

Dynamic Alpha-invariants of Del Pezzo Surfaces

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For every smooth del Pezzo surface S , smooth curve $C \in |-K_S|$ and $\beta \in (0, 1]$, we compute the α -invariant of Tian $\alpha(S, (1 - \beta)C)$ and prove the existence of Kähler–Einstein metrics on S with edge singularities along C of angle $2\pi\beta$ for β in certain interval. In particular, we give lower bounds for the invariant $R(S, C)$, introduced by Donaldson as the supremum of all $\beta \in (0, 1]$ for which such a metric exists. The pairs (S, C) considered are strongly asymptotically log del Pezzo surfaces. We study one of the two classes of such pairs for which such metrics are expected to exist for all small $\beta > 0$.

1 Introduction

In the last 50 years, the existence of *canonical* metrics on complex manifolds has attracted a lot of interest from complex differential geometers. In the case of Fano manifolds, the Yau–Tian–Donaldson conjecture (recently solved by Chen et al. [6] and Tian [22]) predicted that the existence of Kähler–Einstein metrics should be equivalent to the algebro-geometric concept of K -stability.

Received March 27, 2015; Accepted June 9, 2015
Communicated by J.-M. Hwang

In recent decades, many geometric problems have been generalized to a *log setting*, initially in an attempt to solve these problems and later on their own merit. This was indeed the approach successfully suggested by Donaldson to solve the Kähler–Einstein problem (see [9]). In this article, we study the Kähler–Einstein problem for surfaces in a natural log setting.

Definition 1.1 ([4]). Let $(X, D = \sum_{i=1}^r D_i)$ be log smooth, that is, X is non-singular and D is a reduced \mathbb{Z} -divisor whose irreducible components are smooth and intersect with simple normal crossings. Consider $\beta = (\beta_1, \dots, \beta_r) \in (0, 1]^r$. The pair (X, D) is strongly asymptotically log Fano if for all sufficiently small $0 < \|\beta\| \ll 1$, the \mathbb{R} -divisor $-(K_X + \sum(1 - \beta_i)D_i)$ is ample. \square

Strongly asymptotically log Fano varieties are a natural generalization of log Fano varieties (X, D) , for which $-(K_X + D)$ is ample. Furthermore, both concepts generalize Fano varieties. Indeed, a (strongly asymptotically) log Fano variety (X, D) such that $D = 0$ is a Fano variety.

Given a strongly asymptotically log Fano pair (X, D) , we may consider *Kähler–Einstein metrics on X with edge singularities* (KEE metrics) of angles $2\pi\beta_1, \dots, 2\pi\beta_r$ along D_1, \dots, D_r , respectively. In this *log setting*, the KEE problem consists of two parts. On the one hand, we must decide if (X, D) admits a KEE metric for all small values of $\|\beta\|$. On the other hand, it is interesting to describe the geometry of the space of $\beta \in (0, 1]^r$ such that $(X, \sum(1 - \beta_i)D_i)$ admits a KEE metric, and in particular on the boundary of this space.

The particular case when $r = 1$ (i.e., $\text{Supp}(D)$ consists of one smooth component) and $D \in |-K_X|$ is of special interest, since a Kähler–Einstein metric with singularities along D of angle 2π is a Kähler–Einstein metric in the usual sense. Moreover, in this case we can pick any $\beta \in (0, 1]$, since $-(K_X + (1 - \beta)D) \sim_{\mathbb{Q}} -\beta K_X$ is ample. Hence, in that case, studying the geometry of $\beta \in (0, 1]$ is equivalent to computing the following invariant introduced by Donaldson.

Definition 1.2 ([9]). Let X be a smooth Fano variety and let D be a smooth divisor in $|-K_X|$. Then $R(X, D)$ is the supremum of all $\beta \in (0, 1]$ such that X admits a Kähler–Einstein metric with edge singularities along D of angle $2\pi\beta$. \square

It follows from [10] that the smooth Fano variety X admits a Kähler–Einstein metric with edge singularities of angle $2\pi\beta$ along D for every positive $\beta < R(X, D)$.

Remark 1.3. If X is a smooth Fano variety, $D \in |-K_X|$ is a smooth divisor, and X admits a Kähler–Einstein metric, then $R(X, D) = 1$. \square

Cheltsov and Rubinstein [4] have classified all strongly asymptotically log Fano varieties (S, D) in dimension 2. In their classification, (S, D) is conjectured to admit KEE metrics for all small $0 < \beta \ll 1$ [4, Conjecture 1.6.] if and only if $(K_S + D)^2 = 0$. The main subcase of this classification, and the focus of this article, is the case in which S is a del Pezzo surface ($-K_S$ is ample) and $D \in |-K_S|$ is a smooth curve. This is the natural setting in which $R(S, D)$ above is defined. Before explaining our contribution, let us discuss the main tool used: Tian’s α -invariant.

Let (V, Δ) be a log Fano variety, where Δ is an \mathbb{R} -divisor. Its α -invariant can be defined as

$$\alpha(V, \Delta) = \sup \left\{ \lambda \in \mathbb{R} \left| \begin{array}{l} \text{the log pair } (V, \Delta + \lambda B) \text{ is log canonical} \\ \text{for any effective } \mathbb{R} \text{ - divisor } B \sim_{\mathbb{R}} -(K_V + \Delta) \end{array} \right. \right\} \in \mathbb{R}_{>0}.$$

If $\Delta = 0$, we denote $\alpha(V, \Delta)$ by $\alpha(V)$.

Remark 1.4. For every effective \mathbb{R} -Cartier \mathbb{R} -divisor B on V , the number

$$\text{lct}(V, \Delta; B) = \sup\{\lambda \in \mathbb{R} \mid \text{the log pair } (V, \Delta + \lambda B) \text{ is log canonical}\}$$

is called the *log canonical threshold* of B with respect to (V, Δ) . Note that

$$\alpha(V, \Delta) = \inf\{\text{lct}(V, \Delta; B) \mid B \text{ is an effective } \mathbb{R} \text{ - divisor such that } B \sim_{\mathbb{R}} -(K_V + \Delta)\}. \quad \square$$

The role of α -invariants in the study of KEE metrics arises from the following theorem.

Theorem 1.5 ([10, 20, Theorem 2, Lemma 6.13], cf. [5, 15, Theorem 5.4]). Let X be a smooth projective variety of dimension n and let D be a smooth irreducible hypersurface in X . Let $\beta \in (0, 1]$ and suppose that the divisor $-(K_X + (1 - \beta)D)$ is ample. If $\alpha(X, (1 - \beta)D) > \frac{n}{n+1}$, then X admits a Kähler–Einstein metric with edge singularities of angle $2\pi\beta$ along D . \square

The α -invariants of smooth del Pezzo surfaces were computed in [1, Theorem 1.7] (see [18] for toric case, [7, 16] for an analytic approach and [14] for a characteristic free approach). The computation implies the following theorem.

Theorem 1.6. Let S be a smooth del Pezzo surface. Then

$$\alpha(S) = \inf \left\{ \text{lct}(S, 0; B) \mid B \in |-K_S| \text{ and } B = \sum B_i, \right. \\ \left. \text{where } B_i \cong \mathbb{P}^1 \text{ and } -K_S \cdot B_i \leq 3 \forall i \right\}. \quad \square$$

Let X be a smooth Fano variety and $D \in |-K_X|$ be a smooth divisor (see [17]). By the definition of α -invariant, and Theorem 1.5 it follows that there is a KEE metric for all sufficiently small $0 < \beta \ll 1$. On the other hand, it well known that the existence of KEE metrics is a convex property for β . Therefore, (X, D) admits a KEE metric for all $0 < \beta \leq R(X, D)$.

By a theorem of Tian (see [21]), a smooth del Pezzo surface S admits a Kähler-Einstein metric if and only if $S \not\cong \mathbb{F}_1$ and $K_S^2 \neq 7$. Thus, we have the following corollary.

Corollary 1.7 ([21]). Let S be a smooth del Pezzo surface such that $S \not\cong \mathbb{F}_1$ and $K_S^2 \neq 7$, and let C be a smooth curve in $|-K_S|$. Then $R(S, C) = 1$. \square

Unless $R(X, D) = 1$, we do not know a single example for which the invariant $R(X, D)$ is known precisely (cf. [12, Theorem 1.7]).

In this article, we generalize Theorem 1.6 to the case of strongly asymptotically log Fano surfaces (S, C) , where S is a del Pezzo surface and $C \in |-K_S|$ is smooth.

Theorem 1.8 (Main Theorem). Let S be a smooth del Pezzo surface, let C be a smooth curve in $|-K_S|$, and let β be a real number in $(0, 1]$. Then

$$\alpha(S, (1 - \beta)C) = \inf \left\{ \text{lct}(S, (1 - \beta)C; \beta B) \mid \begin{array}{l} B \in |-K_S| \text{ such that } B = C \text{ or } B = \sum B_i, \\ \text{where } B_i \cong \mathbb{P}^1 \text{ and } -K_S \cdot B_i \leq 3 \forall i \end{array} \right\}. \quad \square$$

Moreover, we establish how the α -invariants of del Pezzo surfaces vary under blowups.

Theorem 1.9. Let S_1 and S_2 be smooth del Pezzo surfaces, let C_1 and C_2 be smooth curves in $|-K_{S_1}|$ and $|-K_{S_2}|$, respectively. Suppose that there is a birational morphism $f: S_2 \rightarrow S_1$ such that $f(C_2) = C_1$. Then $\alpha(S_1, (1 - \beta)C_1) \leq \alpha(S_2, (1 - \beta)C_2)$ for every $\beta \in (0, 1]$ except for the following cases:

- (1) $S_1 \cong \mathbb{P}^2$, $S_2 \cong \mathbb{F}_1$, and f is the blowup of an inflection point of the cubic curve $C_1 \subset \mathbb{P}^2$;
- (2) $S_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, $K_{S_2}^2 = 7$, and f is the blowup of a point in C_1 . \square

We will prove Theorems 1.8 and 1.9 in Section 4. In Section 2, we will give very explicit formulas for the invariant $\alpha(S, (1 - \beta)C)$. Instead of presenting them here, let us consider their applications.

Corollary 1.10. Let S be a smooth del Pezzo surface and let C be a smooth curve in $|-K_S|$. Then $\alpha(S, (1 - \beta)C)$ is a decreasing continuous piecewise smooth function for $\beta \in (0, 1]$. □

If S is a smooth del Pezzo surface such that either $S \cong \mathbb{F}_1$ or $K_S^2 = 7$, and C is a smooth curve in $|-K_S|$, then $R(S, C) \geq \frac{1}{6}$ by [4, Proposition 6.10 (i)]. We improve this bound.

Corollary 1.11. Suppose that $S \cong \mathbb{F}_1$. Let C be a smooth curve in $|-K_S|$. Then $R(S, C) \geq \frac{3}{10}$. Furthermore, if C is chosen to be *general* in $|-K_S|$, then $R(S, C) \geq \frac{3}{7}$. □

Corollary 1.12. Let S be a smooth del Pezzo surface such that $K_S^2 = 7$, and let C be a smooth curve in $|-K_S|$. Then $R(S, C) \geq \frac{3}{7}$. Furthermore, if C does not pass through the intersection point of two intersecting (-1) -curves in S , then $R(S, C) \geq \frac{1}{2}$. □

In [19, Theorem 1], Székelyhidi proved that $R(S, C) \leq \frac{4}{5}$ when $S = \mathbb{F}_1$, and $R(S, C) \leq \frac{7}{9}$ when $K_S^2 = 7$ and C passes through the intersection point of two intersecting (-1) -curves in S .

