On rank theorems for morphisms of local rings

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Algebraic Geometry, Lipschitz Geometry and Singularities

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Preliminary

Let $\mathbb{K}$ denote the field of real or complex numbers.

We consider germs of $\mathbb{K}$-analytic mapping :

$$\begin{array}{rccc} \Phi : & (\mathbb{K}^m, 0) & \longrightarrow & (\mathbb{K}^n, 0) \\ & u & \mapsto & \Phi(u) \end{array}$$

Recall that Φ induces a morphism of $\mathbb{K}$-convergent power series:

$$\begin{array}{rccc} \Phi^* := \varphi : & \mathbb{K}\{x\} & \longrightarrow & \mathbb{K}\{u\} \\ & f & \mapsto & f \circ \Phi \end{array}$$

where $u = (u_1, \ldots, u_m)$ and $x = (x_1, \ldots, x_n)$ are indeterminates.

Question: what can be said about $\mathrm{Im}(\Phi)$? Is it an analytic set?

Generic and Analytic ranks

In general, $\mathrm{Im}(\Phi)$ is **not** an analytic subset of $\mathbb{K}^n$.

We consider the germ of the analytic closure of the image, generated by:

$$\mathrm{Ker}(\varphi) = \{f \in \mathbb{K}\{x\}; f \circ \Phi \equiv 0\}.$$

Definition

Let $\varphi : \mathbb{K}\{x\} \longrightarrow \mathbb{K}\{u\}$ be a morphism of convergent power series:

$$\text{the Generic rank:} \quad \mathrm{r}(\varphi) := \mathrm{rank}_{\mathrm{Frac}(\mathbb{K}\{u\})}(\mathrm{Jac}(\Phi)),$$

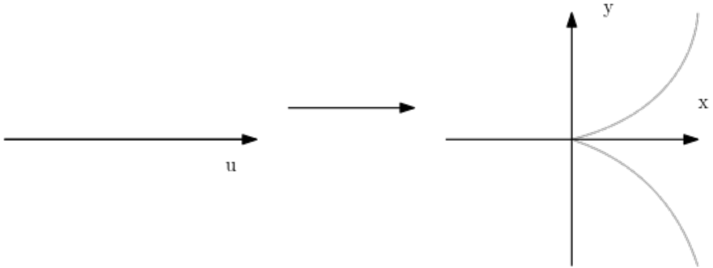
$$\text{the Analytic rank:} \quad \mathrm{r}^{\mathcal{A}}(\varphi) := \dim\left(\frac{\mathbb{K}\{x\}}{\mathrm{Ker}(\varphi)}\right)$$

Remark: $\mathrm{r}(\varphi) \leqslant \mathrm{r}^{\mathcal{A}}(\varphi)$.

Example 1

Consider the morphism:

$$\begin{array}{rccc} \varphi: & \mathbb{K}\{x_1, x_2\} & \longrightarrow & \mathbb{K}\{u\} \\ & (x_1, x_2) & \mapsto & (u^2, u^3) \end{array}$$

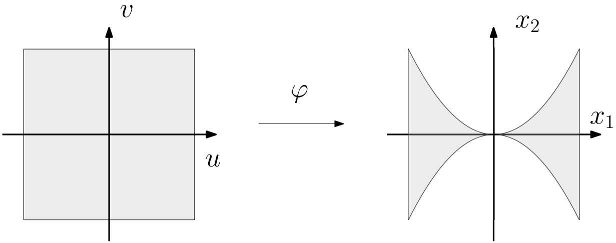


- $r(\varphi) = 1$; its image is generically of dimension 1;
- $r^{\mathcal{A}}(\varphi) = 1$; in fact $\text{Ker}(\varphi^*) = (x^3 - y^2)$.

Example 2

Consider the morphism:

$$\begin{array}{rccc} \varphi : & \mathbb{K}\{x_1, x_2\} & \longrightarrow & \mathbb{K}\{u, v\} \\ & (x_1, x_2) & \mapsto & (u, u^2 v) \end{array}$$



- $\mathrm{r}(\varphi) = 2$; its image is generically of dimension 2;
- $\mathrm{r}^{\mathcal{A}}(\varphi) = 2$; in fact $\mathrm{Ker}(\varphi) = (0)$.

