice dimensions, $n \geq 2$, then

$$\dim_{\mathbb{C}} M(g) = \operatorname{codim}_{\mathscr{A}_e}(X, f) + \dim_{\mathbb{C}} \frac{J(g) + (g)}{J(g)}.$$

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Corollary

$$\dim_{\mathbb{C}} M(g) \geq \operatorname{codim}_{\mathscr{A}_e}(X, f),$$

with equality if (X, f) is w.h.

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with $F(x, z) = (f_z(x), h(x), z)$ such that $h^{-1}(0) = X$ and $f_0|_X = f$.

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The relative generalised Jacobian module is

$$M_{rel}(G) = \frac{(F^*)^{-1}(J_{y,z}(G) \cdot \mathscr{O}_{n+k+r})}{J_y(G)}$$

Theorem

We have an exact sequence:

$$0 \longrightarrow M_{rel}(G) \longrightarrow \frac{F_1(F)}{J_y(G)} \longrightarrow \frac{C(F)}{J_{y,z}(G)\cdot \mathscr{O}_{n+k+r}} \longrightarrow 0.$$

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$F_1(F) :=$ first Fitting ideal of $\mathscr{O}_{n+k+r}$ as $\mathscr{O}_{n+1+k+r}$-module via $F^*\colon \mathscr{O}_{n+1+k+r} \to \mathscr{O}_{n+k+r}$ and $C(F) := F_1(F)\cdot \mathscr{O}_{n+k+r}$.

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Lemma (Piene (1979), Bruce & Marar (1996))

$$\frac{C(F)}{J_{y,z}(G)\cdot \mathscr{O}_{n+k+r}} \cong \frac{\mathscr{O}_{n+k+r}}{R(F)},$$

where $R(F)$ is the ramification ideal, generated by the maximal minors of the Jacobian matrix of F. In particular, it is Cohen-Macaulay of dimension $n+k+r-2$.

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Theorem

Let F be any unfolding such that $\mathscr{B}(F) \subsetneq (\mathbb{C}^{k+r}, 0)$. Then,

$$\mu_I(X, f) = e(\mathfrak{m}_{k+r}; M_{rel}(G)),$$

the Samuel multiplicity of $M_{rel}(G)$ as $\mathscr{O}_{k+r}$-module via the projection onto the parameter space.

Corollary

$$\mu_I(X, f) \leq \dim_{\mathbb{C}} M(g),$$

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If we want to prove the Mond conjecture by means of the Jacobian module, we have to show that $\mu_I(X, f) = \dim_{\mathbb{C}} M(g)$.

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Conjecture (Strong generalised Mond conjecture)

$$\mu_I(X, f) = \dim_{\mathbb{C}} M(g).$$

Proof for $n = 2$

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Theorem (Strong generalised Mond conjecture for $n = 2$)

Let $f\colon (X, S) \to (\mathbb{C}^3, 0)$ be with isolated instability, (X, S) a 2-dimensional ICIS. Then

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Proof: This is equivalent to show that $M_{rel}(G)$ is Cohen-Macaulay for some (and hence for any) unfolding F such that $\mathscr{B}(F) \subsetneq (\mathbb{C}^{k+r}, 0)$.

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We have $\dim_{\mathbb{C}} M(g) < \infty$ and $M(g) \cong M_{rel}(G) \otimes \mathscr{O}_{k+r}/\mathfrak{m}_{k+r}$. This implies $\dim M_{rel}(G) \leq k + r$.

Recall that we have an exact sequence

$$0 \longrightarrow M_{rel}(G) \longrightarrow \frac{F_1(F)}{J_y(G)} \longrightarrow \frac{C(F)}{J_{y,z}(G) \cdot \mathscr{O}_{2+k+r}} \longrightarrow 0$$

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By a lemma of Pellikaan (1988), the three conditions above imply that $F_1(F)/J_y(G)$ is also Cohen-Macaulay.

Finally, we use the depth lemma:

$$\begin{aligned}\operatorname{depth} M_{rel}(G) &\geq \min\left\{\operatorname{depth}\left(\frac{F_1(F)}{J_y(G)}\right), \operatorname{depth}\left(\frac{C(F)}{J_{y,z}(G)\cdot \mathscr{O}_{n+r}}\right)+1\right\}\\ &= \min\{k+r, k+r+1\} = k+r.\end{aligned}$$

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On the other hand,

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therefore

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so $M_{rel}(G)$ is Cohen-Macaulay.

□

Example

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Let $f : (X,0) \to (\mathbb{C}^3,0)$ be the mapping $f(x,y,z) = (x,y,z^3 + xz + y^2)$.

The minimal unfolding with smooth base is $\hat{f} \colon (\mathbb{C}^3,0) \to (\mathbb{C}^4,0)$

$$\hat{f}(x,y,z) = (x,y,z^3 + xz + y^2, x^3 + y^3 - z^2).$$

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With SINGULAR, one easily obtains that $\operatorname{codim}_{\mathscr{A}_e}(X, f) = 6$ and
$\mu_I(X, f) = \dim_{\mathbb{C}} M(g) = 6$.