1.1 Structure of the article

In Section 2, we define explicit functions $\hat{\alpha}(S, (1 - \beta)C): (0, 1] \rightarrow \mathbb{R}$ for all smooth del Pezzo surfaces S and all smooth curves $C \in |-K_S|$. These functions coincide with specific values of $\text{lct}(S, (1 - \beta)C, \beta B)$ where $B \in |-K_S|$ and $B = C$ or $B = \sum B_i$ where $B_i \cong \mathbb{P}^1$ and $-K_S \cdot B_i \leq 3$ for all i .

The goal of this article is to prove Theorems 1.8 and 1.9. Both results follow from showing that $\alpha(S, (1 - \beta)C) = \hat{\alpha}(S, (1 - \beta)C)$ (see Theorems 4.1 and 4.10). Proving that this equality holds boils down to show that, given *any* effective \mathbb{R} -divisor $D \sim_{\mathbb{R}} -K_S$ and *any* point $p \in S$, the pair

$$(S, (1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta D)$$

is log canonical at p for all $\beta \in (0, 1]$.

The proof consists of several cases according to the pair (S, C) and the position of $p \in S$. These are covered in Section 4. A particularly involved case is when

$4 \leq K_S^2 \leq 7$, p belongs to a unique (-1) -curve \mathcal{L} and $p = \mathcal{L} \cap C$. This case is treated separately in Section 5. In Section 3, we provide a few local inequalities for pairs (S, D) which are not log canonical. We use these inequalities in the proofs in Sections 4 and 5.

Throughout this article, we assume that all considered varieties are projective and defined over \mathbb{C} .

2 Explicit Formulas

Let S be a smooth del Pezzo surface. If $K_S^2 \geq 3$, then $-K_S$ is very ample (see [11, Proposition III.3.4]). In this case, we will identify S with its anticanonical image, and we will call a curve $Z \subset S$ such that $Z \cdot (-K_S) = 1, 2, 3$ a line, conic, cubic, respectively. Let C be a smooth curve in $|-K_S|$, and let β be a positive real number in $(0, 1]$. Let

$$\check{\alpha}(S, (1 - \beta)C) = \inf \left\{ \text{lct}(S, (1 - \beta)C; \beta B) \left| \begin{array}{l} B \in |-K_S| \text{ such that } B = C \text{ or } B = \sum B_i, \\ \text{where } B_i \cong \mathbb{P}^1 \text{ and } -K_S \cdot B_i \leq 3 \forall i \end{array} \right. \right\}.$$

Then $\alpha(S, (1 - \beta)C) \leq \check{\alpha}(S, (1 - \beta)C)$. Theorem 1.8 states that $\alpha(S, (1 - \beta)C) = \check{\alpha}(S, (1 - \beta)C)$. In this section, we will define a number $\hat{\alpha}(S, (1 - \beta)C)$ such that $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$. In Section 4, we will prove that $\alpha(S, (1 - \beta)C) \geq \hat{\alpha}(S, (1 - \beta)C)$. The latter inequality implies Theorem 1.8, since $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C) \geq \alpha(S, (1 - \beta)C)$.

2.1 Projective plane

Suppose that $S \cong \mathbb{P}^2$. Then C is a smooth cubic curve on S . Let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1 + 3\beta}{9\beta}, \frac{1}{3\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{6}, \\ \frac{1 + 3\beta}{9\beta} & \text{for } \frac{1}{6} \leq \beta \leq \frac{2}{3}, \\ \frac{1}{3\beta} & \text{for } \frac{2}{3} \leq \beta \leq 1. \end{cases}$$

Let P be an inflection point of the curve C and let T be the line in \mathbb{P}^2 that is tangent to C at the point P . Then $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$, since

$$\hat{\alpha}(S, (1 - \beta)C) = \min\{\text{lct}(S, (1 - \beta)C; \beta C), \text{lct}(S, (1 - \beta)C; 3\beta T)\}.$$

2.2 Smooth quadric surface

Suppose that $S \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1 + 2\beta}{6\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{4}, \\ \frac{1 + 2\beta}{6\beta} & \text{for } \frac{1}{4} \leq \beta \leq 1. \end{cases}$$

Let T be a divisor of bi-degree $(1, 1)$ on S that is a union of two fibers of each projection from S to \mathbb{P}^1 . Suppose in addition that one component of T is tangent to C at some point, and another component of T passes through this point. Then $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$, since

$$\hat{\alpha}(S, (1 - \beta)C) = \min\{\text{lct}(S, (1 - \beta)C; \beta C), \text{lct}(S, (1 - \beta)C; 2\beta T)\}.$$

2.3 First Hirzebruch surface

Suppose that $S \cong \mathbb{F}_1$. Let Z be the unique (-1) -curve in S , and let F be the fiber of the natural projection $S \rightarrow \mathbb{P}^1$ that passes through the point $C \cap Z$. Then $C \sim 2Z + 3F$. If F is tangent to C at the point $C \cap Z$, let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1 + 2\beta}{8\beta}, \frac{1}{3\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{6}, \\ \frac{1 + 2\beta}{8\beta} & \text{for } \frac{1}{6} \leq \beta \leq \frac{5}{6}, \\ \frac{1}{3\beta} & \text{for } \frac{5}{6} \leq \beta \leq 1. \end{cases}$$

If F is not tangent to C at the point $C \cap Z$, let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1 + \beta}{5\beta}, \frac{1}{3\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{4}, \\ \frac{1 + \beta}{5\beta} & \text{for } \frac{1}{4} \leq \beta \leq \frac{2}{3}, \\ \frac{1}{3\beta} & \text{for } \frac{2}{3} \leq \beta \leq 1. \end{cases}$$

In both cases, we have $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$, because

$$\hat{\alpha}(S, (1 - \beta)C) = \min\{\text{lct}(S, (1 - \beta)C; \beta C), \text{lct}(S, (1 - \beta)C; \beta(2Z + 3F))\}.$$

2.4 Blowup of \mathbb{P}^2 at two points

Suppose that $K_S^2 = 7$. Then there exists a birational morphism $\pi : S \rightarrow \mathbb{P}^2$ that is the blowup of two distinct points in \mathbb{P}^2 . Denote by E_1 and E_2 two π -exceptional curves, and denote by L the proper transform of the line in \mathbb{P}^2 that passes through $\pi(E_1)$ and $\pi(E_2)$. Then E_1 , E_2 , and L are all (-1) -curves in S .

The pencil $|E_2 + L|$ contains a unique curve that passes through $C \cap E_1$. Similarly, $|E_1 + L|$ contains a unique curve that passes through $C \cap E_2$. Denote these curves by L_1 and L_2 , respectively. Then L_1 is irreducible and smooth unless $L_1 = E_2 + L$ (in this case $E_1 \cap L \in C$). Similarly, the curve L_2 is irreducible and smooth unless $L_2 = E_1 + L$ and $L \cap E_2 \in C$.

If C does not contain the points $E_1 \cap L$ nor $E_2 \cap L$, then there exists a unique smooth irreducible curve $R \in |E_1 + E_2 + L|$ such that R passes through $C \cap L$ and is tangent to C at the point $C \cap L$. If either $E_1 \cap L \in C$ or $E_2 \cap L \in C$, we let $R = E_1 + E_2 + L$. In the former case, either R and C have simple tangency at the point $C \cap L$ or the curve R is tangent to C at the point $C \cap L$ with multiplicity 3 (in this case, we must have $R \cap C = C \cap L$, because $R \cdot C = 3$).

If either $E_1 \cap L \in C$ or $E_2 \cap L \in C$ (but not both, since $C \cdot L = 1$), then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1 + \beta}{5\beta}, \frac{1}{3\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{4}, \\ \frac{1 + \beta}{5\beta} & \text{for } \frac{1}{4} \leq \beta \leq \frac{2}{3}, \\ \frac{1}{3\beta} & \text{for } \frac{2}{3} \leq \beta \leq 1. \end{cases}$$

If the curve C does not contain the points $E_1 \cap L$ nor $E_2 \cap L$, and either L_1 is tangent to C at the point $C \cap E_1$ or L_2 is tangent to C at the point $C \cap E_2$, then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1 + 2\beta}{6\beta}, \frac{1}{3\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{4}, \\ \frac{1 + 2\beta}{6\beta} & \text{for } \frac{1}{4} \leq \beta \leq \frac{1}{2}, \\ \frac{1}{3\beta} & \text{for } \frac{1}{2} \leq \beta \leq 1. \end{cases}$$

If the curve C does not contain the points $E_1 \cap L$ nor $E_2 \cap L$ (this implies that the curve R is smooth), neither L_1 is tangent to C at the point $C \cap E_1$ nor L_2 is tangent to C at the point $C \cap E_2$, and the curve R is tangent to C at the point $C \cap L$ with multiplicity 3, then

we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1 + 3\beta}{7\beta}, \frac{1}{3\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{4}, \\ \frac{1 + 3\beta}{7\beta} & \text{for } \frac{1}{4} \leq \beta \leq \frac{4}{9}, \\ \frac{1}{3\beta} & \text{for } \frac{4}{9} \leq \beta \leq 1. \end{cases}$$

Finally, if the curve C does not contain the points $E_1 \cap L$ nor $E_2 \cap L$ (and hence the curve R is smooth), neither L_1 is tangent to C at the point $C \cap E_1$ nor L_2 is tangent to C at the point $C \cap E_2$, and R is tangent to C at the point $C \cap L$ with multiplicity 2, then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1}{3\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{3}, \\ \frac{1}{3\beta} & \text{for } \frac{1}{3} \leq \beta \leq 1. \end{cases}$$

We have $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$. Indeed, if either $E_1 \cap L \in C$ or $E_2 \cap L \in C$, then

$$\hat{\alpha}(S, (1 - \beta)C) = \min\{\text{lct}(S, (1 - \beta)C; \beta C), \text{lct}(S, (1 - \beta)C; \beta(3L + 2E_1 + 2E_2))\},$$

which implies that $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$. If neither $E_1 \cap L \in C$ nor $E_2 \cap L \in C$, then

$$\min\{\text{lct}(S, (1 - \beta)C; \beta C), \text{lct}(S, (1 - \beta)C; \beta(3L + 2E_1 + 2E_2))\} = \min \left\{ 1, \frac{1}{3\beta} \right\}.$$

If the curve C does not contain the points $E_1 \cap L$ nor $E_2 \cap L$, and L_1 is tangent to C at the point $C \cap E_1$, then

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1}{3\beta}, \text{lct}(S, (1 - \beta)C; \beta(2L_1 + 2E_1 + L)) \right\},$$

and similarly if L_2 is tangent to C at the point $C \cap E_2$. If the curve C does not contain the points $E_1 \cap L$ nor $E_2 \cap L$ (this implies that the curve R is smooth), neither L_1 is tangent to C at the point $C \cap E_1$ nor L_2 is tangent to C at the point $C \cap E_2$, and the curve R is tangent to C at the point $C \cap L$ with multiplicity 3, then

$$\min\{\text{lct}(S, (1 - \beta)C; \beta C), \text{lct}(S, (1 - \beta)C; \beta(3L + 2E_1 + 2E_2)), \text{lct}(S, (1 - \beta)C; \beta(L + 2R))\}$$

equals $\hat{\alpha}(S, (1 - \beta)C)$. We conclude that $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$ in every case.

2.5 Blowup of \mathbb{P}^2 at three points

Suppose that $K_S^2 = 6$. Then there exists a birational morphism $\pi : S \rightarrow \mathbb{P}^2$ that is the blowup of three non-collinear points. Denote the π -exceptional curves by E_1, E_2, E_3 , denote the proper transform on S of the line in \mathbb{P}^2 that passes through $\pi(E_1)$ and $\pi(E_2)$ by L_{12} , denote the proper transform on S of the line in \mathbb{P}^2 that passes through $\pi(E_1)$ and $\pi(E_3)$ by L_{13} , and denote the proper transform on S of the line in \mathbb{P}^2 that passes through $\pi(E_2)$ and $\pi(E_3)$ by L_{23} . Then $E_1, E_2, E_3, L_{12}, L_{13}$, and L_{23} are all the lines in S .

