Normal form of a holomorphic lagrangian submanifold

Talk at Pipa conference

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December 12, 2023

Set-up:

X is a holomorphic symplectic manifold (not necessarily compact or algebraic), that is, X is a complex manifold with a closed holomorphic 2-form σ which is non-degenerate at every point. Consequently, $dim(X) = 2n$ is even and the canonical class $K_X = 0$.

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Weinstein's neighbourhood theorem: A neighbourhood U of Z in X is symplectomorphic to a neighbourhood of the zero section of Z in its cotangent bundle.

Main Result

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We assume that Z is contractible, that is, there exists a proper holomorphic map $f : X \to X'$ which is an isomorphism outside Z, and $f(Z)$ is a point.

By Grauert-Riemenschneider theorem, $R^i f_* \mathcal{O}(K_X) = 0$ for $i \geq 1$, so $R^i f_* \mathcal{O}_X = 0$ for $i \geq 1$: X' has a rational singularity.

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MAIN RESULT: $Z \cong \mathbb{P}^n$. Moreover, a neighbourhood of Z in X is holomorphically symplectomorphic to a neighbourhood of the zero section in T^*Z with its standard symplectic structure.

Remarks

REMARKS 1. $\mathbb{P}^n \subset X$ is always contractible: indeed it is lagrangian so $N^*_{\mathbb{P}^n/X} \cong T\mathbb{P}^n$ is ample. Grauert 1962: a compact submanifold with ample conormal bundle is contractible. In the same paper, Grauert remarks that a contractible submanifold does not need to have ample conormal bundle.

2. $Z \cong \mathbb{P}^n$ is well-known when X is projective.

3. Once we know that $Z \cong \mathbb{P}^n$, Grauert's criterion implies that a neighbourhood of Z in X is biholomorphic to a neighbourhood of the zero section in T^*Z.

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Grauert's conditions:

$$H^1(Z, Sym^r N^*_{Z/X}) = H^1(Z, TZ \otimes Sym^r N^*_{Z/X}) = 0 \ \forall r > 0$$

Easy to check because when Z is lagrangian, the symplectic form induces an isomorphism $N^*_{Z/X} \cong TZ$ (see e.g. Cho–Miyaoka–Shepherd-Barron)

Reminder about the proof in the projective case (Kebekus, C-M-SB)

Step 1. By MMP, Z is covered by rational curves (Kawamata, Nakayama).

Step 2. Estimate for the dimension of the parameter space T of maps $f : \mathbb{P}^1 \to X^{2n}$, $K_X = 0$:

$$dim(T) \geq \chi(\mathbb{P}^1, f^* TX) = 0 - 2n(0-1) = 2n$$

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Z. Ran '85: when X is holomorphic symplectic, the dimension is at least $2n - 2$!

Remark: We don't even need that X is projective here.

Projective case (end). Goal

Step 3. Since Z is contractible, the deformations of the rational curves in Z remain in Z.
Z (n-dimensional) has really many rational curves ($\geq 2n - 2$-dimensional families)!

Take a family of minimal rational curves (i.e. such that through a general point, all irreducible). There is an $n - 1$-dimensional family through a general point $\implies Z \cong \mathbb{P}^n$ (Kebekus).

TO DO TODAY

1) Prove that $Z \cong \mathbb{P}^n$ in general (X not necessarily compact or algebraic);

2) Prove that a neighbourhood of Z is holomorphically symplectomorphic to a neighbourhood of the zero section in $T^*\mathbb{P}^n$.

Grauert's question

OBSERVATION (Grauert 62): if Z is contractible, $N^*_{Z/X}$ does not have to be ample (even if Z is a divisor), nor satisfy "weak positivity" (that is, the zero-section is not necessarily contractible in the total space of $N^*_{Z/X}$).

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QUESTION (Grauert 62): If $Z \subset X$ is contractible, is it true that there is an ideal $J \subset \mathcal{O}_X$, such that $Supp(\mathcal{O}_X/J) = Z$, and J/J^2 is ample?

In other words, if Z is contractible, is there *some* scheme/analytic space structure on Z such that its conormal sheaf is positive? This question has been answered in the affirmative by Ancona and Vo Van Tan (around 1980).

Positivity properties of coherent sheaves

The following results are due to Vo Van Tan and Ancona:

THEOREM 1 The following properties are equivalent for a coherent sheaf $\mathcal{A}$ on a compact irreducible complex analytic space Y:

a) Ampleness: for any coherent sheaf $\mathcal{F}$ and $k \geq k_0(\mathcal{F})$, $S^k\mathcal{A} \otimes \mathcal{F}$ is globally generated;

b) Cohomological positivity: for any coherent sheaf $\mathcal{F}$ and $k \geq k_0(\mathcal{F})$, $H^i(Y, S^k\mathcal{A} \otimes F)$ is globally generated;

c) Weak positivity: The zero section of the relative spectrum of $\sum_k S^k(\mathcal{A})$ is contractible.