Classical results

Theorem (Chevalley 43, Tarski 48)

If $\varphi : \mathbb{K}[x] \longrightarrow \mathbb{K}[u]$ *is polynomial or algebraic, then:*

$$\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{A}}(\varphi)$$

Theorem (Remmert's proper mapping, 58)

Let $\Phi : X \to Y$ *be a proper analytic morphism between complex analytic reduced spaces. Then the image* $\Phi(X)$ *is an analytic space.*

Osgood Example (1916)

Theorem (Osgood, 1916)

There exists a morphism $\varphi : \mathbb{K}\{x_1, x_2, x_3\} \longrightarrow \mathbb{K}\{u, v\}$ *such that:*

$$\mathrm{r}(\varphi) = 2, \quad \mathrm{r}^{\mathcal{A}}(\varphi) = 3$$

Consider the morphism:

$$\begin{array}{rcl} \varphi : \mathbb{K}\{x_1, x_2, x_3\} & \longrightarrow & \mathbb{K}\{u, v\} \\ (x_1, x_2, x_3) & \mapsto & (u, uv, uve^v) \end{array}$$

and note that $\mathrm{Ker}(\varphi) = (f)$ where:

$$f(x) = x_3 - x_2 \exp(x_2/x_1)$$

but f is not analytic; it does not even admit a Taylor expansion at 0!

$$\mathrm{r}(\varphi) = 2, \quad \mathrm{r}^{\mathcal{A}}(\varphi) = 3$$

Formal rank and a question of Grothendieck (1960)

Definition

Let $\varphi : \mathbb{K}\{x\} \longrightarrow \mathbb{K}\{u\}$ be a morphism of $\mathbb{K}$-convergent power series.
Let $\widehat{\varphi} : \mathbb{K}[\![x]\!] \longrightarrow \mathbb{K}[\![u]\!]$ be the extension of φ to the completion.

$$\text{Formal rank:} \quad \mathrm{r}^{\mathcal{F}}(\varphi) := \dim \left(\frac{\mathbb{K}[\![x]\!]}{\mathrm{Ker}(\widehat{\varphi})} \right)$$

Remark: $\mathrm{r}(\varphi) \leqslant \mathrm{r}^{\mathcal{F}}(\varphi) \leqslant \mathrm{r}^{\mathcal{A}}(\varphi)$.

Question (Grothendieck, 60): Is it true that $\mathrm{r}^{\mathcal{F}}(\varphi) = \mathrm{r}^{\mathcal{A}}(\varphi)$? In particular,

$$\text{If } \exists\, F \in \mathbb{K}[\![x]\!] \text{ such that } F \circ \widehat{\varphi} \equiv 0,$$
$$\text{does it } \exists\, f \in \mathbb{K}\{x\} \text{ such that } f \circ \varphi \equiv 0?$$

Context

This type of results includes the Newton-Puisseux's Theorem and:

Theorem (Artin Approximation, 1958/59)

Let $f \in (\mathbb{K}\{x\})^m$ be an analytic function.

Suppose that $\exists$ formal power series $H_1(u), \ldots, H_n(u) \in \mathbb{K}[\![u]\!]$ such that:

$$\widehat{f}(H_1(u), \ldots, H_n(u)) \equiv 0.$$

Then, $\forall\, c \in \mathbb{N}$, $\exists$ analytic functions $h_1^{(c)}(u), \ldots, h_n^{(c)}(u) \in \mathbb{K}\{u\}$ such that:

$$f(h_1^{(c)}(u), \ldots, h_n^{(c)}(u)) \equiv 0 \text{ and } \widehat{h_i^{(c)}}(u) - H_i(u) \in (u)^c.$$

Grothendieck's question can be seen as a dual to Artin's approximation.

Gabrielov's rank Theorem

Question (Grothendieck, 60): Is it true that $\mathrm{r}^{\mathcal{F}}(\varphi) = \mathrm{r}^{\mathcal{A}}(\varphi)$?

Theorem (Gabrielov's rank Theorem)

Let $\varphi : \mathbb{K}\{x\} \longrightarrow \mathbb{K}\{u\}$ *be a morphism of* $\mathbb{K}$*-convergent power series.*

$$\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) \implies \mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) = \mathrm{r}^{\mathcal{A}}(\varphi).$$

But no in general: Gabrielov (1971) provides a map

$$\psi : \mathbb{K}\{x_1, x_2, x_3, x_4\} \quad \longrightarrow \quad \mathbb{K}\{u, v\}$$

such that

$$\mathrm{r}(\psi) = 2, \quad \mathrm{r}^{\mathcal{F}}(\psi) = 3, \quad \mathrm{r}^{\mathcal{A}}(\psi) = 4.$$

History and Interest

Proofs in the literature:

1. Gabrielov, Izv. Akad. Naut. SSSR. (73);
2. Tougeron, Lectures Notes in Math. Trento (90);
3. Belotto, Curmi, Rond, JEP (21).

Applications and/or connected works:

1. **Study of map germs**:
   Eakin, Harris (77); Izumi (86, 89);
2. **Foliation Theory**:
   Malgrange (77), Cerveau, Mattei (82);
3. **Subanalytic geometry**:
   Bierstone, Schwarz (82), Bierstone, Milman (82), Pawlucki (90, 92).
4. **Counter-examples in real-analytic geometry**:
   Pawlucki (89), Bierstone, Parusinski (20), Belotto, Bierstone (23).