If the curve C contains an intersection point of two intersecting lines in S , then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1 + \beta}{4\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{3}, \\ \frac{1 + \beta}{4\beta} & \text{for } \frac{1}{3} \leq \beta \leq 1. \end{cases}$$

If the curve C does not contain the intersection points of any two intersecting lines and, there are a line Z_1 and an irreducible conic Z_2 in S such that Z_2 is tangent to C at the point $C \cap Z_1$, then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1 + 2\beta}{5\beta}, \frac{1}{2\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{3}, \\ \frac{1 + 2\beta}{5\beta} & \text{for } \frac{1}{3} \leq \beta \leq \frac{3}{4}, \\ \frac{1}{2\beta} & \text{for } \frac{3}{4} \leq \beta \leq 1. \end{cases}$$

If C does not contain the intersection point of any two intersecting lines, and for every line Z_1 in S , there exists no irreducible conic Z_2 in S such that Z_2 is tangent to C at $C \cap Z_1$, then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1}{2\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{2}, \\ \frac{1}{2\beta} & \text{for } \frac{1}{2} \leq \beta \leq 1. \end{cases}$$

One has $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$. Indeed, we have $2E_1 + 2L_{12} + L_{13} + E_2 \sim -K_S$. Thus, if $E_1 \cap L_{12} \notin C$, $E_1 \cap L_{13} \notin C$, and $E_2 \cap L_{12} \notin C$, then

$$\begin{aligned} & \min\{\text{lct}(S, (1 - \beta)C; \beta C), \text{lct}(S, (1 - \beta)C; \beta(2E_1 + 2L_{12} + L_{13} + E_2))\} \\ &= \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{2}, \\ \frac{1}{2\beta} & \text{for } \frac{1}{2} \leq \beta \leq 1. \end{cases} \end{aligned}$$

Otherwise, this minimum is $\hat{\alpha}(S, (1 - \beta)C)$. This shows that $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$ except for the case when C does not contain the intersection point of any two intersecting lines, but there are a line Z_1 and a conic Z_2 in S such that Z_2 is tangent to C at the point $C \cap Z_1$. In the latter case, we may assume that $Z_1 = E_1$ and $Z_2 \in |L_{12} + E_2|$, which implies that

$$\hat{\alpha}(S, (1 - \beta)C) = \min\{\text{lct}(S, (1 - \beta)C; \beta C), \text{lct}(S, (1 - \beta)C; \beta(2Z_2 + E_1 + L_{23}))\},$$

since $2Z_2 + E_1 + L_{23} \sim -K_S$. Thus, in all cases we have $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$.

2.6 Blowup of \mathbb{P}^2 at four points

Suppose that $K_S^2 = 5$. Then there exists a birational morphism $\pi : S \rightarrow \mathbb{P}^2$ that contracts four smooth rational curves to four points such that no three of them are collinear. Denote these curves by E_1, E_2, E_3, E_4 . For any integers i and j such that $1 \leq i < j \leq 4$, denote by L_{ij} the proper transform on S via π of the line in \mathbb{P}^2 that passes through $\pi(E_i)$ and $\pi(E_j)$. This gives us six lines $L_{12}, L_{13}, L_{14}, L_{23}, L_{24}$, and L_{34} . Moreover, $E_1, E_2, E_3, E_4, L_{12}, L_{13}, L_{14}, L_{23}, L_{24}$, and L_{34} are all the lines in S . Let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1}{2\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{2}, \\ \frac{1}{2\beta} & \text{for } \frac{1}{2} \leq \beta \leq 1. \end{cases}$$

Then $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$, since $2E_1 + L_{12} + L_{13} + L_{14} \sim -K_S$ and

$$\hat{\alpha}(S, (1 - \beta)C) = \min\{\text{lct}(S, (1 - \beta)C; \beta C), \text{lct}(S, (1 - \beta)C; \beta(2E_1 + L_{12} + L_{13} + L_{14}))\}.$$

2.7 Complete intersections of two quadrics

Suppose that $K_S^2 = 4$. Then there exists a birational morphism $\pi : S \rightarrow \mathbb{P}^2$ that is the blowup of five points such that no three of them are collinear. Denote by E_1, E_2, E_3, E_4 , and E_5 the π -exceptional curves. For any integers i and j such that $1 \leq i < j \leq 5$, denote by L_{ij} the proper transform via π on S of the line in \mathbb{P}^2 that passes through $\pi(E_i)$ and $\pi(E_j)$. Denote by E the proper transform on S of the unique smooth conic in \mathbb{P}^2 that passes through $\pi(E_1), \pi(E_2), \pi(E_3), \pi(E_4)$, and $\pi(E_5)$. Then $E_1, E_2, E_3, E_4, E_5, L_{12}, L_{13}, L_{14}, L_{15}, L_{23}, L_{24}, L_{25}, L_{34}, L_{35}, L_{45}$, and E are all the lines in S .

If the curve C contains the intersection point of any two intersecting lines, then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1 + \beta}{3\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{2}, \\ \frac{1 + \beta}{3\beta} & \text{for } \frac{1}{2} \leq \beta \leq 1. \end{cases}$$

If the curve C does not contain the intersection point of any two intersecting lines, but there are two conics C_1 and C_2 in S such that $C_1 + C_2 \sim -K_S$, and C_1 and C_2 both tangent C at one point, then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1 + 2\beta}{4\beta}, \frac{2}{3\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{2}, \\ \frac{1 + 2\beta}{4\beta} & \text{for } \frac{1}{2} \leq \beta \leq \frac{5}{6}, \\ \frac{2}{3\beta} & \text{for } \frac{5}{6} \leq \beta \leq 1. \end{cases}$$

Finally, if the curve C does not contain the intersection point of any two intersecting lines, and for every two conics C_1 and C_2 in S such that $C_1 + C_2 \sim -K_S$, the conics C_1 and C_2 do not tangent C at one point, then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{2}{3\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{2}{3}, \\ \frac{2}{3\beta} & \text{for } \frac{2}{3} \leq \beta \leq 1. \end{cases}$$

We claim that $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$. Indeed, the lines L_{12} and L_{34} intersect at a single point. Let Z be the proper transform on S of the line in \mathbb{P}^2 that passes through $\pi(E_5)$ and $\pi(L_{12} \cap L_{34})$. Then $L_{12} + L_{34} + Z \sim -K_S$. Moreover, if $L_{12} \cap L_{34} \in C$, then

$$\min\{\text{lct}(S, (1 - \beta)C; \beta C), \text{lct}(S, (1 - \beta)C; \beta(L_{12} + L_{34} + Z))\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{2}, \\ \frac{1 + \beta}{3\beta} & \text{for } \frac{1}{2} \leq \beta \leq 1. \end{cases}$$

However, if $L_{12} \cap L_{34} \notin C$, then this minimum equals $\min\{1, \frac{2}{3\beta}\}$. Since we can repeat these computations for any pair of intersecting lines in S , we see that $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$ except possibly the case when C does not contain the intersection point of any two intersecting lines, but there are two conics C_1 and C_2 in S such that $C_1 + C_2 \sim -K_S$,

and C_1 and C_2 both tangent C at one point. In the latter case, $\hat{\alpha}(S, (1 - \beta)C)$ is equal to $\min\{\text{lct}(S, (1 - \beta)C; \beta C), \text{lct}(S, (1 - \beta)C; \beta(L_{12} + L_{34} + Z)), \text{lct}(S, (1 - \beta)C; \beta(C_1 + C_2))\}$, since $C_1 + C_2 \sim -K_S$. This shows that $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$ in all three cases.

2.8 Cubic surfaces

Suppose that $K_S^2 = 3$. Then S is a smooth cubic surface in \mathbb{P}^3 . Recall that an Eckardt point in S is a point of intersection of three lines contained in S . A general cubic surface contains no Eckardt points. If S contains an Eckardt point that is contained in C , then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{1 + \beta}{3\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{1}{2}, \\ \frac{1 + \beta}{3\beta} & \text{for } \frac{1}{2} \leq \beta \leq 1. \end{cases}$$

If S contains an Eckardt point and C contains no Eckardt points, then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{2}{3\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{2}{3}, \\ \frac{2}{3\beta} & \text{for } \frac{2}{3} \leq \beta \leq 1. \end{cases}$$

If S contains no Eckardt points, but S contains a line L and a conic M such that L is tangent to M and $L \cap M \in C$, then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{2 + \beta}{4\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{2}{3}, \\ \frac{2 + \beta}{4\beta} & \text{for } \frac{2}{3} \leq \beta \leq 1. \end{cases}$$

If S contains no Eckardt points, and for every line L and every conic M on S such that L is tangent to M , we have $L \cap M \notin C$, but there is a cuspidal curve $T \in |-K_S|$ such that $T \cap C = \text{Sing}(T)$, then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{2 + 3\beta}{6\beta}, \frac{3}{4\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{2}{3}, \\ \frac{2 + 3\beta}{6\beta} & \text{for } \frac{2}{3} \leq \beta \leq \frac{5}{6}, \\ \frac{3}{4\beta} & \text{for } \frac{5}{6} \leq \beta \leq 1. \end{cases}$$

Finally, if S contains no Eckardt points, and for every line L and every conic M on S such that L is tangent to M , we have $L \cap M \not\subset C$, and every irreducible cuspidal curve $T \in |-K_S|$ intersects C by at least two point, then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{3}{4\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{3}{4}, \\ \frac{3}{4\beta} & \text{for } \frac{3}{4} \leq \beta \leq 1. \end{cases}$$

One can easily check that $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$ (see [13, Theorem 4.9.1]).

2.9 Double covers of \mathbb{P}^2

Suppose that $K_S^2 = 2$. Recall that C is a non-singular curve in the linear system $|-K_S|$. Different choices of C will give rise to different invariants $\hat{\alpha}(S, (1 - \beta)C)$ whose values depend on the existence of certain singular irreducible curves $Z \in |-K_S|$ such that C contains the singular point of Z . We give the different values of $\hat{\alpha}(S, (1 - \beta)C)$ for all smooth $C \in |-K_S|$ distinguishing four possible cases for the curve Z .

If $|-K_S|$ contains a tacnodal curve Z whose singular point is contained in C , then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{2 + \beta}{4\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{2}{3}, \\ \frac{2 + \beta}{4\beta} & \text{for } \frac{2}{3} \leq \beta \leq 1. \end{cases}$$

If $|-K_S|$ contains at least one tacnodal curve Z , but C does not contain singular points of any tacnodal curve in $|-K_S|$, then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{3}{4\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{3}{4}, \\ \frac{3}{4\beta} & \text{for } \frac{3}{4} \leq \beta \leq 1. \end{cases}$$

If $|-K_S|$ contains no curves with tacnodal singularities but there is at least one cuspidal rational point $Z \in |-K_S|$ such that C contains the cuspidal singular point of Z , then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{3 + 2\beta}{6\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{3}{4}, \\ \frac{3 + 2\beta}{6\beta} & \text{for } \frac{3}{4} \leq \beta \leq 1. \end{cases}$$

Finally, if $| -K_S |$ contains no curves with tacnodal singularities, and C does not contain the cuspidal singular point of any cuspidal rational curves in $| -K_S |$, then we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{5}{6\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{5}{6}, \\ \frac{5}{6\beta} & \text{for } \frac{5}{6} \leq \beta \leq 1. \end{cases}$$

Clearly, these four cases exhaust all possibilities for smooth curves $C \in | -K_S |$. One can easily check that $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$ (see [13, Theorem 4.10.1]).

