# Log canonical thresholds on hypersurfaces



2001 Sb. Math. 192 1241

(http://iopscience.iop.org/1064-5616/192/8/A08)

*Sbornik*: *Mathematics* **192**:8 1241–1257

*Matematicheskiĭ Sbornik* **192**:8 155–172
DOI 10.1070/SM2001v192n08ABEH000591

# Log canonical thresholds on hypersurfaces

I. A. Cheltsov

**Abstract.** A lower bound for global log canonical thresholds on smooth hypersurfaces is found. This bound cannot be improved for the fixed degree and dimension of the hypersurface.

Bibliography: 12 titles.

## § 1. Introduction

In this paper we generalize results obtained in [1].

Let $X$ be an arbitrary smooth hypersurface of degree $K$ in $\mathbb{P}^N$ for $N \neq 1$ and let

$$(X, B_X) = \left(X, \sum_{i=0}^{n} b_i B_i\right)$$

be a log pair on $X$, where all the $b_i$ are real positive numbers and all the $B_i$ are irreducible reduced effective divisors on $X$.

All varieties in this paper are complex projective and all boundaries are non-trivial.

**Definition 1.1.** The largest real number $\lambda$ such that the singularities of the log pair $(X, \lambda B_X)$ are log canonical is called the *log canonical threshold* of the log pair $(X, B_X)$ and denoted by $\lambda(X, B_X)$.

Assume that for some $r \in \mathbb{R}_{>0}$,

$$B_X \equiv rH, \tag{1.1}$$

where $H$ is a hyperplane section of the hypersurface $X$.

*Remark* 1.2. The Picard group of the hypersurface $X$ is $\mathbb{Z}$ for $N \geqslant 4$. Hence condition (1.1) holds for all log pairs on $X$ if $N \neq 3$.

The main result of this paper is as follows.

**Theorem 1.3.** *The inequality*

$$\lambda(X, B_X) \geqslant \min\left(\frac{N-1}{rK}, \frac{1}{r}\right)$$

*holds.*

---

*AMS* 2000 *Mathematics Subject Classification.* Primary 14J17; Secondary 14B05,14E30.

We shall prove Theorem 1.3 in § 3 and apply Theorem 1.3 in § 7 to obtain a similar result for canonical thresholds.

Note that for all possible values of the degree $K$ and dimension $N$ one can always find a smooth hypersurface with boundary satisfying condition (1.1) such that the inequality in Theorem 1.3 becomes an equality.

**Example 1.4.** Let $X \subset \mathbb{P}^N$ be a hypersurface of degree $K$ described by the equation

$$x_0^K = \sum_{i=1}^{N} x_i^K$$

and let $B_X$ be equal to $D$, the hyperplane section $x_0 = x_1$ of $X$. Then we have the equality $\lambda(X, B_X) = \min((N-1)/K, 1)$.

Example 1.4 enables us to expect the following result.

**Conjecture 1.5.** *The equality* $\lambda(X, B_X) = \min((N-1)/(rK), 1/r)$ *holds if and only if*

$$B_X = \begin{cases} rS, & \text{where } S \text{ is hyperplane section} \\ & \text{that is a cone of degree } K \geqslant N, \\ rS, & \text{where } S \text{ is a hyperplane section} \\ & \text{of } X \text{ and } N > K, \\ rS + \Sigma, & \text{where } S \text{ is a subvariety of } X, \\ & \Sigma \text{ is a boundary in } X,\ N \leqslant 3, \text{ and } N > K. \end{cases}$$

In this paper we prove Conjecture 1.5 in the following cases: in § 4 for $N = 3$ and $K \geqslant N$, in § 5 for $N = 4$ and $K \geqslant N$, in § 6 for all $N > K$.

*Remark* 1.6. The results of § 5 show that if $N > 4$ and $K \geqslant N$, then to prove Conjecture 1.5 one may assume that $B_X = D$ for some hyperplane section $D$ of the hypersurface $X$.

*Remark* 1.7. It is possible to generalize the methods of § 5 to higher-dimensional cases by making essential use of the Log Minimal Model Program.

In the next section we consider several concepts and preliminary results used in the proof of Theorem 1.3 and several cases of Conjecture 1.5.

## § 2. Locus of log canonical singularities

In this section we consider properties of the so-called locus of log canonical singularities introduced originally by Shokurov.

We fix a log pair

$$(X, B_X) = \left(X, \sum_{i=1}^{n} b_i B_i\right),$$

where all the $b_i$ are real positive numbers and all the $B_i$ are irreduced and reducible effective Weil divisors on $X$. Assume also that the divisor $K_X + B_X$ is $\mathbb{R}$-Cartier.

**Definition 2.1.** A proper irreducible subvariety $Y \subset X$ is a *centre of the log canonical singularities* of a log pair $(X, B_X)$ if there exists a birational morphism $f\colon W \to X$ and a divisor $E$ on $W$ such that $f(E) = Y$ and $a(X, B_X, E) \leqslant -1$.

**Example 2.2.** Let $X \cong \mathbb{P}^2$ and let $B_X = C$ be a reduced cubic curve in $\mathbb{P}^2$ with unique singular point $O$. Then both $O$ and $C$ are centres of log canonical singularities of the log pair $(X, B_X)$.

A centre of the log canonical singularities is of a local nature. Nevertheless, we can consider its global analogue.

**Definition 2.3.** We denote by $\mathrm{LCS}(X, B_X)$ the set of centres of the log canonical singularities of a log pair $(X, B_X)$.

On the set $\mathrm{LCS}(X, B_X)$ we can introduce a scheme structure. To this end we consider some birational morphism $f\colon W \to X$ such that $W$ is smooth and the union of all strict transforms of the divisors $B_i$ on $W$ and all $f$-exceptional divisors makes up a divisor with simple normal crossing singularities. Let $(W, B^W)$ be a log pair on $W$ such that $f(B^W) = B_X$ and

$$K_W + B^W \sim_{\mathbb{Q}} f^*(K_X + B_X).$$

*Remark* 2.4. Usually, a birational morphism $f$ is called a log resolution of the log pair $(X, B_X)$ and the log pair $(W, B^W)$ is called a log pull back of the log pair $(X, B_X)$.

**Definition 2.5.** The subscheme associated with the ideal sheaf

$$\mathcal{I}(X, B_X) = f_*(\lceil -B^Y \rceil),$$

is called the *log canonical singularity subscheme* of the log pair $(X, B_X)$ and denoted by $\mathcal{L}(X, B_X)$.