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c) Weak positivity: The zero section of the relative spectrum of $\sum_k S^k(\mathcal{A})$ is contractible.

THEOREM 2 There exists a torsion-free ample coherent sheaf on Y if and only if Y is Moishezon.

(Compare to Grauert's result that Y has a positive vector bundle if and only if Y is projective.)

Contractibility

THEOREM 3, answer to Grauert's question A compact subvariety $Z \subset X$ in a complex analytic space is contractible if and only if J/J^2 is positive for some coherent ideal sheaf defining Z as a set.

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Once we know this, there are two ways to prove that $Z \cong \mathbb{P}^n$.
WAY 1: BACK TO MMP

THEOREM (Shokurov, Villalobos-Paz) Let Z be Moishezon, then either Z is projective, or Z has rational curves.

If Z has rational curves, go back to steps 1-3 (note that Kebekus' arguments carry over verbatim to Moishezon case).

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CLAIM If $f : X \to X'$ is a contraction of a projective Z to a point p, then some ample line bundle on Z extends together with its sections to a neighbourhood $f^{-1}(U)$, $p \in U$.

Hence f is projective and one gets rational curves on Z from MMP.

Uniruledness from Campana-Paun

Another approach (our original one): denote by J_Z the ideal sheaf of Z, and let J be the ideal sheaf obtained from contractibility. Then $J \subset J_Z$. Let k be the maximal number such that $J \subset J_Z^k$. The natural map $J \subset J_Z^k / J_Z^{k+1}$ is zero on J^2 and so induces a non-trivial morphism

$$J/J^2 \to S^k(J_Z/J_Z^2) = S^k(N^*_{Z/X}) = S^k(TZ).$$

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Define a movable curve as a member of a dominating family of irreducible curves.

THEOREM (Campana-Paun) If $S^k TZ$ has a subsheaf E which is of positive degree on a movable curve, then Z is uniruled.

Remark Campana and Paun formulate the theorem in the projective setting but stress its birational nature.

Moser's isotopy lemma

We turn to the question about normal forms. One way to construct symplectomorphisms is the following result by Moser.

LEMMA (Moser) Let ω_t be a smooth family of symplectic forms on a compact manifold M. If the class of ω_t in $H^2(M)$ is constant, there is a flow of diffeomorphisms ϕ_t, $\phi_t^*\omega_0 = \omega_t$.

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Verbitsky and Soldatenkov have discovered that Moser's lemma has a holomorphic analogue. Their point is to consider C-symplectic forms on a smooth $4n$-manifold M: there are 2-forms Ω such that $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega^n}$ is a volume form.

THEOREM (Bogomolov-Deev-Verbitsky) The kernel of Ω defines a complex structure on M. In this complex structure, Ω is holomorphic symplectic.

Holomorphic Moser lemma

Holomorphic Moser lemma needs one extra assumption.

THEOREM (Verbitsky-Soldatenkov) Let Ω_t be a family of C-symplectic form on M, with constant cohomology class, and I_t the corresponding complex structures. Assume $H^1(M, \mathcal{O}_{I_t}) = 0$ for all t. Then there exists a family of holomorphic diffeomorphisms ϕ_t, $\phi_t^* \Omega_0 = \Omega_t$.

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We have an adaptation to a family of neighbourhoods of lagrangian submanifolds.

THEOREM (Soldatenkov-Verbitsky) In the situation of the preceding theorem, take M not necessarily compact and let $E_t \subset (M, I_t)$ be a family of lagrangian submanifolds. Then E_t have open neighbourhoods U_t such that (U_t, Ω_t, E_t) is trivialized by a flow of holomorphic diffeomorphisms.

Deformation to the normal cone

Note that the condition on cohomologies is satisfied when M is a small neighbourhood of a lagrangian $\mathbb{P}^n$ (note that this $\mathbb{P}^n$ is always contractible, then use Grauert-Riemenschneider).
So to prove the "holomorphic Weinstein theorem", we just need to construct a family of holomorphic symplectic manifolds including a neighbourhood of Z in X and a neighbourhood of the zero section in $T^*Z = N_{Z/X}$ as members. This is a classical construction called deformation to the normal cone.

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Consider $M = X \times \Delta$ where Δ is the unit disk, and let M' be its blow-up up along $Z \times 0$. The fiber of this blow-up over 0 has two components, the proper preimage D_1 of $X \times 0$, and the exceptional D_2. It is easy to see that $D_2 \backslash (D_1 \cap D_2)$ is isomorphic to the total space of $N_{Z/X}$. So discarding D_1 from the blow-up gives the desired deformation $\tilde{M} \to \Delta$, and we conclude with the following

PROPOSITION: If t is the coordinate on Δ and σ the symplectic form on X, σ/t is a holomorphic form on the central fiber.