2.10 Double covers of quadric cones

Suppose that $K_S^2 = 1$. As above, C is a non-singular curve in the pencil $| -K_S |$. If $| -K_S |$ contains no cuspidal curves, then we let $\hat{\alpha}(S, (1 - \beta)C) = 1$ for every $\beta \in (0, 1]$. Otherwise, we let

$$\hat{\alpha}(S, (1 - \beta)C) = \min \left\{ 1, \frac{5}{6\beta} \right\} = \begin{cases} 1 & \text{for } 0 < \beta \leq \frac{5}{6}, \\ \frac{5}{6\beta} & \text{for } \frac{5}{6} \leq \beta \leq 1. \end{cases}$$

In the former case, we have $\hat{\alpha}(S, (1 - \beta)C) = \text{lct}(S, (1 - \beta)C; \beta C)$. In the latter case, we have

$$\hat{\alpha}(S, (1 - \beta)C) = \min\{\text{lct}(S, (1 - \beta)C; \beta C), \text{lct}(S, (1 - \beta)C; \beta Z)\},$$

where Z is a cuspidal curve in $| -K_S |$. Thus, $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$ in both cases.

3 Local Inequalities

Let S be a smooth surface, let D be an effective \mathbb{R} -divisor on S , and let P be a point in S .

Lemma 3.1. Suppose that (S, D) is not log canonical at P . Then $\text{mult}_P(D) > 1$. □

Proof. This is a well-known fact. See [8, Exercise 6.18], for instance. ■

Lemma 3.2. Suppose that (S, D) is not log canonical at P . Let B be an effective \mathbb{R} -divisor on S such that (S, B) is log canonical and $B \sim_{\mathbb{R}} D$. Then there exists an effective \mathbb{R} -divisor D' on S such that $D' \sim_{\mathbb{R}} D$, the log pair (S, D') is not log canonical at P , and $\text{Supp}(D')$ does not contain at least one irreducible component of $\text{Supp}(B)$. □

Proof. Let μ be the greatest real number such that $D' := (1 + \mu)D - \mu B$ is effective. Since $D \neq B$, the number μ does exist. Then $D' \sim_{\mathbb{R}} D$, the log pair (S, D') is not log canonical at P , and $\text{Supp}(D')$ does not contain at least one irreducible component of $\text{Supp}(B)$. ■

Let $\pi_1: S_1 \rightarrow S$ be a blowup of the point P , let F_1 be the π -exceptional curve, and let D^1 be the proper transform of D via π_1 . Then $K_{S_1} + D^1 + (\text{mult}_P(D) - 1)F_1 \sim_{\mathbb{R}} \pi_1^*(K_S + D)$.

Lemma 3.3. Suppose that (S, D) is not log canonical at P . Then $\text{mult}_P(D) > 1$ and there exists a point $P_1 \in F_1$ such that $(S_1, D^1 + (\text{mult}_P(D) - 1)F_1)$ is not log canonical at P_1 . Moreover, one has $\text{mult}_P(D) + \text{mult}_{P_1}(D^1) > 2$. If, in addition, $\text{mult}_P(D) \leq 2$, then such point P_1 is unique. □

Proof. This is a well-known fact. See, for example, [3, Remark 2.5]. ■

Let C be an irreducible curve on S that contains P . Suppose that C is smooth at P . Write $D = aC + \Omega$, where $a \in \mathbb{R}_{\geq 0}$, and Ω is an effective \mathbb{R} -divisor on S with $C \not\subset \text{Supp}(\Omega)$.

Theorem 3.4. If $(S, aC + \Omega)$ is not log canonical at P and $a \leq 1$, then $\text{mult}_P(\Omega \cdot C) > 1$. □

Proof. For the proof, see, for example, [8, Exercise 6.31], [14, Lemma 2.5], or [2, Theorem 7]. ■

Denote the proper transform of the curve C on the surface S_1 by C^1 , and denote the proper transform of the \mathbb{R} -divisor Ω on the surface S_1 by Ω^1 .

Lemma 3.5. Suppose that $a \leq 1$, the log pair $(S, aC + \Omega)$ is not log canonical at the point P , and $\text{mult}_P(\Omega) \leq 1$. Then $(S_1, aC^1 + \Omega^1 + (a + \text{mult}_P(\Omega) - 1)F_1)$ is not log canonical at $C^1 \cap F_1$, it is log canonical at every point in $E_1 \setminus (C^1 \cap F_1)$, and $\text{mult}_P(\Omega \cdot C) > 2 - a$. □

Proof. Since $a \leq 1$ and $\text{mult}_P(\Omega) \leq 1$, we have $\text{mult}_P(D) \leq 2$. By Lemma 3.3, there exists a unique point $P_1 \in F_1$ such that the log pair $(S_1, aC^1 + \Omega^1 + (a + \text{mult}_P(\Omega) - 1)F_1)$ is not log canonical at P_1 . If $P_1 \notin C^1$, then $\text{mult}_P(\Omega) = F_1 \cdot \Omega^1 \geq \text{mult}_{P_1}(\Omega^1 \cdot F_1) > 1$ by Theorem 3.4, which is impossible, since $\text{mult}_P(\Omega) \leq 1$. Thus, $P_1 \in C^1$. Then, by Theorem 3.4 again,

$$\text{mult}_P(\Omega \cdot C) \geq \text{mult}_P(\Omega) + \text{mult}_{P_1}(\Omega^1 \cdot C^1) > 2 - a. \quad \blacksquare$$

Let us consider an *infinite* sequence of blowups

$$\cdots \xrightarrow{\pi_{n+1}} S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_3} S_2 \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S$$

such that each π_n is the blowup of the point in the proper transform of the curve C on the surface S_{n-1} that dominates P . Denote the π_n -exceptional curve by F_n , and denote the proper transform of C on S_n by C^n . For every $n \geq 1$, write $P_n = C^n \cap F_n$, denote the proper transform of the divisor Ω on S_n by Ω^n , let $m_n = \text{mult}_{P_n}(\Omega^n)$, and let $m_0 = \text{mult}_P(\Omega)$. For every positive integers $k \leq n$, denote the proper transform of the curve F_k on S_n by F_k^n . Finally, we let

$$D^{S_n} = aC^n + \Omega^n + \sum_{k=1}^n \left(ka - k + \sum_{i=0}^{k-1} m_i \right) F_k^n$$

for every $n \geq 1$. Then $K_{S_n} + D^{S_n} \sim_{\mathbb{R}} (\pi_1 \circ \pi_2 \circ \cdots \circ \pi_n)^*(K_S + D)$ for every $n \geq 1$.

Theorem 3.6. Suppose that $(S, aC + \Omega)$ is not log canonical at P and $a \leq 1$. Then $m_0 + a > 1$ and $\text{mult}_P(\Omega \cdot C) > 1$. Moreover, the following additional assertions hold:

- (i) if $m_0 \leq 1$, then the log pair (S_1, D^{S_1}) is not log canonical at P_1 ;
- (ii) if (S_n, D^{S_n}) is not log canonical at some point in F_n , then D^{S_n} is an effective divisor;
- (iii) if (S_n, D^{S_n}) is not log canonical at some point in F_n and $\sum_{i=0}^{n-1} m_i \leq n + 1 - na$, then such point in F_n is unique;
- (iv) if (S_n, D^{S_n}) is not log canonical at P_n , then $(n + 1)a + \sum_{i=0}^n m_i > n + 2$, the log pair $(S_{n+1}, D^{S_{n+1}})$ is not log canonical at some point in F_{n+1} , and $\text{mult}_P(\Omega \cdot C) > n + 1 - na$;
- (v) if $n \geq 2$, $m_{n-1} \leq 1$ and $\sum_{i=0}^{n-1} m_i \leq n + 1 - na$, then (S_n, D^{S_n}) is log canonical at every point of F_n different from P_n and $F_n \cap F_{n-1}^n$;
- (vi) if $n \geq 2$ and $\sum_{i=0}^{n-1} m_i \leq n - (n - 1)a$, then (S_n, D^{S_n}) is log canonical at $F_n \cap F_{n-1}^n$;
- (vii) if $n \geq 2$, $\sum_{i=0}^{n-2} m_i \leq n - (n - 1)a$ and $\sum_{i=0}^{n-3} m_i + 2m_{n-2} \leq n + 1 - na$, then (S_n, D^{S_n}) is log canonical at $F_n \cap F_{n-1}^n$. □

Proof. By Lemma 3.1, we have $m_0 + a > 1$. By Theorem 3.4, we have $\text{mult}_P(\Omega \cdot C) > 1 - a$. Assertion (i) follows from Lemma 3.5.

For assertion (ii), it is enough to show that $k(a - 1) + \sum_{i=0}^{k-1} m_i \geq 0$ for $1 \leq k \leq n$. If $a \geq 1$, this is clear. If $0 \leq a \leq 1$, then $m_0 > 1$ by Lemma 3.1 and hence D^{S_1} is effective. Assertion (ii) follows by induction. Indeed, since (S_n, D^{S_n}) is not log canonical at some

point in F_n , then $(S_{l-1}, D^{S_{l-1}})$ is not log canonical at P_{l-1} for $1 \leq l \leq n$. If $(l-1)(a-1) + \sum_{i=0}^{l-2} m_i \geq 0$, then

$$l(a-1) + \sum_{i=0}^{l-1} m_i \geq (a-1) + m_{l-1} \geq 0.$$

For the last inequality, we note that the induction hypothesis gives us that D_k is effective and (S_k, D^{S_k}) is not log canonical for $0 \leq k \leq l-1$. Hence, Lemma 3.1 implies that $m_{l-1} > 1$, proving assertion (ii).

Inequality $\sum_{i=0}^{n-1} m_i \leq n+1-na$ is equivalent to $\text{mult}_{P_{n-1}}(D^{S_{n-1}}) \leq 2$. Thus, assertion (iii) follows from Lemma 3.3. If (S_n, D^{S_n}) is not log canonical at P_n , then $(n+1)a + \sum_{i=0}^n m_i > n+2$ by Lemma 3.1, the pair $(S_{n+1}, D^{S_{n+1}})$ is not log canonical at some point in F_{n+1} by Lemma 3.3, and

$$\text{mult}_P(\Omega \cdot C) - \sum_{i=0}^{n-1} m_i = \text{mult}_{P_n}(\Omega^n \cdot C^n) > 1 - \left(na - n + \sum_{i=0}^{n-1} m_i \right),$$

by Theorem 3.4. This proves assertion (iv).

Suppose that $n \geq 2$. Let $O = F_n \cap F_{n-1}^n$. If $\sum_{i=0}^{n-1} m_i \leq n+1-na$ and (S_n, D^{S_n}) is not log canonical at some point in $F_n \setminus (P_n \cup O)$, then $m_{n-1} = F_n \cdot \Omega^n > 1$ by Theorem 3.4, which implies assertion (v). If (S_n, D^{S_n}) is not log canonical at O and $\sum_{i=0}^{n-1} m_i \leq n+1-na$, then

$$m_{n-1} = F_n \cdot \Omega^n \geq \text{mult}_O(F_n \cdot \Omega^n) > 1 - \left((n-1)a - n + 1 + \sum_{i=0}^{n-2} m_i \right)$$

by Theorem 3.4. If (S_n, D^{S_n}) is not log canonical at O and $\sum_{i=0}^{n-2} m_i \leq n - (n-1)a$, then

$$m_{n-2} - m_{n-1} = F_{n-1}^n \cdot \Omega^n \geq \text{mult}_O(F_{n-1}^n \cdot \Omega^n) > 1 - \left(na - n + \sum_{i=0}^{n-1} m_i \right)$$

by Theorem 3.4. This proves assertions (vi) and (vii). ■

Corollary 3.7. Suppose that $(S, aC + \Omega)$ is not log canonical at P , $C \notin \text{Supp}(\Omega)$, $a \leq 1$ and $m_0 \leq \min\{1, 1 + \frac{1}{n} - na\}$ for some integer $n \geq 1$. Then $\text{mult}_P(\Omega \cdot C) > n+1-na$. □

Corollary 3.8. Suppose that $(S, aC + \Omega)$ is not log canonical at P , $a \leq 1$ and $m_0 \leq 1$. Suppose that $2m_0 \leq 3 - 2a$ or $m_0 + m_1 \leq 2 - a$. Suppose that $m_0 + 2m_1 \leq 4 - 3a$ or $m_0 + m_1 + m_2 \leq 3 - 2a$. Then $\text{mult}_P(\Omega \cdot C) > 4 - 3a$. If $m_0 + m_1 + 2m_2 \leq 5 - 4a$ or $m_0 + m_1 + m_2 + m_3 \leq 4 - 3a$, then $\text{mult}_P(\Omega \cdot C) > 5 - 4a$. □

Let us conclude this section by recalling the following theorem.