We point out also the obvious fact that each centre of log canonical singularities of a log pair $(X, B_X)$ lies in the support of the subscheme $\mathcal{L}(X, B_X)$.

**Definition 2.6.** The support of the subscheme $\mathcal{L}(X, B_X)$ is called the *locus of log canonical singularities* of the log pair $(X, B_X)$ and denoted by $\mathrm{LCS}(X, B_X)$.

Note that there is a slight ambiguity in our regarding $\mathrm{LCS}(X, B_X)$ as a subvariety and a set of subvarieties at the same time. We hope that this will not lead to confusion.

The next result is known as the Shokurov vanishing.

**Theorem 2.7.** *Let $H$ be a nef and big divisor on $X$ such that $K_X + B_X + H$ is numerically equivalent to a Cartier divisor $D$. Then $H^i(X, \mathcal{I}(X, B_X) \otimes D) = 0$ for all $i > 0$.*

*Proof.* By the Kawamata–Vieweg vanishing, for $i > 0$,

$$R^i f_*(f^*(D) + \lceil -B^W \rceil) = 0.$$

Hence the degeneration of the spectral sequence

$$E_2^{p,q} = H^p(R^q f_*(f^*(D) + \lceil -B^W \rceil)) \implies E^{p,q} = H^{p+q}(f_*(f^*(D) + \lceil -B^W \rceil))$$

and the equality

$$f_*(f^*(D) + \lceil -B^W \rceil) = \mathcal{I}(X, B_X) \otimes D$$

yield for all $i$ the relation

$$H^i(X, \mathcal{I}(X, B_X) \otimes D) = H^i(W, f^*(D) + \lceil -B^W \rceil).$$

On the other hand, for all $i > 0$,

$$H^i(W, f^*(D) + \lceil -B^W \rceil) = 0$$

by the Kawamata–Vieweg vanishing.

For each Cartier divisor $D$ on $X$ we have the exact sequence

$$0 \to \mathcal{I}(X, B_X) \otimes D \to \mathcal{O}_X(D) \to \mathcal{O}_{\mathcal{L}(X,B_X)}(D) \to 0.$$

Applying Theorem 2.7 we arrive at the following result, which is also well known as the Shokurov connectedness theorem.

**Theorem 2.8.** *Suppose that the divisor $-(K_X + B_X)$ is nef and big. Then the locus of canonical singularities* $\mathrm{LCS}(X, B_X)$ *is connected.*

The next statement is a relative version of Theorem 2.8.

**Theorem 2.9.** *Let $g\colon X \to Z$ be a morphism with connected fibres such that the divisor $-(K_X + B_X)$ is $g$-nef and $g$-big. Then the locus* $\mathrm{LCS}(X, B_X)$ *is connected in the neighbourhood of each fibre of $g$.*

One application of Theorem 2.9 is an inductive connection between centres of log canonical singularities and so-called centres of canonical singularities.

**Definition 2.10.** A proper irreducible subvariety $Y \subset X$ is a *centre of canonical singularities* of a log pair $(X, B_X)$ if there exists a birational morphism $f\colon W \to X$ and an $f$-exceptional divisor $E$ on $W$ such that $f(E) = Y$ and $a(X, M_X, E) \leqslant 0$.

In § 7 we describe an application of Theorem 1.3 to finding lower bounds for canonical thresholds on smooth hypersurfaces. To this end we shall require the following result, a special case of the so-called *Inverse of adjunction*.

**Theorem 2.11.** *Let $Z$ be a centre of canonical singularities of a log pair $(X, B_X)$, and $H$ an effective irreducible and smooth Cartier divisor on $X$ containing $Z$ and not lying in the support of the boundary $B_X$. Then* $\mathrm{LCS}(H, B_X|_H) \neq \varnothing$.

*Proof.* Let $f\colon W \to X$ be a log resolution of $(X, B_X + H)$ and let $\widehat{H} = f^{-1}(H)$. Then on $W$ we have

$$K_W + \widehat{H} \sim_{\mathbb{Q}} f^*(K_X + B_X + H) + \sum_{E \neq \widehat{H}} a(X, B_X + H, E)E.$$

Note that

$$\{Z, H\} \subset \mathrm{LCS}(X, B_X + H).$$

Hence applying Theorem 2.9 to the log pullback of the log pair $(X, B_X + H)$ on $W$ we obtain

$$\widehat{H} \cap E \neq \varnothing$$

for some divisor $E \neq \widehat{H}$ on $W$ such that $a(X, B_X, E) \leqslant -1$. Our claim now follows from the relations

$$K_{\widehat{H}} \sim (K_W + \widehat{H})|_{\widehat{H}} \sim_{\mathbb{Q}} f|_{\widehat{H}}^{*}(K_H + B_X|_H) + \sum_{E \neq \widehat{H}} a(X, B_X + H, E)E|_{\widehat{H}}.$$

## § 3. Proof of the main result

In this section we prove Theorem 1.3.

Consider a smooth hypersurface $X$ of degree $K$ in $\mathbb{P}^N$, where $N \neq 1$, and a log pair $(X, B_X)$ such that $B_X \equiv H$, where $H$ is a hyperplane section of $X$. Let $\mu$ be an arbitrary real positive number that is smaller than $\min((N-1)/K, 1)$.

**Lemma 3.1.** *The singularities of the log pair $(X, \mu B_X)$ are log terminal.*

Note that Lemma 3.1 yields the inequality

$$\lambda(X, B_X) \geqslant \min\biggl(\frac{N-1}{K}, 1\biggr).$$

*Remark* 3.2. Theorem 1.3 follows from Lemma 3.1.

We shall use the following result of Pukhlikov (see [2]).

**Statement 3.3.** *For each curve $C$ on the hypersurface $X$ the inequality $\operatorname{mult}_C(B_X) \leqslant r$ holds.*

*Proof.* We consider a sufficiently general cone $R_C$ over the curve $C$. Then

$$R_C \cap X = C \cup \widetilde{C},$$

where $\widetilde{C}$ is some curve on $X$ of degree $(K-1)\deg(C)$. The generality of the cone $R_C$ means that the intersection $C \cap \widetilde{C}$ is transversal and consists of precisely $(K-1)\deg(C)$ distinct points (see [2]). On the other hand, the curve $\widetilde{C}$ does not lie in the support of the boundary $B_X$ for the same reasons. Thus,

$$(K-1)\deg(C)\operatorname{mult}_C(B_X) \leqslant B_X \cdot \widetilde{C} = rH \cdot \deg(\widetilde{C}) = r(K-1)\deg(S).$$

Hence $\operatorname{mult}_C(B_X) \leqslant r$.