Weierstrass local rings

Let $\mathcal{K}$ be an uncountable algebraically closed field of characteristic zero. A Weierstrass family (over $\mathcal{K}$), denoted by $\mathcal{K}\{\{x\}\}$, is a family

$$\mathcal{K}[x_1,\ldots,x_n] \subset \mathcal{K}\{\{x_1,\ldots,x_n\}\} \subset \mathcal{K}[\![x_1,\ldots,x_n]\!], \quad \forall\, n \in \mathbb{N},$$

of $\mathcal{K}$-algebras closed by permutation of coordinates, intersection with hyper-planes, division by units and Weierstrass division.

Basic properties: Henselien, Noetherien, UFD, closed by Weierstrass Preparation and Noether normalization.

Theorem (Denef, Lipschitz, 1980)

Artin's approximation holds true for Weierstrass families.

W-temperate families

A Weierstrass family $\mathcal{K}\{\{x\}\}$ is said to be temperate if it is:

1. Closed by local blowings-down of points;
2. Closed by the Abhyankhar-Moh generic hyperplane section criteria;
3. Temperateness:
   Hypothesis: Let $\Gamma(t,z) \in \mathcal{K}[t,z]$ be an irreducible polynomial:

$$\Gamma(t,z) = z^d + \sum_{i=0}^{d-1} a_i(t) z^i = \prod_{i=1}^{d} (z - \gamma_i(t)), \text{ where } \gamma(t) \in \mathcal{K}\{\{t\}\}$$

Let $x = (x_1, x_2)$, $\alpha \in \mathbb{N}^*$ and consider

$$P(x,z) = \sum_{k \in \mathbb{N}} x_1^k p_k(x_2, z) \in \mathcal{K}[\![x]\!][z],$$

where $p_k(x_2, z) \in \mathcal{K}[x_2, z]$ is such that $\deg_{x_2}(p_k) \leqslant \alpha k$, $\forall\, k \in \mathbb{N}$.

Thesis: $P(x, \gamma(x_2)) \in \mathcal{K}\{\{x\}\} \Longrightarrow P(x, \gamma'(x_2)) \in \mathcal{K}\{\{x\}\}$.

Rank Theorems for morphisms of local rings

Let $\mathcal{K}$ be an uncountable algebraically closed field of characteristic zero.

Theorem (Belotto, Curmi, Rond, preprint)

Let $\varphi : \mathcal{K}\{\{x\}\} \longrightarrow \mathcal{K}\{\{u\}\}$ be a morphism of W-temperate power series.

$$\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) \implies \mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) = \mathrm{r}^{W}(\varphi).$$

1. The proof follows from commutative algebra and algebraic geometry.
2. Complex analysis is only used to show that $\mathbb{C}\{x\}$ is W-temperate.
3. We provide examples of non Weierstrass families (local rings of C^{∞}-definable functions) which do not satisfy the rank Theorem.

Corollary 1: Rank Theorem for convergent power series

Let $\mathcal{L}$ be a complete valued field of characteristic zero. For example:

- $\mathbb{K}$ (that is, $\mathbb{R}$ or $\mathbb{C}$);
- non-archimedean fields such as $\mathbb{Q}_p$, $\mathbb{C}_p$, $\mathbb{C}((t))$.

Recall that $f \in \mathcal{L}[\![x]\!]$ is $\mathcal{L}$-convergent, if $\exists\, A, B > 0$ such that:

$$f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha \implies |f_\alpha| < A^{|\alpha|} B \,\forall\, \alpha.$$

Corollary (Belotto, Curmi, Rond, preprint)

Let $\varphi : \mathcal{L}\{x\} \longrightarrow \mathcal{L}\{u\}$ be a morphism of $\mathcal{L}$-convergent power series.

$$\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) \implies \mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) = \mathrm{r}^{\mathcal{A}}(\varphi).$$

Remark: The proof depends on complex (or non-archimedean) analysis.

Corollary 2: Rank Theorems in families

We consider Eisenstein power series, going back to Zariski (79).

Let $\mathcal{O}$ be an UFD (such as $\mathcal{O}(D)$, where $D \subset \mathbb{C}^n$ is a polydisc, Dales 74).

Let $\mathcal{K} = \overline{\mathrm{Frac}(\mathcal{K})}$ and:

$$\mathcal{K}\{\{x\}\} := \bigcup_{c \in \mathcal{K}} \bigcup_{f \in \mathcal{O}} \mathcal{O}_f[\![x]\!][c]$$

where $\mathcal{O}_f$ is the localization by $\{1, f, f^2, \ldots\}$. $\mathcal{K}\{\{x\}\}$ is W-temperate.

As a Corollary, we provide a new proof of:

Theorem (Pawłucki 92)

Given an analytic map $\Phi : M \to N$, let Z be the set of non-regular points (in the sense of Gabrielov). Then Z is a proper analytic subset of M.

Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine."

Sir Michael Atiyah, (Collected works. Vol. 6. Oxford Science Publications, 2004).

Thank you for your attention!