Theorem 3.9 ([2, Theorem 13]). Let C_1 and C_2 be two irreducible curves on S that are both smooth at P and intersect transversally at P . Let $D = a_1C_1 + a_2C_2 + \Delta$, where a_1 and a_2 are non-negative real numbers, and Δ is an effective \mathbb{R} -divisor on S whose support does not contain the curves C_1 and C_2 . If (S, D) is not log canonical at P and $\text{mult}_P(\Delta) \leq 1$, then $\text{mult}_P(\Delta \cdot C_1) > 2(1 - a_2)$ or $\text{mult}_P(\Delta \cdot C_2) > 2(1 - a_1)$. \square

4 The Proof

Let us use the notation of Section 2. The goal of this section is to prove the following theorem.

Theorem 4.1. One has $\alpha(S, (1 - \beta)C) = \hat{\alpha}(S, (1 - \beta)C)$ for every $\beta \in (0, 1]$. \square

This theorem implies Theorem 1.8, since $\hat{\alpha}(S, (1 - \beta)C) \geq \check{\alpha}(S, (1 - \beta)C)$ (see Section 2) and $\check{\alpha}(S, (1 - \beta)C) \geq \alpha(S, (1 - \beta)C)$ (by definition) for every $\beta \in (0, 1]$.

Let D be any effective \mathbb{R} -divisor such that $D \sim_{\mathbb{R}} -K_S$, and let P be any point in S . Since $\alpha(S, (1 - \beta)C) \leq \hat{\alpha}(S, (1 - \beta)C)$, to prove Theorem 4.1, it is enough to show that the log pair

$$(S, (1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta D) \tag{4.1}$$

is log canonical at P for every $\beta \in (0, 1]$. We will do this in several steps.

Lemma 4.2. Suppose that (4.1) is not log canonical at P . Then $P \in C$, we have

$$\text{mult}_P(D) > \frac{1}{\hat{\alpha}(S, (1 - \beta)C)} \geq 1,$$

and (4.1) is log canonical outside of the point P . Moreover, if there exists a (-1) -curve $Z \subset S$ such that $P \in Z$, then $Z \subset \text{Supp}(D)$. Furthermore, there exists an effective \mathbb{R} -divisor $D' \sim_{\mathbb{R}} D$ such that $C \not\subset \text{Supp}(D')$ and $(S, (1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta D')$ is not log canonical at P . \square

Proof. If $P \notin C$, then $(S, \hat{\alpha}(S, (1 - \beta)C)\beta D)$ is not log canonical at P , which is impossible, since $\alpha(S) \leq \beta \hat{\alpha}(S, (1 - \beta)C)$ by [1, Theorem 1.7]. We have $\hat{\alpha}(S, (1 - \beta)C) \text{mult}_P(D) > 1$ by Lemma 3.1. In particular, if there exists a (-1) -curve $Z \subset S$ such that $P \in Z$, then Z must be contained in $\text{Supp}(D)$, because otherwise we would have $1 = Z \cdot D \geq \text{mult}_P(D) > 1$.

We see that (4.1) is log canonical outside of the curve C . Moreover, the coefficient of the curve C in the divisor $(1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta D$ does not exceed 1, since $D \sim_{\mathbb{R}} C$. Hence, the log pair (4.1) is log canonical outside of finitely many points. Now the

connectedness principle (see, e.g., [8, Theorem 6.32]) implies that (4.1) is log canonical outside of P .

Since $(S, (1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta C)$ is log canonical, it follows from Lemma 3.2 that there is an effective \mathbb{R} -divisor $D' \sim_{\mathbb{R}} D$ such that $C \not\subset \text{Supp}(D')$ and $(S, (1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta D')$ is not log canonical at P . ■

Thus, to prove that (4.1) is log canonical at P , we may assume that $P \in C \not\subset \text{Supp}(D)$.

Lemma 4.3. If $S \cong \mathbb{P}^2$, then (4.1) is log canonical at P . □

Proof. Suppose that (4.1) is not log canonical at P . Let L be a general line in S that contains P . Then $\text{mult}_P(D) \leq D \cdot L = 3$. But $3\hat{\alpha}(S, (1 - \beta)C)\beta \leq \frac{1}{3} + \beta$ (see Section 2.1). Thus, if $\beta \leq \frac{2}{3}$, then

$$\hat{\alpha}(S, (1 - \beta)C)\beta \text{mult}_P(D) \leq 3\hat{\alpha}(S, (1 - \beta)C)\beta \leq \frac{1}{3} + \beta \leq 1.$$

Similarly, if $\frac{2}{3} \leq \beta \leq 1$, then $\hat{\alpha}(S, (1 - \beta)C)\beta \text{mult}_P(D) \leq \frac{1}{3} \text{mult}_P(D) \leq 1$. Applying Corollary 3.7 with $n=3$ to (4.1), we get

$$9\beta\hat{\alpha}(S, (1 - \beta)C) = \hat{\alpha}(S, (1 - \beta)C)\beta(C \cdot D) \geq \hat{\alpha}(S, (1 - \beta)C)\beta \text{mult}_P(C \cdot D) > 1 + 3\beta,$$

which contradicts the definition of $\hat{\alpha}(S, (1 - \beta)C)$ in Section 2.1. ■

Lemma 4.4. Suppose that $S \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then (4.1) is log canonical at P . □

Proof. Suppose that (4.1) is not log canonical at P . Let L_1 and L_2 be the fibers of two different projections $S \rightarrow \mathbb{P}^1$ that both pass through P . Since $(S, (1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta(2L_1 + 2L_2))$ is log canonical and $2L_1 + 2L_2 \sim_{\mathbb{R}} D$, we may assume that either $L_1 \not\subset \text{Supp}(D)$ or $L_2 \not\subset \text{Supp}(D)$ by Lemma 3.2. This implies that $\text{mult}_P(D) \leq 2$, since $D \cdot L_1 = D \cdot L_2 = 2$. Then

$$\hat{\alpha}(S, (1 - \beta)C)\beta \text{mult}_P(D) \leq 2\hat{\alpha}(S, (1 - \beta)C)\beta \leq \min\{1, \frac{1}{4} + \beta\}$$

(see Section 2.2). Applying Corollary 3.7 with $n=4$, we get

$$8\hat{\alpha}(S, (1 - \beta)C)\beta = \hat{\alpha}(S, (1 - \beta)C)\beta(C \cdot D) \geq \hat{\alpha}(S, (1 - \beta)C)\beta \text{mult}_P(C \cdot D) > 1 + 4\beta,$$

which contradicts the definition of $\hat{\alpha}(S, (1 - \beta)C)$ in Section 2.2. ■

Lemma 4.5. Suppose that $K_S^2 \leq 3$. Then (4.1) is log canonical at P . □

Proof. Suppose that (4.1) is not log canonical at P . By [3, Theorem 1.12], there is $T \in |-K_S|$ such that (S, T) is not log canonical at P , and *all* irreducible components of the curve T are contained in the support of the divisor D . Moreover, such T is unique.

Since (S, T) is not log canonical at P , we have very limited number of choices for $T \in |-K_S|$. Going through all of them, we see that $(S, (1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta T)$ is log canonical at P (for details, see the proofs of [13, Theorems 4.9.1, 4.10.1, and 4.11.1]).

By Lemma 3.2, there is an effective \mathbb{R} -divisor D' on the surface S such that $D' \sim_{\mathbb{R}} D$, the log pair $(S, (1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta D')$ is not log canonical at P , and $\text{Supp}(D')$ does not contain at least one irreducible component of T . The latter contradicts [3, Theorem 1.12]. ■

Corollary 4.6. Theorem 4.1 holds in the following cases: $S \cong \mathbb{P}^2$, $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $K_S^2 \leq 3$. □

Lemma 4.7. Suppose that $4 \leq K_S^2 \leq 7$, and P is the intersection point of two intersecting (-1) -curves in S . Then (4.1) is log canonical at P . □

Proof. Suppose that (4.1) is not log canonical at P . Denote by Z_1 and Z_2 the two (-1) -curves in S that contain P . We write $D = aZ_1 + bZ_2 + \Omega$, where a and b are non-negative real numbers, and Ω is an effective \mathbb{R} -divisor whose support does not contain Z_1 and Z_2 . By Lemma 4.2, one has $a > 0$ and $b > 0$. Let $x = \text{mult}_P(\Omega)$. Then $1 - b + a = \Omega \cdot Z_1 \geq x$, which gives $b - a + x \leq 1$. Similarly, we obtain $a - b + x \leq 1$. Then $a \leq 1 + b$, $b \leq 1 + a$, and $x \leq 1$. Thus, we have

$$\begin{aligned} \text{mult}_P((1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta\Omega) &= 1 - \beta + \hat{\alpha}(S, (1 - \beta)C)\beta x \\ &\leq 1 - \beta + \hat{\alpha}(S, (1 - \beta)C)\beta \leq 1, \end{aligned}$$

because $\hat{\alpha}(S, (1 - \beta)C) \leq 1$. Applying Theorem 3.9 to (4.1), we see that

$$\begin{aligned} 2(1 - \hat{\alpha}(S, (1 - \beta)C)\beta a) &< Z_1 \cdot (\hat{\alpha}(S, (1 - \beta)C)\beta\Omega + (1 - \beta)C) \\ &= \hat{\alpha}(S, (1 - \beta)C)\beta(1 - a + b) + 1 - \beta, \end{aligned}$$

or

$$\begin{aligned} 2(1 - \hat{\alpha}(S, (1 - \beta)C)\beta b) &< Z_2 \cdot (\hat{\alpha}(S, (1 - \beta)C)\beta\Omega + (1 - \beta)C) \\ &= \hat{\alpha}(S, (1 - \beta)C)\beta(1 - b + a) + 1 - \beta. \end{aligned}$$

In both cases, we obtain $\hat{\alpha}(S, (1 - \beta)C)\beta(1 + a + b) > 1 + \beta$.

Suppose that $K_S^2 = 7$. Let us use the notation of Section 2.4. We may assume that $Z_1 = E_1$ and $Z_2 = L$. Since $3L + 2E_1 + 2E_2 \sim -K_S$ and $(S, (1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta(3L + 2E_1 + 2E_2))$ is log canonical, we may also assume that $E_2 \notin \text{Supp}(\Omega)$ by Lemma 3.2. Then $1 - b = E_2 \cdot \Omega \geq 0$, which gives $b \leq 1$. Since $a \leq 1 + b$, we get $a + b \leq 3$. Thus, we have

$$4\beta\hat{\alpha}(S, (1 - \beta)C) \geq \hat{\alpha}(S, (1 - \beta)C)\beta(1 + a + b) > 1 + \beta,$$

which contradicts the definition of $\hat{\alpha}(S, (1 - \beta)C)$.