Note that Statement 3.3 yields the following result.

**Corollary 3.4.** *A subvariety of $X$ having positive dimension cannot be a centre of log canonical singularities of the log pair $(X, \mu B_X)$.*

*Proof of Lemma* 3.1. Assume that the singularities of the log pair $(X, \mu B_X)$ are worse than log terminal. We shall show that this assumption leads to a contradiction.

This assumption and Corollary 3.4 yield the existence of a point $O$ in $X$ such that $O$ is a centre of log canonical singularities of $(X, \mu B_X)$. We choose a projection $\gamma\colon X \to \mathbb{P}^{N-1}$ such that the morphism $\gamma$ is étale in a neighbourhood of $O$ and for all $B_i$ containing $O$ the restrictions of $\gamma$

$$\gamma|_{B_i}\colon B_i \to \gamma(B_i)$$

are one-to-one in a neighbourhood of $O \in B_i$. Let

$$(\mathbb{P}^{N-1}, B_{\mathbb{P}^{N-1}}) = \biggl(\mathbb{P}^{N-1}, \sum_{i=0}^{n} b_i \gamma(B_i)\biggr).$$

Then the point $\gamma(O)$ is an isolated centre of log canonical singularities of the log pair $(\mathbb{P}^{N-1}, \mu B_{\mathbb{P}^{N-1}})$.

Let $\mathcal{L}$ be a log canonical subscheme of the log pair $(\mathbb{P}^{N-1}, \mu B_{\mathbb{P}^{N-1}})$, let $\mathcal{I}$ be the ideal sheaf associated with the subscheme $\mathcal{L}$, and $D$ a Cartier divisor on $\mathbb{P}^{N-1}$. Then the exact sequence of sheaves

$$0 \to \mathcal{I} \otimes D \to \mathcal{O}_{\mathbb{P}^{N-1}}(D) \to \mathcal{O}_{\mathcal{L}}(D) \to 0$$

induces the exact sequence of groups

$$0 \to H^0(\mathcal{I} \otimes D) \to H^0(\mathcal{O}_{\mathbb{P}^{N-1}}(D)) \to H^0(\mathcal{O}_{\mathcal{L}}(D)) \to H^1(\mathcal{I} \otimes D) \to \cdots.$$

In addition, $H^0(\mathcal{L}(D)) \neq 0$ because $\gamma(O)$ is the support of one isolated component of the subscheme $\mathcal{L}$.

Now let $D = \mathcal{O}_{\mathbb{P}^{N-1}}(-1)$. Then the inequality $\mu < \min((N-1)/K, 1)$ shows that for some ample divisor $L$ on $\mathbb{P}^{N-1}$,

$$D \equiv K_{\mathbb{P}^{N-1}} + \mu B_{\mathbb{P}^{N-1}} + L.$$

Thus, it follows from Theorem 2.7 that the map

$$H^0(\mathcal{O}_{\mathbb{P}^{N-1}}(D)) \to H^0(\mathcal{O}_{\mathcal{L}}(D))$$

is surjective. Hence $H^0(\mathcal{O}_{\mathbb{P}^{N-1}}(-1)) \neq 0$.

## § 4. Lines on smooth surfaces in $\mathbb{P}^3$

In this section we prove Conjecture 1.5 for surfaces of degree greater than 2.

Let $X$ be a smooth surface of degree $K \geqslant 3$ in $\mathbb{P}^3$ and let $(X, B_X)$ be a log pair on $X$ such that

$$B_X = \sum_{i=0}^{n} b_i B_i \equiv rH,$$

where $b_i \in \mathbb{R}_{>0}$, all the curves $B_i$ are irreducible and reduced, $r \in \mathbb{R}_{>0}$, and $H$ is a hyperplane section of $X$.

**Theorem 4.1.** *The equality* $\lambda(X,B_X)=2/(rK)$ *holds if and only if* $B_X=r\sum_{i=1}^{K}L_i$, *where the* $L_i$ *are distinct lines on* $X$ *passing through some point* $O$ *in* $X$.

*Remark* 4.2. Obviously, $\lambda(X,B_X)=2/(rK)$ in the case when the boundary $B_X$ is a multiple of a sum of $K$ lines on $X$ passing through some common point.

Thus, we only have to prove the "only if" part of Theorem 4.1. Assume that the singularities of the log pair $(X,(2/(rK))B_X)$ are not terminal. We claim that in this case the boundary $B_X$ is proportional to a sum of $K$ lines on $X$ passing through some common point.

*Remark* 4.3. It follows from Corollary 3.4 that a curve on $X$ cannot be a centre of log canonical singularities of the log pair $(X,(2/(rK))B_X)$.

Hence there exists a point $O$ on the hypersurface $X$ that is a centre of log canonical singularities of the log pair $(X,(2/(rK))B_X)$.

**Lemma 4.4.** *All the curves* $B_i$ *are components of* $X\cap T$, *where* $T$ *is the hyperplane in* $\mathbb{P}^3$ *tangent to* $X$ *at the point* $O$.

*Proof.* Arguing by contradiction assume that some curve $B_j$ is not a component of $X\cap T$. Then we can find a projection $\gamma\colon X\dashrightarrow\mathbb{P}^2$ from some point in $B_j$ such that the rational map $\gamma$ is étale in a neighbourhood of $O$ and for all $B_i$ containing $O$ the restrictions

$$\gamma|_{B_i}\colon B_i\dashrightarrow\gamma(B_i)$$

are one-to-one in a neighbourhood of $O$.

Let

$$B_{\mathbb{P}^2}=\sum_{i=0}^{n}b_i\gamma(B_i).$$

Then the point $\gamma(O)$ is an isolated centre of log canonical singularities of the log pair $(\mathbb{P}^2,(2/(rK))B_{\mathbb{P}^2})$ and

$$\mathcal{O}_{\mathbb{P}^2}(-1)\equiv K_{\mathbb{P}^2}+\frac{2}{rK}B_{\mathbb{P}^2}+L,$$

where $L$ is some ample divisor on $\mathbb{P}^2$.

Let $\mathcal{L}$ be the log canonical singularity subscheme of $(\mathbb{P}^2,(2/(rK))B_{\mathbb{P}^2})$, and $\mathcal{I}$ the ideal sheaf associated with the subscheme $\mathcal{L}$. Then the exact sequence

$$0\to\mathcal{I}(-1)\to\mathcal{O}_{\mathbb{P}^2}(-1)\to\mathcal{O}_{\mathcal{L}}(-1)\to 0$$

gives us an exact sequence

$$0\to H^0(\mathcal{O}_{\mathcal{L}}(-1))\to H^1(\mathcal{I}(-1))\to\cdots$$

such that $H^0(\mathcal{L}(-1))=H^0(\mathcal{L})\neq 0$ because the point $\gamma(O)$ is the support of an isolated component of the subscheme $\mathcal{L}$. By Theorem 2.7, $H^1(\mathcal{I}(-1))=0$.

We have thus proved that all the $B_i$ are irreducible reduced components of $X\cap T$, where $T$ is the hyperplane in $\mathbb{P}^3$ tangent to $X$ at the point $O$. Note that by Statement 3.3,

$$T\cap X=\sum_{i=0}^{n}B_i.$$

**Lemma 4.5.** *For each $B_j$ the intersection form of the curves $\bigcup_{l\neq j} B_l$ on $X$ is seminegative-definite.*

*Proof.* Let $D=\sum_{l=0}^{n} B_l$. Then for each $B_i\neq B_j$ on the surface $X$,

$$(D-B_j)\cdot B_i=\deg(B_i)-B_j\cdot B_i,$$

while on the surface $T\cong\mathbb{P}^2$,

$$B_j\cdot B_i=\deg(B_j)\deg(B_i).$$

However, both $T$ and $X$ are smooth surfaces, therefore

$$(B_j\cdot B_i)_X=(B_j\cdot B_i)_T,$$

and on $X$ we have

$$\biggl(\sum_{l\neq j} B_l\biggr)\cdot B_i=(D-B_j)\cdot B_i=\deg(B_i)-B_j\cdot B_i=\deg(B_i)(1-\deg(B_j))\leqslant 0.$$

The required result follows now from [3].

We require Lemma 4.5 for the demonstration of the following property of the boundary $B_X$.

**Statement 4.6.** $B_X=r\sum_{i=0}^{n} B_i$.

*Proof.* By assumption, $B_X\equiv r\sum_{i=0}^{n} B_i$. Hence

$$\sum_{b_i>r}(b_i-r)B_i\equiv\sum_{b_l<r}(r-b_l)B_l.$$

Thus, if $b_i\neq r$, then

$$\biggl(\sum_{b_i>r}(b_i-r)B_i\biggr)^2=\sum_{b_i>r>b_l}(b_i-r)(r-b_l)B_i\cdot B_l>0.$$

However, it follows from Lemma 4.5 that $\bigl(\sum_{b_i>r}(b_i-r)B_i\bigr)^2$ is non-positive.

We now regard all the curves $B_i$ and the point $O$ as subvarieties of $T\cong\mathbb{P}^2$ and denote the boundary $\sum_{i=0}^{n} B_i$ by $S$. Note that $O$ is an isolated centre of log canonical singularities of the log pair $(\mathbb{P}^2,(2/K)S)$.

*Remark* 4.7. It follows from Theorem 1.3 that the log pair $(X,(2/(rK))B_X)$ has log canonical singularities. Hence the log pair $(\mathbb{P}^2,(2/K)S)$ also has log canonical singularities.

Let $\mathcal{L}$ be the log canonical singularity subscheme of $(\mathbb{P}^2,(2/K)S)$. Then the natural map

$$H^0(\mathcal{O}_{\mathbb{P}^2})\to H^0(\mathcal{O}_{\mathcal{L}})$$

is surjective by Theorem 2.7.

**Corollary 4.8.** *$O$ is the unique centre of log canonical singularities of* $(\mathbb{P}^2, (2/K)S)$.

Consider now a birational morphism $f\colon V \to \mathbb{P}^2$ such that $f$ is an isomorphism everywhere outside $O$, the surface $V$ is smooth, and

$$K_V + \frac{2}{K}\widetilde{S} \equiv f^*\left(K_{\mathbb{P}^2} + \frac{2}{K}S\right) - E + G,$$

where $\widetilde{S} = f^{-1}(S)$, $E$ and $G$ are effective $f$-exceptional divisors, and $\lfloor E \rfloor \neq 0$.

Note that the negativity of the intersection form of curves in the support of $G$ and the strict inequality $K_V \cdot G < 0$ mean that at least one component of $G$ can be contracted to a smooth point. Hence we can assume that $G = \varnothing$.

Let $C$ be an $f$-exceptional curve on $V$ that is not a component of the support of $E$ and has a non-trivial intersection with $E$. Then $K_V \cdot C < 0$. Hence $C$ can be contracted to a smooth point.

*Remark* 4.9. We can assume that the birational morphism $f\colon V \to \mathbb{P}^2$ is an isomorphism outside $O$, the surface $V$ is smooth, and

$$K_V + \frac{2}{K}\widetilde{S} \equiv f^*\left(K_{\mathbb{P}^2} + \frac{2}{K}S\right) - E,$$

where $E$ is an effective $f$-exceptional divisor on $V$ such that $\lfloor E \rfloor \neq 0$ and the support of $E$ contains all $f$-exceptional divisors on $V$.

The singularities of the log pair $(V, (2/K)\widetilde{S}+E)$ are log canonical and the divisor

$$K_V + \frac{2}{K}\widetilde{S} + E \equiv f^*(\mathcal{O}_{\mathbb{P}^2}(-1))$$

is not nef. Hence we can apply the Log Minimal Model Program to the log pair $(V, (2/K)\widetilde{S} + E)$, and therefore there exists a morphism $g\colon V \to W$ such that the divisor $-(K_V + (2/K)\widetilde{S} + E)$ is $g$-ample and $g$ is either a $\mathbb{P}^1$-bundle or a contraction of an irreducible smooth rational curve.