Suppose that $K_S^2 = 6$. Let us use the notation of Section 2.5. Without loss of generality, we may assume that $Z_1 = E_1$ and $Z_2 = L_{12}$. Since $(S, (1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta(2L_{12} + 2E_1 + L_{13} + E_2))$ is log canonical and $2L_{12} + 2E_1 + L_{13} + E_2 \sim -K_S$, we may assume that $\text{Supp}(\Omega)$ does not contain L_{13} or E_2 by Lemma 3.2. If $L_{13} \notin \text{Supp}(\Omega)$, then $1 - a = \Omega \cdot L_{13} \geq 0$, which implies that $a \leq 1$. Similarly, if $E_2 \notin \text{Supp}(\Omega)$, then $b \leq 1$. Since $a \leq 1 + b$ and $b \leq 1 + a$, we see that $a + b \leq 3$. Thus, we have

$$4\beta\hat{\alpha}(S, (1 - \beta)C) \geq \hat{\alpha}(S, (1 - \beta)C)\beta(1 + a + b) > 1 + \beta,$$

which contradicts the definition of $\hat{\alpha}(S, (1 - \beta)C)$.

Suppose that $K_S^2 = 5$. Let us use the notation of Section 2.6. Without loss of generality, we may assume that $Z_1 = E_1$ and $Z_2 = L_{12}$. Since $(S, (1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta(2E_1 + L_{12} + L_{13} + L_{14}))$ is log canonical and $2E_1 + L_{12} + L_{13} + L_{14} \sim -K_S$, we may assume that $\text{Supp}(\Omega)$ does not contain L_{13} or L_{14} by Lemma 3.2. Since $(S, (1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta(E_1 + 2L_{12} + E_2 + L_{34}))$ is log canonical and $E_1 + 2L_{12} + E_2 + L_{34} \sim -K_S$, we may assume that $\text{Supp}(\Omega)$ does not contain E_2 or L_{34} by Lemma 3.2. If $L_{13} \notin \text{Supp}(\Omega)$, then $1 - a = \Omega \cdot L_{13} \geq 0$, which gives $a \leq 1$. Similarly, if $L_{14} \notin \text{Supp}(\Omega)$, then $a \leq 1$. If $E_2 \notin \text{Supp}(\Omega)$, then $1 - b = \Omega \cdot E_2 \geq 0$, which gives $b \leq 1$. Similarly, if $L_{34} \notin \text{Supp}(\Omega)$, then $b \leq 1$. Thus, we have $a \leq 1$ and $b \leq 1$. Then

$$3\beta\hat{\alpha}(S, (1 - \beta)C) \geq \hat{\alpha}(S, (1 - \beta)C)\beta(1 + a + b) > 1 + \beta,$$

which contradicts the definition of $\hat{\alpha}(S, (1 - \beta)C)$.

We have $K_S^2 = 4$. Let us use the notation of Section 2.7. Without loss of generality, we may assume that $Z_1 = L_{12}$ and $Z_2 = L_{34}$. Let Z be the proper transform on S of the line in \mathbb{P}^2 that passes through $\pi(E_5)$ and $\pi(L_{12} \cap L_{34})$. Since $(S, (1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta(L_{12} + L_{34} + Z))$ is log canonical and $L_{12} + L_{34} + Z \sim -K_S$, we may assume

that $Z \notin \text{Supp}(\Omega)$ by Lemma 3.2. Then $2 - a - b = \Omega \cdot Z \geq 0$, which implies that $3\beta\hat{\alpha}(S, (1 - \beta)C) \geq \hat{\alpha}(S, (1 - \beta)C)\beta(1 + a + b) > 1 + \beta$. The latter contradicts the definition of $\hat{\alpha}(S, (1 - \beta)C)$. ■

Lemma 4.8. Suppose that $S \cong \mathbb{F}_1$, and P is contained in the unique (-1) -curve in S . Then (4.1) is log canonical at P . □

Proof. Let us use the notation of Section 2.3. Then $P = Z \cap C$, since $P \in C$. Suppose that (4.1) is not log canonical at P . By Lemma 4.2, we have $Z \subset \text{Supp}(D)$. By Lemma 3.2, we may assume that $F \notin \text{Supp}(D)$, since $(S, (1 - \beta)C + \hat{\alpha}(S, (1 - \beta)C)\beta(2Z + 3F))$ is log canonical and $2Z + 3F \sim -K_S$. Then $\text{mult}_P(D) \leq F \cdot D = 2$. On the other hand, we have $2\hat{\alpha}(S, (1 - \beta)C)\beta \leq \frac{1}{4} + \beta$ and $2\hat{\alpha}(S, (1 - \beta)C)\beta \leq 1$. Applying Corollary 3.7 with $n = 4$ to (4.1), we get

$$8\hat{\alpha}(S, (1 - \beta)C)\beta = \hat{\alpha}(S, (1 - \beta)C)\beta(C \cdot D) \geq \hat{\alpha}(S, (1 - \beta)C)\beta \text{mult}_P(C \cdot D) > 1 + 4\beta,$$

which contradicts the definition of $\hat{\alpha}(S, (1 - \beta)C)$. ■

Lemma 4.9. Suppose that $4 \leq K_S^2 \leq 7$, and P is contained in a (-1) -curve in S . Then (4.1) is log canonical at P . □

Proof. For the proof, see Section 5. ■

The following result implies Theorem 1.9 *modulo* Theorem 4.1.

Theorem 4.10. Let S_1 and S_2 be smooth del Pezzo surfaces and let C_1 and C_2 be smooth curves in $|-K_{S_1}|$ and $|-K_{S_2}|$, respectively. Suppose that there exists a birational morphism $f: S_2 \rightarrow S_1$ such that $f(C_2) = C_1$. Then $\hat{\alpha}(S_1, (1 - \beta)C_1) \leq \hat{\alpha}(S_2, (1 - \beta)C_2)$ for every $\beta \in (0, 1]$ except for the following cases:

- (1) $S_1 \cong \mathbb{P}^2$, $S_2 \cong \mathbb{F}_1$, and f is the blowup of an inflection points of the cubic curve $C_1 \subset \mathbb{P}^2$;
- (2) $S_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, $K_{S_2}^2 = 7$, and f is the blowup of a point in C_1 . □

Proof. Since $f(C_2) = C_1$, the morphism f is the blowup of $K_{S_1}^2 - K_{S_2}^2 \geq 0$ distinct points on the curve C_2 . Suppose that $\hat{\alpha}(S_1, (1 - \beta)C_1) > \hat{\alpha}(S_2, (1 - \beta)C_2)$. Going through all possible cases considered in Section 2, we end up with the following possibilities:

- (1) $S_1 \cong \mathbb{P}^2$, $S_2 \cong \mathbb{F}_1$, and f is the blowup of an inflection points of the cubic curve $C_1 \subset \mathbb{P}^2$,

- (2) $S_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, $K_{S_2}^2 = 7$, and f is the blowup of a point in C_1 ,
- (3) $K_{S_1}^2 = 4$, $K_{S_2}^2 = 3$, the morphism f is the blowup of a point in C_1 , the curve C_1 does not contain intersection points of any two lines, for every two conics Z_1 and Z_2 in S_1 such that $Z_1 + Z_2 \sim -K_{S_1}$, the conics Z_1 and Z_2 do not tangent C_1 at one point, and S_2 contains an Eckardt point and this point is contained in C_2 ,
- (4) $K_{S_1}^2 = 3$, $K_{S_2}^2 = 2$, the morphism f is the blowup of a point in C_1 , the surface S_1 contains no Eckardt points, for every line L and every conic M on S_1 such that L is tangent to M we have $L \cap M \notin C_1$, and every irreducible cuspidal curve $T \in |-K_{S_1}|$ intersects C_1 by at least two points, the linear system $|-K_{S_2}|$ contains a curve with a tacnodal singularity and this tacnodal singular point is contained in C_2 .

The first two cases are indeed possible. Let us show that the last two cases are impossible. Denote by E the f -exceptional curve. Then $f(E) \in C_1$.

Suppose that $K_{S_1}^2 = 4$ and $K_{S_2}^2 = 3$. Then C_2 contains an Eckardt point O . Denote by L_1, L_2, L_3 the lines in S_2 that passes through O . Then either E is one of these three lines, or E intersects exactly one of them. Without loss of generality, we may assume that either $E = L_3$ or $E \cap L_1 = E \cap L_3 = \emptyset$. In the former case, $f(L_1)$ and $f(L_2)$ are two conics in S_1 such that $f(L_1) + f(L_2) \sim -K_{S_2}$, and both $f(L_1)$ and $f(L_2)$ tangent the curve $C_1 = f(C_2)$ at the point $f(P) \in C_1$. Since we know that such conics do not exist by assumption, we conclude that $E \cap L_1 = E \cap L_3 = \emptyset$. Then $f(L_1)$ and $f(L_2)$ are two lines in S_1 that both pass through the point $f(P) \in C_1$. Such lines do not exist either. Thus, this case is impossible.

Now we suppose that $K_{S_1}^2 = 3$ and $K_{S_2}^2 = 2$. Let Z be a curve in $|-K_{S_2}|$ such that Z has tacnodal singularity $Q \in C_2$. Then $Z = L_1 + L_2$, where L_1 and L_2 are two (-1) -curves in S_2 that are tangent to each other at the point $Q \in C_2$. Then either E is one of these two curves, or E intersects exactly one of them. Without loss of generality, we may assume that either $E = L_2$ or $E \cap L_1 = \emptyset$. In the former case, $f(L_1)$ is a cuspidal curve in $|-K_{S_1}|$ whose intersection with the curve C_1 consists of the point $f(Q) = \text{Sing}(f(L_1))$. By assumption, such a cuspidal curve does not exist. Thus, $E \cap L_1 = \emptyset$. Then $f(L_1)$ is a line and $f(L_2)$ is a conic. Moreover, the line $f(L_1)$ tangents to $f(L_2)$ at the point $f(Q) \in C_1$. The latter is impossible by assumption. ■

To prove Theorem 4.1, we have to prove that (4.1) is log canonical at P , where P is a point in $C \not\subset \text{Supp}(D)$. The latter follows from Corollary 4.6, Lemmas 4.7–4.9, and the following lemma.

Lemma 4.11. Suppose that $K_S^2 \geq 3$, and neither $S \cong \mathbb{P}^2$ nor $S \cong \mathbb{P}^1 \times \mathbb{P}^1$. Suppose that P is not contained in any (-1) -curve in S . If Theorem 4.1 holds for all smooth del Pezzo surfaces of degree $K_S^2 - 1$, then (4.1) is log canonical at P . \square

Proof. Suppose that (4.1) is not log canonical at P . Let $f: \tilde{S} \rightarrow S$ be a blowup of P . Then \tilde{S} is a smooth del Pezzo surface of degree $K_{\tilde{S}}^2 = K_S^2 - 1$, since P is not contained in any (-1) -curve in S . Denote the f -exceptional curve by E , denote the proper transform of C on \tilde{S} by \tilde{C} , and denote the proper transform of D on \tilde{S} by \tilde{D} . Then $\tilde{C} \in |-K_{\tilde{S}}|$, since $P \in C$. The log pair

$$\left(\tilde{S}, (1 - \beta)\tilde{C} + \hat{\alpha}(S, (1 - \beta)C)\beta \left(\tilde{D} + \left(\text{mult}_P(D) - \frac{1}{\hat{\alpha}(S, (1 - \beta)C)} \right) E \right) \right) \quad (4.2)$$

is not log canonical by Lemma 3.3. Let $\tilde{D}' = \tilde{D} + (\text{mult}_P(D) - 1)E$. Then $\tilde{D}' \sim_{\mathbb{R}} -K_{\tilde{S}}$, and \tilde{D}' is effective by Lemma 4.2. Furthermore, the log pair $(\tilde{S}, (1 - \beta)\tilde{C} + \hat{\alpha}(S, (1 - \beta)C)\beta\tilde{D}')$ is not log canonical because (4.2) is not log canonical. This shows that $\hat{\alpha}(S, (1 - \beta)C) > \alpha(\tilde{S}, (1 - \beta)\tilde{C})$. But it follows from Theorem 4.10 that $\hat{\alpha}(\tilde{S}, (1 - \beta)\tilde{C}) \geq \hat{\alpha}(S, (1 - \beta)C)$. Thus, we see that $\hat{\alpha}(\tilde{S}, (1 - \beta)\tilde{C}) > \alpha(\tilde{S}, (1 - \beta)\tilde{C})$. Hence, Theorem 4.1 does not hold for \tilde{S} . \blacksquare

This completes the proof of Theorem 4.1 modulo Lemma 4.9.