**Lemma 4.10.** *The curves $B_i$ are lines passing through $O$ if the morphism $g$ is a $\mathbb{P}^1$-bundle.*

*Proof.* Let $g$ be a $\mathbb{P}^1$-bundle and let $C$ be a sufficiently general fibre of $g$. Then

$$\left(K_V + \frac{2}{K}\widetilde{S}\right) \cdot C = (f^*(\mathcal{O}_{\mathbb{P}^2}(-1)) - E) \cdot C = -\deg(f(C)) - E \cdot C \leqslant -\deg(f(C)) - 1$$

because $\lfloor E \rfloor \cap C \neq \varnothing$. On the other hand,

$$\left(K_V + \frac{2}{K}\widetilde{S}\right) \cdot C = -2 + \frac{2}{K}\widetilde{S} \cdot C.$$

Hence $f(C)$ is a line on $\mathbb{P}^2$ passing through the point $O$ and all the components of $\widetilde{S}$ are fibres of the morphism $g$. In particular, all the curves $B_i$ are lines on $X$ passing through $O$.

Thus, we can assume that $g$ is a contraction of an irreducible smooth rational curve $C$.

**Lemma 4.11.** *The image $f(C)$ is a line on $\mathbb{P}^2$ passing through the point $O$, the curve $C$ is a component of $\widetilde{S}$, and either $C^2=-1$, or $K=3$ and $C^2=-2$.*

*Proof.* The required result follows from the inequalities

$$\left(K_V+\frac{2}{K}\widetilde{S}\right)\cdot C=-\deg(f(C))-E\cdot C<-\deg(f(C))$$

and

$$\left(K_V+\frac{2}{K}\widetilde{S}\right)\cdot C=-2-C^2+\frac{2}{K}\widetilde{S}\cdot C\geqslant\begin{cases}-2-C^2 & \text{for } C\not\subset\widetilde{S},\\ -2-\dfrac{K-2}{K}C^2 & \text{for } C\subset\widetilde{S}.\end{cases}$$

*Remark* 4.12. We shall assume in what follows that $K\geqslant 4$ because for $K=3$ the proof of Theorem 4.1 can be completed in an obvious way.

We set $\rho=f\circ g^{-1}$, $\overline{S}=g(\widetilde{S})$, $\overline{E}=g(E)$, and $\overline{D}=g(D)$, where $D$ is a sufficiently general irreducible reduced curve in the linear system $|f^*(\mathcal{O}_{\mathbb{P}^2}(1))|$.

*Remark* 4.13. The set $\{W,\rho,\overline{S},\overline{E},\overline{D}\}$ has the following properties:

$$\Lambda(W,\rho,\overline{S},\overline{E},\overline{D})=\begin{cases}W \text{ is a smooth surface}\\ K_W+\dfrac{2}{K}\overline{S}\equiv-\overline{D}-\overline{E};\\ \overline{D} \text{ is an irreducible reduced curve with } D^2>0;\\ \overline{D}\cdot Z\neq 0 \text{ for each irreducible curve on } W \text{ with } Z^2=-1;\\ \overline{D}\cdot Z\geqslant\deg(\rho(Z)) \text{ for each curve } Z \text{ on } W;\\ \left(W,\dfrac{2}{K}\overline{S}\right) \text{ has log canonical singularities;}\\ \overline{S} \text{ is a reduced curve;}\\ Z\cap\overline{E}\neq\varnothing \text{ for each irreducible curve } Z\subset W\\ \text{with } Z^2<0;\\ \overline{E} \text{ is an effective divisor, } \lfloor\overline{E}\rfloor\neq\varnothing, \text{ and } f\circ g^{-1}(E)=O;\\ \text{the support of } \overline{E} \text{ contains all } \rho\text{-exceptional curves on } W;\\ \text{the support of } \overline{E} \text{ does not contain components of } \overline{S};\\ Z^2<-1 \text{ for each irreducible curve } Z \text{ in the support of } \lfloor\overline{E}\rfloor.\end{cases}$$

The properties $\Lambda(W,\rho,\overline{S},\overline{E},\overline{D})$ ensure the existence of a morphism $h\colon W\to U$ such that the divisor $-(K_W+(2/K)\overline{S})$ is $h$-ample and $h$ is either a $\mathbb{P}^1$-bundle or a contraction of a smooth rational curve.

**Lemma 4.14.** *The set of properties $\Lambda(W,\rho,\overline{S},\overline{E},\overline{D})$ ensures that in the case when $h$ is not birational all components of $S$ are lines on $\mathbb{P}^2$ passing through the point $O$.*

The assertion of the lemma follows easily from the proof of Lemma 4.10 and the properties $\Lambda(W,\rho,\overline{S},\overline{E},\overline{D})$.

**Lemma 4.15.** *Suppose that a morphism $h$ contracts one smooth rational curve $Z$. Then $Z^2 = -1$, $Z \cap \lfloor E \rfloor = \varnothing$, and either $Z$ lies in the support of $\overline{E}$ or $Z$ is a component of $\overline{S}$ and $f \circ g^{-1}(Z)$ is a line passing through the point $O$.*

*Proof.* In the case when $Z$ does not lie in the support of $\overline{E}$,

$$\left(K_W + \frac{2}{K}\overline{S}\right) \cdot Z \leqslant -\deg(f \circ g^{-1}(Z)) - \overline{E} \cdot Z$$

and

$$\left(K_W + \frac{2}{K}\overline{S}\right) \cdot Z = -2 - Z^2 + \frac{2}{K}\overline{S} \cdot Z \geqslant \begin{cases} -2 - Z^2 & \text{for } Z \not\subset \overline{S}, \\ -2 - \dfrac{K-2}{K}Z^2 & \text{for } Z \subset \overline{S}. \end{cases}$$

Hence $Z \cap \lfloor \overline{E} \rfloor = \varnothing$, $f \circ g^{-1}(Z)$ is a line on $\mathbb{P}^2$ passing through $O$, and the curve $Z$ is a component of $\overline{S}$ such that $Z^2 = -1$.

In the case when $Z$ lies in the support of the divisor $\overline{E}$,

$$\left(K_W + \frac{2}{K}\overline{S}\right) \cdot Z = -2 - Z^2 + \frac{2}{K}\overline{S} \cdot Z \geqslant -2 - Z^2.$$

Hence $Z^2 = -1$ and

$$\left(K_W + \frac{2}{K}\overline{S}\right) \cdot Z \geqslant -1.$$

On the other hand,

$$\left(K_W + \frac{2}{K}\overline{S}\right) \cdot Z = -\overline{D} \cdot Z - \lfloor \overline{E} \rfloor \cdot Z - \{\overline{E}\} \cdot Z < -1 - \lfloor \overline{E} \rfloor \cdot Z - Z^2 = -\lfloor \overline{E} \rfloor \cdot Z.$$

Hence in this case the inequality $Z \cap \lfloor E \rfloor \neq \varnothing$ leads to a contradiction.

Thus, the set of properties $\Lambda(W, \rho, \overline{S}, \overline{E}, \overline{D})$ ensures the existence of a morphism $h \colon W \to U$ such that $h$ is either a $\mathbb{P}^1$-bundle or birational. Moreover, Lemma 4.15 asserts that Theorem 4.1 follows from the properties $\Lambda(W, \rho, \overline{S}, \overline{E}, \overline{D})$ in the case when $h$ is not birational. On the other hand, if $h$ is birational, then Lemma 4.16 asserts that the properties $\Lambda(W, \rho, \overline{S}, \overline{E}, \overline{D})$ ensure the properties

$$\Lambda(U, \rho \circ h^{-1}, h(\overline{S}), h(\overline{E}), h(\overline{D}))$$

of the set $\{U, \rho \circ h^{-1}, h(\overline{S}), h(\overline{E}), h(\overline{D})\}$. In other words, birational morphisms induced by the properties $\Lambda$ preserve $\Lambda$. Hence we can prove Theorem 4.1 by repeating our construction finitely many times.

## § 5. Cones on smooth 3-folds in $\mathbb{P}^4$

In this section we prove Conjecture 1.5 for 3-folds of degree not lower than 4.

For a smooth hypersurface $X$ of degree $K$ in $\mathbb{P}^4$ with $K \geqslant 4$ we consider a log pair

$$(X, B_X) = \left(X, \sum_{i=0}^{n} b_i B_i\right)$$

such that $b_i \in \mathbb{R}_{>0}$ and all the divisors $B_i$ are irreducible and reduced.

*Remark* 5.1. It follows from the equality $\operatorname{Pic}(X) = \mathbb{Z}$ that $B_X \equiv rH$ for some $r \in \mathbb{R}_{>0}$, where $H$ is a hyperplane section of $X$.

**Theorem 5.2.** *The equality $\lambda(X, B_X) = 3/(rK)$ holds if and only if $B_X = rS$, where $S$ is a hyperplane section of $X$ that is a cone over a smooth plane curve of degree $K$.*

Note that the "if" part of Theorem 5.2 is trivial. Hence to prove the "only if" part of the theorem we assume that

$$\lambda(X, B_X) = \frac{3}{rK}\,.$$

In particular, the singularities of the log pair $(X, (3/(rK))B_X)$ are not log terminal. We merely need to show that in this case $B_X = rS$, where $S$ is a cone, because the other properties of $S$ are consequences of Statement 3.3.

*Remark* 5.3. It follows from Statement 3.3 that each centre of log canonical singularities of the log pair $(X, (3/(rK))B_X)$ is a point in $X$.

We can thus assume that there exists a point $O$ in $X$ that is an isolated centre of log canonical singularities of the log pair $(X, (3/(rK))B_X)$.

**Lemma 5.4.** *$B_X = rS$, where $S$ is a hyperplane section of $X$ that is singular at the point $O$.*

*Proof.* We choose a projection $\gamma\colon X \dashrightarrow \mathbb{P}^3$ such that the rational map $\gamma$ is étale in a neighbourhood of $O$ and for all divisors $B_i$ containing $O$ the restrictions

$$\gamma|_{B_i} : B_i \dashrightarrow \gamma(B_i)$$

are one-to-one in a neighbourhood of $O \in B_i$. Let

$$B_{\mathbb{P}^3} = \sum_{i=0}^{n} b_i \gamma(B_i).$$

Then the point $\gamma(O)$ is an isolated centre of log canonical singularities of the log pair $(\mathbb{P}^3, (3/(rK))B_{\mathbb{P}^3})$.

Let $\mathcal{L}$ be the log canonical singularity subscheme of the pair $(\mathbb{P}^3, (3/(rK))B_{\mathbb{P}^3})$, let $\mathcal{I}$ be the ideal sheaf associated with $\mathcal{L}$, and $D$ a Cartier divisor on $\mathbb{P}^3$. Then the exact sequence

$$0 \to \mathcal{I} \otimes D \to \mathcal{O}_{\mathbb{P}^3}(D) \to \mathcal{O}_{\mathcal{L}}(D) \to 0$$

induces the exact sequence of groups

$$0 \to H^0(\mathcal{I} \otimes D) \to H^0(\mathcal{O}_{\mathbb{P}^3}(D)) \to H^0(\mathcal{O}_{\mathcal{L}}(D)) \to H^1(\mathcal{I} \otimes D) \to \cdots\,.$$

Note that the point $\gamma(O)$ is the support of an isolated component of $\mathcal{L}$. Hence

$$H^0(\mathcal{L}(D)) \neq 0.$$

Now let $D = \mathcal{O}_{\mathbb{P}^3}(-1)$. Then

$$D \equiv K_{\mathbb{P}^3} + \frac{3}{rK} B_{\mathbb{P}^3}$$

in the case when all the restrictions $\gamma|_{B_i}$ are morphisms. However, if some rational map $\gamma|_{B_i}$ is not a morphism, then

$$D \equiv K_{\mathbb{P}^3} + \frac{3}{rK} B_{\mathbb{P}^3} + L$$

where $L$ is an ample divisor on $\mathbb{P}^3$, and Theorem 2.7 shows that $H^1(\mathcal{I} \otimes D) = 0$ and the map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(D)) \to H^0(\mathcal{O}_{\mathcal{L}}(D))$$

is surjective, which contradicts the inequality $H^0(\mathcal{L}(D)) \neq 0$.

Let $S = X \cap T$, where $T$ is the hyperplane in $\mathbb{P}^4$ tangent to $X$ at $O$. Then $S$ is an irreducible sequence because $\mathrm{Pic}(X) = \mathbb{Z}$. Assume that $B_i \neq S$ for some $i$. Then we can choose $\gamma$ to be the projection from some point $P$ in $B_i$. For if $P \notin S$, then the projection from $P$ is étale in a neighbourhood of $O$ and we may choose $P$ in $B_i$ such that all restrictions $\gamma|_{B_i}$ are one-to-one in a neighbourhood of $O$ for all surfaces $B_i$ containing $O$. Hence our previous arguments and the relation $B_X \equiv rH$ show that $B_X = rS$.

We have thus proved that $B_X = rS$, where $S$ is a hyperplane section of $X$ singular at $O$.