5 The Proof of Lemma 4.9

In this section, we will prove Lemma 4.9. Let us use its notation and assumptions. Then $4 \leq K_S^2 \leq 7$ and P is a point in $C \not\subset \text{Supp}(D)$ that is contained in a (-1) -curve in S . Let us denote this (-1) -curve by \mathcal{L} . We must prove that (4.1) is log canonical at P . By Lemma 4.7, we may assume that \mathcal{L} is the only (-1) -curve in S that contains P . We write $D = a\mathcal{L} + \Omega$, where a is a non-negative real number, and Ω is an effective \mathbb{R} -divisor such that $\mathcal{L} \not\subset \text{Supp}(\Omega)$. By Lemma 4.2, we have $a > 0$. Let $x = \text{mult}_P(\Omega)$. Then $1 + a = \mathcal{L} \cdot \Omega \geq x$.

Corollary 5.1. One has $x \leq 1 + a$. \square

Let $\lambda = \hat{\alpha}(S, (1 - \beta)C)$. Consider a sequence of four blowups

$$S_4 \xrightarrow{\pi_4} S_3 \xrightarrow{\pi_3} S_2 \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S$$

such that π_1 is the blowup of the point P , π_2 is the blowup of the intersection point of the π_1 -exceptional curve and the proper transform of the curve C on S_1 , π_3 is the blowup of the intersection point of the π_2 -exceptional curve and the proper transform of the curve

C on S_2 , and π_4 is the blowup of the intersection point of the π_3 -exceptional curve and the proper transform of the curve C on S_3 . Denote by F_1, F_2, F_3 , and F_4 the exceptional curves of the blowups π_1, π_2, π_3 , and π_4 , respectively. Denote by C^1, C^2, C^3 , and C^4 the proper transforms of the curve C on the surfaces S_1, S_2, S_3 , and S_4 , respectively. Let $P_1 = C^1 \cap F_1, P_2 = C^2 \cap F_2, P_3 = C^3 \cap F_3$, and $P_4 = C^4 \cap F_4$. Denote the proper transform of the divisor Ω on the surfaces S_1, S_2, S_3 , and S_4 by $\Omega^1, \Omega^2, \Omega^3$, and Ω^4 , respectively. Let $x_1 = \text{mult}_{P_1}(\Omega), x_2 = \text{mult}_{P_2}(\Omega)$, and $x_3 = \text{mult}_{P_3}(\Omega)$.

Lemma 5.2. Suppose that (4.1) is not log canonical at P . Then at least one of the following four conditions is not satisfied:

- (i) $\lambda\beta(a + x) \leq 1$;
- (ii) $2\lambda\beta(a + x) - 2\beta \leq 1$ or $\lambda\beta(a + x + x_1) - \beta \leq 1$;
- (iii) $\lambda\beta(a + x + 2x_1) - 3\beta \leq 1$ or $\lambda\beta(a + x + x_1 + x_2) - 2\beta \leq 1$;
- (iv) $\lambda\beta(a + x + x_1 + 2x_2) - 4\beta \leq 1$ or $\lambda\beta(a + x + x_1 + x_2 + x_3) - 3\beta \leq 1$.

If $\lambda\beta K_S^2 \leq 1 + 3\beta$, then at least one of the conditions (i), (ii), or (iii) is not satisfied. □

Proof. If conditions (i)–(iv) are satisfied, then Corollary 3.8 gives

$$K_S^2 = D \cdot C \geq \text{mult}_P(D \cdot C) > \frac{1 + 4\beta}{\lambda\beta},$$

which is impossible, since $\lambda\beta K_S^2 \leq 1 + 4\beta$ by the definition of $\lambda = \hat{\alpha}(S, (1 - \beta)C)$ for $4 \leq K_S^2 \leq 7$. Similarly, if conditions (i)–(iii) are satisfied, then $\lambda\beta K_S^2 > 1 + 3\beta$ by Corollary 3.8. ■

Lemma 5.3. Suppose that $K_S^2 = 7$. Then (4.1) is log canonical at P . □

Proof. Suppose that (4.1) is not log canonical at P . Let us use the notation of Section 2.4. Without loss of generality, we may assume that either $\mathcal{L} = E_1$ or $\mathcal{L} = L$ (but not both).

Suppose that $\mathcal{L} = L$. Since $P \notin E_1 \cup E_2$, the curve R is smooth and irreducible. Since $(S, (1 - \beta)C, \lambda\beta(L + 2R))$ is log canonical and $L + 2R \sim -K_S$, we may assume that $R \not\subset \text{Supp}(\Omega)$. Denote the proper transform of the curve R on S_1 by R^1 , and denote its proper transform on S_2 by R^2 . Then $3 - a - x - x_1 = R^2 \cdot \Omega^2 \geq 0$, which gives $a + x + x_1 \leq 3$. Since $x - a \leq 1$ by Corollary 5.1, then $x_1 \leq \frac{4}{3}$ and all conditions of Lemma 5.2 are satisfied, giving a contradiction.

We have $\mathcal{L} = E_1$. Then L_1 is irreducible, since $P \notin L$. Since $(S, (1 - \beta)C, \lambda\beta(2L_1 + 2E_1 + L))$ is log canonical and $2L_1 + 2E_1 + L \sim -K_S$, we may assume that L_1

or L is not contained in $\text{Supp}(\Omega)$ by Lemma 3.2. We write $\Omega = bL_1 + \Delta$, where b is a non-negative real number, and Δ is an effective \mathbb{R} -divisor on S such that $L_1 \not\subset \text{Supp}(\Delta)$ and $E_1 \not\subset \text{Supp}(\Delta)$. Then $1 - b + a = E_1 \cdot \Delta \geq y$, which gives $b + y \leq 1 + a$. If $b > 0$, then $a \leq 1$. Indeed, if $L \not\subset \text{Supp}(\Delta)$, then $1 - a = L \cdot \Delta \geq 0$.

Denote the proper transform of the divisor Δ on S_1 by Δ^1 , denote the proper transform of the divisor Δ on S_2 by Δ^2 , and denote the proper transform of the divisor Δ on S_3 by Δ^3 . Let $y = \text{mult}_P(\Delta)$, $y_1 = \text{mult}_{P_1}(\Delta^1)$, $y_2 = \text{mult}_{P_2}(\Delta^2)$, and $y_3 = \text{mult}_{P_3}(\Delta^3)$. Then $x = b + y$. Since $L_1 \cdot C = 2$, either $\text{mult}_P(L_1 \cdot C) = 1$ or $\text{mult}_P(L_1 \cdot C) = 2$. Thus, we have $x_2 = y_2$ and $x_3 = y_3$.

Suppose that $\text{mult}_P(L_1 \cdot C) = 1$. Then $x_1 = y_1$ and $2 - a = L_1 \cdot \Delta \geq y$. We have $b + y \leq 1 + a$ by Corollary 5.1. If $b > 0$, then $a \leq 1$. Therefore, we have $\lambda\beta(a + x) \leq 1$, $\lambda\beta(a + x + x_1) - \beta \leq 1$, $\lambda\beta(a + x + 2x_1) - 3\beta \leq 1$, and $\lambda\beta(a + x + x_1 + 2x_2) - 4\beta \leq 1$, which contradicts Lemma 5.2.

Thus, we see that $\text{mult}_P(L_1 \cdot C) = 2$. Then $x_1 = y_1 + b$ and $2 - a = L_1 \cdot \Delta \geq y + y_1$, which gives $a + y + y_1 \leq 2$. Since L_1 is tangent to C at the point P , we have

$$\lambda = \hat{\alpha}(S, (1 - \beta)C) \leq \min \left\{ 1, \frac{1 + 2\beta}{7\beta}, \frac{1}{3\beta} \right\}.$$

Moreover, we have $b + y \leq 1 + a$ by Corollary 5.1. Furthermore, if $b > 0$, then $a \leq 1$. This gives $\lambda\beta(a + x) \leq 1$, $2\lambda\beta(a + x) - 2\beta \leq 1$, $\lambda\beta(a + x + x_1 + x_2) - 2\beta \leq 1$, and $\lambda\beta(a + x + x_1 + 2x_2) - 4\beta \leq 1$, which is impossible by Lemma 5.2. ■

Lemma 5.4. Suppose that $K_S^2 = 6$. Then (4.1) is log canonical at P . □

Proof. Suppose that (4.1) is not log canonical at P . Let us use the notation of Section 2.5. Without loss of generality, we may assume that $\mathcal{L} = E_1$. Denote the proper transform of the curve E_1 on the surface S_1 by E_1^1 . Let L be the proper transform on S of the line in \mathbb{P}^2 that is tangent to $\pi(C)$ at the point $\pi(P)$. Then $-K_S \cdot L = 2$, since $P \notin L_{12} \cup L_{13} \cup L_{23}$. Denote the proper transform of the curve L on S_1 by L^1 , denote the proper transform of the curve L on S_2 by L^2 , and denote the proper transform of the curve L on S_3 by L^3 .

We claim that $L \subset \text{Supp}(\Omega)$. Indeed, suppose that $L \not\subset \text{Supp}(\Omega)$. Then $a + x \leq 2$, since $2 - a = \Omega \cdot L \geq x$. But $x \leq 1 + a$ by Corollary 5.1. Therefore, we have $x_1 \leq x \leq \frac{3}{2}$. These inequalities give $\lambda\beta(a + x) \leq 1$, $2\lambda\beta(a + x) - \beta \leq 1$, and $\lambda\beta(a + x + 2x_1) - 3\beta \leq 1$. Therefore, $\lambda\beta(a + x + x_1 + 2x_2) - 4\beta > 1$ and $6\lambda\beta > 1 + 3\beta$ by Lemma 5.2. The former inequality implies $a + x + x_1 + 2x_2 > 6$. The latter inequality implies that L is not tangent to C at the point P (see Section 2.5).

Let Z be the proper transform on S of the conic in \mathbb{P}^2 that passes through the points $\pi(E_1)$, $\pi(E_2)$, $\pi(E_3)$, and is tangent to $\pi(C)$ at the point $\pi(P)$. Then Z is irreducible, $E_1 + L + Z \sim -K_S$, and $-K_S \cdot Z = 3$, since L is not tangent to C at P . Then $\text{mult}_P(Z \cdot C) \leq 3$, since $-K_S \cdot Z = 3$.

We write $\Omega = cZ + \gamma$, where c is a non-negative real number, and γ is an effective \mathbb{R} -divisor on S whose support does not contain Z . Denote the proper transform of the divisor γ on S_1 by γ^1 , denote the proper transform of the divisor γ on S_2 by γ^2 , and denote the proper transform of the divisor γ on S_3 by γ^3 . Let $z = \text{mult}_P(\gamma)$, $z_1 = \text{mult}_{P_1}(\gamma^1)$, $z_2 = \text{mult}_{P_2}(\gamma^2)$, $z_3 = \text{mult}_{P_3}(\gamma^3)$. Then $x = c + z$, $x_1 = c + z_1$, $x_3 = z_3$. If $\text{mult}_P(Z \cdot C) = 2$, then $x_2 = z_2$ and $3 - a - c - z = Z^1 \cdot \gamma^1 \geq \text{mult}_{P_1}(Z^1 \cdot \gamma^1) \geq z_1$, which implies that

$$6 < a + x + x_1 + 2x_2 = a + z + z_1 + 2z_2 + 2c \leq 3 + 2z_2 + c \leq 3 + 2z_2 + 2c \leq 3 + 2x \leq 6,$$

since $z + c \leq \frac{3}{2}$ and $a + c + z \leq 2$. Thus, we see that $\text{mult}_P(Z \cdot C) = 3$. Then $x_2 = c + z_2$ and $3 - a - c - z - z_1 = Z^2 \cdot \gamma^2 \geq \text{mult}_{P_2}(Z^2 \cdot \gamma^2) \geq z_2$, which gives $a + c + z + z_1 + z_2 \leq 3$. Then

$$6 < a + x + x_1 + 2x_2 = a + z + z_1 + 2z_2 + 3c < 3 + z_2 + 2c \leq 3 + 2z_2 + 2c \leq 3 + 2x \leq 6,$$

which is absurd. This shows that $L \subset \text{Supp}(\Omega)$.