*Remark* 5.5. It follows from Statement 3.3 that the singularities of $S$ are isolated. On the other hand $S$ has hypersurface singularities. Hence the surface $S$ is normal.

We now regard the surface $S$ and the point $O$ as subvarieties of $\mathbb{P}^3$. Then $O$ is an isolated centre of log canonical singularities of the log pair $(\mathbb{P}^3, (3/K)S)$. Moreover, the log pair $(\mathbb{P}^3, (3/K)S)$ has log canonical singularities by Theorem 1.3.

Let $\mathcal{L}$ be the log canonical singularities subscheme of the log pair $(\mathbb{P}^3, (3/K)S)$. Then the map

$$H^0(\mathcal{O}_{\mathbb{P}^3}) \to H^0(\mathcal{O}_{\mathcal{L}})$$

is surjective by Theorem 2.7.

**Corollary 5.6.** *The point $O$ is the unique centre of log canonical singularities of the log pair* $(\mathbb{P}^3, (3/K)S)$.

To complete the proof of Theorem 5.2 we log generalize the main idea of [4].

Let $h\colon Y \to \mathbb{P}^3$ be a log terminal modification of the log pair $(\mathbb{P}^3, (3/K)S)$ and let $t\colon V \to Y$ be a terminal modification of $Y$. Then the birational morphism $f = h \circ t$ is an isomorphism outside $O$, the 3-fold $V$ has terminal $\mathbb{Q}$-factorial singularities, and

$$K_V + \frac{3}{K}\widetilde{S} \equiv f^*\left(K_{\mathbb{P}^3} + \frac{3}{K}S\right) - E,$$

where $\widetilde{S} = f^{-1}(S)$. Note that the $f$-exceptional divisor $E$ is effective and its support coincides with the support of the exceptional locus of the birational morphism $f$. Moreover, $\lfloor E \rfloor \neq 0$.

The singularities of the log pair $(V, (3/K)\widetilde{S} + E)$ are log canonical. In particular, we can apply the Log Minimal Model Program to this log pair. Thus, the equivalence

$$K_V + \frac{3}{K}\widetilde{S} + E \equiv f^*(\mathcal{O}_{\mathbb{P}^3}(-1))$$

ensures the existence of an extremal contraction $g\colon V\to W$ such that the divisor $-(K_V+(3/K)\widetilde{S}+E)$ is $g$-ample. Note that $W$ is not a point.

*Remark* 5.7. There exist no curves on $V$ contractible by both $g$ and $f$.

**Lemma 5.8.** *The extremal contraction $g$ is neither a del Pezzo fibration nor a contraction of a divisor to a point.*

*Proof.* Assume that $g$ is either a del Pezzo fibration or a contraction of a divisor to a point. Then $g$ maps some surface $F$ on $V$ into a point in $W$ and

$$F\cap E\neq\varnothing.$$

By the $\mathbb{Q}$-factoriality of $V$ the intersection $F\cap E$ contains a curve contracted by both morphisms, $f$ and $g$.

**Lemma 5.9.** *The morphism $g$ is not a small contraction.*

*Proof.* Assume that $g$ is a small contraction and let $C$ be a curve contracted by $g$. Then it is well known that $K_V\cdot C>-1$ (see [5]). On the other hand,

$$\left(K_V+\frac{3}{K}\widetilde{S}\right)\cdot C=(f^*(\mathcal{O}_{\mathbb{P}^3}(-1))-E)\cdot C<-1.$$

Hence $C\subset\widetilde{S}$ and $\widetilde{S}\cdot C<0$. The last inequality yields

$$(K_V+\widetilde{S})\cdot C<\left(K_V+\frac{3}{K}\widetilde{S}\right)\cdot C.$$

Let $h\colon\widehat{S}\to\widetilde{S}$ be a normalization of $\widetilde{S}$. By the adjunction formula (see [6]),

$$K_{\widehat{S}}+\mathrm{Diff}_{\widehat{S}}(0)\equiv h^*((K_V+\widetilde{S})|_{\widetilde{S}}).$$

However, the surface $\widetilde{S}$ is non-singular at a general point of $C$ and, in particular, $C$ does not lie in $\mathrm{Diff}_{\widehat{S}}(0)$. Thus,

$$K_{\widehat{S}}\cdot h^{-1}(C)<-1.$$

On the other hand the curve $h^{-1}(C)$ is contractible on the surface $\widehat{S}$. Hence it follows easily that $K_{\widehat{S}}\cdot h^{-1}(C)\geqslant-1$.

*Proof of Theorem* 5.2. We have already proved that $g$ is either a contraction of the surface to a curve or a conic bundle.

Let $C$ be a sufficiently general fibre of $g$. Then

$$\left(K_V+\frac{3}{K}\widetilde{S}\right)\cdot C=(f^*(\mathcal{O}_{\mathbb{P}^3}(-1))-E)\cdot C=-\deg(f(C))-E\cdot C<-\deg(f(C)).$$

Assume that $g$ is a conic bundle. Then

$$\lfloor E\rfloor\cap C\neq\varnothing,$$

because components of the divisor $E$ cannot lie in fibres of $g$. Thus,

$$\left(K_V+\frac{3}{K}\widetilde{S}\right)\cdot C\leqslant -\deg(f(C))-E\cdot C\leqslant -\deg(f(C))-1.$$

On the other hand,

$$\left(K_V+\frac{3}{K}\widetilde{S}\right)\cdot C=-2+\frac{3}{K}\widetilde{S}\cdot C.$$

Hence $f(C)$ is a line passing through $O$ and $\widetilde{S}$ lies in fibres of $g$. It immediately follows from the second result that $S$ is a cone in $\mathbb{P}^3$.

To complete the proof we assume that the morphism $g$ is a contraction of some surface to a curve. In this case the inequality

$$-1\leqslant K_V\cdot C=\left(K_V+\frac{3}{K}\widetilde{S}\right)\cdot C-\frac{3}{K}\widetilde{S}\cdot C<-\deg(f(C))-\frac{3}{K}\widetilde{S}\cdot C$$

shows that $g$ contracts the surface $\widetilde{S}$. Thus,

$$-\deg(f(C))>\left(K_V+\frac{3}{K}\widetilde{S}\right)\cdot C=(K_V+\widetilde{S})\cdot C-\frac{K-3}{K}\widetilde{S}\cdot C\geqslant -2+\frac{K-3}{K}.$$

Hence $f(C)$ is a line in $S$, and therefore $S$ is a cone in $\mathbb{P}^3$.

## § 6. Hypersurfaces of small degree

In this section we prove Conjecture 1.5 for hypersurfaces of degrees not exceeding their dimensions.

Let $X$ be a smooth hypersurface of degree $K$ in $\mathbb{P}^N$ with $K<N$ and let $(X,B_X)$ be a log pair on $X$ such that $B_X\equiv rH$ for some positive real number $r\in\mathbb{R}_{>0}$, where $H$ is a hyperplane section of the hypersurface $X$.

**Theorem 6.1.** *The equality $\lambda(X,B_X)=1/r$ holds if and only if*

$$B_X=\begin{cases} rS, & \text{where } S \text{ is a hyperplane section of } X \text{ and } N>K,\\ rS+\Sigma, & \text{where } S \text{ is a subvariety of } X,\ \Sigma \text{ is a boundary, } N\leqslant 3, \text{ and } N>K.\end{cases}$$

*Proof.* It follows from Statement 3.3 that the log pair $(X,(1/r)B_X)$ has no centres of log canonical singularities of positive dimension with the only possible exception of components of $B_X$. Moreover, in the case when no components of the boundary $B_X$ are centres of log canonical singularities of the log pair $(X,(1/r)B_X)$ we can repeat the proof of Lemma 3.1 verbatim to show that the singularities of the log pair $(X,(1/r)B_X)$ are log terminal.

Thus, either $\lambda(X,B_X)>1/r$ or $B_X=rS+\Sigma$, where $S$ is a subvariety of $X$ and $\Sigma$ a boundary. To complete the proof we must now show that $\Sigma=\varnothing$ for $N\geqslant 4$. In this case, however, $\operatorname{Pic}(X)=\mathbb{Z}$, which brings us to the required result.

## § 7. Canonical thresholds

Let $X$ be a smooth hypersurface of degree $N$ in $\mathbb{P}^N$ for $N \geqslant 4$, $(X, B_X)$ a log pair on $X$, and $r$ a positive real number such that $B_X \equiv rH$, where $H$ is a hyperplane section of the hypersurface $X$.

**Definition 7.1.** The largest real number $\mu$ such that the singularities of $(X, \mu B_X)$ are canonical is called the *canonical threshold* of the log pair $(X, B_X)$ and denoted by $\mu(X, B_X)$.

**Theorem 7.2.** *The following inequality holds*:

$$\mu(X, B_X) \geqslant \frac{N-2}{rN}.$$

*Proof.* Assume that the reverse inequality holds. Then for some positive $\mu$ that is smaller than $(N-2)/(rN)$ the singularities of the log pair $(X, \mu B_X)$ are not terminal and it follows from Statement 3.3 that there exists a point $O$ in $X$ that is a centre of log canonical singularities of the log pair $(X, \mu B_X)$.

Let $H$ be a sufficiently general hyperplane section of $X$ passing through $O$. Then $O$ is a centre of log canonical singularities of the log pair $(H, \mu B_X|_H)$ by Theorem 2.11, which is impossible in view of Theorem 1.3.

Note that Theorem 7.2 is significant in view of a relation existing between so-called birational rigidity and the canonical thresholds of movable log pairs (see [7]).

**Statement 7.3.** *Suppose that the inequality $\mu(X, M_X) \geqslant 1$ holds for all movable log pairs $(X, M_X)$ on a hypersurface $X$ with $K_X + M_X \equiv 0$. Then*

$$\operatorname{Bir}(X) = \operatorname{Aut}(X)$$

*and $X$ is not birationally isomorphic to a Mori fibration not biregular to $X$.*

This follows from Theorem 4.2 of [8].

The canonical threshold of an arbitrary movable log pair $(X, M_X)$ such that $K_X + M_X \equiv 0$ has been proved to be not less than 1 for $N = 4, 5, 6, 7, 8$ (see [9]–[11]); the same has been proved for all $N \geqslant 4$ if the hypersurface $X$ is general (see [12]).

The author would like to thank V. A. Iskovskikh, A. V. Pukhlikov, and V. V. Shokurov for helpful conversations.

## Bibliography


[1] I. A. Cheltsov and J. Park, "Log canonical thresholds and generalized Eckardt points", Preprint, 2000; math.AG/0003121.

[2] A. V. Pukhlikov, "A remark on the theorem on a three-dimensional quartic", *Trudy Mat. Inst. Steklov.* **208** (1995), 278–289; English transl. in *Proc. Steklov Inst. Math.* **208** (1995).

[3] M. Artin, "On isolated rational singularities of surfaces", *Amer. J. Math.* **88** (1966), 129–136.

[4] I. A. Chel'tsov, "Three-dimensional algebraic manifolds having a divisor with a numerically trivial canonical class", *Uspekhi Mat. Nauk* **51**:1 (1996), 177–178; English transl. in *Russian Math. Surveys* **51** (1996).

[5] S. Mori, "Flip theorem and the existence of minimal models for 3-folds", *J. Amer. Math. Soc.* **1**:1 (1988), 117–253.
[6] V. V. Shokurov, "3-fold log flips", *Izv. Ross. Akad. Nauk* **56**:1 (1992), 105–203; English transl. in *Russian Acad. Sci. Izv. Math.* **40** (1993).
[7] I. A. Cheltsov, "Log models of birationally rigid varieties", *J. Math. Sci.* **102** (2000), 3843–3875.
[8] A. Corti, "Factorizing birational maps of threefolds after Sarkisov", *J. Algebraic Geom.* **4** (1995), 223–254.
[9] V. A. Iskovskikh and Yu. I. Manin, "Three-dimensional quartics and counterexamples to the Lüroth problem", *Mat. Sb.* **86 (128)** (1971), 140–166; English transl. in *Math. USSR-Sb.* **15** (1971).
[10] A. V. Pukhlikov, "Birational automorphisms of four-dimensional quintic", *Invent. Math.* **87** (1987), 303–329.
[11] I. A. Cheltsov, "On a smooth quintic 4-fold", *Mat. Sb.* **191**:9 (2000), 139–160; English transl. in *Sb. Math.* **191** (2000).
[12] A. V. Pukhlikov, "Birational automorphisms of Fano hypersurfaces", *Invent. Math.* **134** (1998), 401–426.

Moscow
*E-mail address*: cheltsov@yahoo.com

Received 26/DEC/00
Translated by I. CHELTSOV