We write $\Omega = bL + \Delta$, where b is a positive real number, and Δ is an effective \mathbb{R} -divisor on S such that $L \not\subset \text{Supp}(\Delta)$. Let $y = \text{mult}_P(\Delta)$. Then $2 - a = \Delta \cdot L \geq y$. Denote the proper transform of the divisor Δ on S_1 by Δ^1 , denote the proper transform of the divisor Δ on S_2 by Δ^2 , and denote the proper transform of the divisor Δ on S_3 by Δ^3 . Let $y_1 = \text{mult}_{P_1}(\Delta^1)$, $y_2 = \text{mult}_{P_2}(\Delta^2)$, and $y_3 = \text{mult}_{P_3}(\Delta^3)$. Then $x = b + y$, $x_2 = y_2$, and $x_3 = y_3$, which implies that $b + y \leq 1 + a$ by Corollary 5.1. Then

$$(S_1, (1 - \beta)C^1 + \lambda\beta aE_1^1 + \lambda\beta bL^1 + \lambda\beta\Delta^1 + (\lambda\beta(a + b + y) - \beta)F_1) \tag{5.1}$$

is not log canonical at some point $Q_1 \in F_1$ by Lemma 3.3.

We claim that L is tangent to C at the point P . Indeed, suppose that L is not tangent to C at P . Then $x_1 = y_1$. Let Z be the proper transform on S of the conic in \mathbb{P}^2 that passes through $\pi(E_1)$, $\pi(E_2)$, $\pi(E_3)$, and is tangent to $\pi(C)$ at $\pi(P)$. Then Z is irreducible and $-K_S \cdot Z = 3$. Moreover, we have $E_1 + L + Z \sim -K_S$, and the log pair $(S, (1 - \beta)C + \lambda\beta(E_1 + L + Z))$ is log canonical. Thus, we may assume that $Z \not\subset \text{Supp}(D)$ by Lemmas 3.2. Then $3 - a - b - y = Z^1 \cdot \Delta^1 \geq \text{mult}_{P_1}(Z^1 \cdot \Delta^1) \geq y_1$. Since we also have

$b + y \leq 1 + a$, $a + y \leq 2$, $x = y + b$, $x_1 = y_1$, and $x_2 = y_2$, we see that

$$\begin{aligned} \lambda\beta y_1 \leq 1, \quad \lambda\beta(a + b + y) - \beta \leq \lambda\beta(a + b + y + y_1) - \beta \leq 1, \\ \lambda\beta(a + b + y + 2y_1) - 3\beta \leq 1, \quad \lambda\beta(a + b + y_1 + 2y_2) - 4\beta \leq 1. \end{aligned} \tag{5.2}$$

In particular, (5.1) is log canonical at every point of F_1 that is different from Q_1 by Lemma 3.3. If $Q_1 \neq L^1 \cap F_1$ and $Q_1 \neq P_1$, then $\lambda\beta(a + y) = F_1 \cdot (\lambda\beta(aE_1 + \Delta^1)) > 1$, by Theorem 3.4. But $\lambda\beta(a + y) \leq 1$, since $a + y \leq 2$. This shows that $Q_1 = L^1 \cap F_1$ or $Q_1 = P_1$. Since $b - a + y \leq 1$ and $a + b + y + y_1 \leq 3$, we have $b + y \leq 2$. So, if $Q_1 = L^1 \cap F_1$, then

$$1 < \lambda\beta F_1 \cdot (bL^1 + \Delta^1) = \lambda\beta(b + y) \leq 2\lambda\beta \leq 1,$$

by Theorem 3.4. If $Q_1 = P_1$, then $6 = D \cdot C > \frac{1+4\beta}{\lambda\beta}$ by (5.2) and Theorem 3.6. The latter contradicts $6\lambda\beta \leq 1 + 4\beta$.

We see that L is tangent to C at the point P . Then $x_1 = y_1 + b$ and

$$\lambda \leq \min \left\{ 1, \frac{1 + 2\beta}{5\beta}, \frac{1}{2\beta} \right\},$$

which gives $6\lambda\beta \leq 1 + 3\beta$. Moreover, we have $a + y + y_1 \leq 2$, because $2 - a - y - y_1 = L^2 \cdot \Delta^2 \geq 0$. Furthermore, since $2L + L_{23} + E_1 \sim -K_S$ and $(S, (1 - \beta)C + \lambda\beta(2L + L_{23} + E_1))$ is log canonical, we may assume that $L_{23} \not\subset \text{Supp}(\Delta)$ by Lemma 3.2. This gives us $b \leq 1$, because $1 - b = \Delta \cdot L_{23} \geq 0$. Since $L + L_{12} + L_{13} + 2E_1 \sim -K_S$ and $(S, (1 - \beta)C + \lambda\beta(L + L_{12} + L_{13} + 2E_1))$ is log canonical, we may assume that $L_{12} \not\subset \text{Supp}(\Delta)$ or $L_{13} \not\subset \text{Supp}(\Delta)$ by Lemma 3.2. If $L_{12} \not\subset \text{Supp}(\Delta)$, then $1 - a = \Delta \cdot L_{12} \geq 0$, which gives $a \leq 1$. Similarly, we get $a \leq 1$ if $L_{13} \not\subset \text{Supp}(\Delta)$. Thus, we have

$$a \leq 1, \quad b \leq 1, \quad b - a + y \leq 1, \quad a + y + y_1 \leq 2, \tag{5.3}$$

which implies that $\lambda\beta(a + b + y) - \beta \leq 1$. In particular, (5.1) is log canonical at every point of F_1 that is different from Q_1 by Lemma 3.3. If $Q_1 \neq P_1$ and $Q_1 \neq E_1^1 \cap F_1$, then $\lambda\beta y = \lambda\beta \Delta^1 \cdot F_1 > 1$ by Theorem 3.4. The latter is impossible, since $\lambda\beta y \leq 2\lambda\beta \leq 1$ by (5.3). If $Q_1 = E_1^1 \cap F_1$, then

$$1 < E_1^1 \cdot (\lambda\beta \Delta^1 + (\lambda\beta(a + b + y) - \beta)F_1) = \lambda\beta(1 + 2a) - \beta$$

by Theorem 3.4. The latter is impossible, since $\lambda\beta(1 + 2a) - \beta \leq 3\lambda\beta - \beta \leq 1$ by (5.3). Thus, we see that $Q_1 = P_1$.

By (5.3), one has $a + 2b + y + y_1 \leq 4$. This implies that $\lambda\beta(a + 2b + y + y_1) - 2\beta \leq 1$. Then

$$(S_2, (1 - \beta)C^2 + \lambda\beta bL^2 + \lambda\beta\Delta^2 + (\lambda\beta(a + b + y) - \beta)F_1^2 + (\lambda\beta(a + 2b + y + y_1) - 2\beta)F_2)$$

is not log canonical at a unique point $Q_2 \in F_2$ by Lemma 3.3. If $Q_2 \notin L^2 \cup F_1^2 \cup C^2$, then $\lambda\beta y_2 = \lambda\beta\Delta^2 \cdot F_2 > 1$ by Theorem 3.4, which is impossible, since $\lambda\beta y_2 \leq 1$ by (5.3). Similarly, if $Q_2 = F_2 \cap L^2$, then $\lambda\beta(b + y_2) = \lambda\beta(bL^2 + \Delta^2) \cdot F_2 > 1$ by Theorem 3.4, which is impossible because $b + y_2 \leq b + y \leq 2$ by (5.3). If $Q_2 = F_2 \cap F_1^2$, then

$$\lambda\beta(y + y_1 + a + b) - \beta = (\lambda\beta\Delta^2 + (\lambda\beta(a + b + y) - \beta)F_1^2) \cdot F_2 > 1$$

by Theorem 3.4, which is impossible, since $y + y_1 + a + b \leq 3$ by (5.3). Then $Q_2 = P_2$.

We have $\lambda\beta(a + 2b + y + y_1 + y_2) - 3\beta \leq 1$, since $a + 2b + y + y_1 + y_2 \leq 5$ by (5.3).

Then

$$(S_3, (1 - \beta)C^3 + \lambda\beta\Delta^3 + (\lambda\beta(a + 2b + y + y_1) - 2\beta)F_2^3 + (\lambda\beta(a + 2b + y + y_1 + y_2) - 3\beta)F_3)$$

is not log canonical at a *unique* point $Q_3 \in F_3$ by Lemma 3.3. If $Q_3 \notin F_2^3 \cup C^3$, then $\lambda\beta y_3 = \lambda\beta\Delta^3 \cdot F_3 > 1$ by Theorem 3.4, which is impossible, because $\lambda\beta y_3 \leq 1$ by (5.3). If $Q_3 = F_3 \cap F_2^3$, then Theorem 3.4 gives

$$1 < F_2^3 \cdot (\lambda\beta\Delta^3 + (\lambda\beta(a + 2b + y + y_1 + y_2) - 3\beta)F_3) = \lambda\beta(a + 2b + y + 2y_1) - 3\beta \leq 5\lambda\beta - 3\beta,$$

which is impossible, since $a + 2b + y + 2y_1 \leq 5$ by (5.3). Thus, we see that $Q_3 = P_3$. By Theorem 3.6(iv), we have $6 = D \cdot C \geq \text{mult}_P(D \cdot C) > \frac{1+3\beta}{\lambda\beta}$. The latter is impossible, since we already proved earlier that $6\lambda\beta \leq 1 + 3\beta$. ■

Lemma 5.5. Suppose that $K_S^2 = 5$. Then (4.1) is log canonical at P . □

Proof. Suppose that (4.1) is not log canonical at P . Let us use the notation of Section 2.5. Then $\lambda = \min\{1, \frac{1}{2\beta}\}$. This implies that $5\lambda\beta \leq 1 + 3\beta$. By Lemma 5.2, at least one of the conditions (i)–(iii) in Lemma 5.2 is not satisfied. In particular, if $a + x \leq 2$, then $\lambda\beta(a + x + 2x_1) - 3\beta > 1$.

Without loss of generality, we may assume that $\mathcal{L} = L_{12}$. Let B_3 be the proper transform on S of the line in \mathbb{P}^2 that passes through $\pi(P)$ and $\pi(E_3)$, and let B_4 be the proper transform on S of the line in \mathbb{P}^2 that passes through $\pi(P)$ and $\pi(E_4)$. Since $L_{12} + B_3 + B_4 \sim -K_S$ and $(S, (1 - \beta)C + \lambda\beta(L_{12} + B_3 + B_4))$ is log canonical, we may

assume that at least one curve among B_3 and B_4 is not contained in $\text{Supp}(\Omega)$. Intersecting this curve with Ω , we get $a + x \leq 2$. Then $\lambda\beta(a + x + 2x_1) - 3\beta > 1$. This implies that $a + x + 2x_1 > 5$.

Denote the proper transform of the curve B_3 on the surface S_1 by B_3^1 , and denote the proper transform of the curve B_4 on the surface S_1 by B_4^1 . Recall $P_1 = C^1 \cap F_1$.

Suppose that $P_1 \notin B_3^1 \cup B_4^1$. Then B_3 and B_4 do not tangent C at P . Let R be the proper transform on S of the line in \mathbb{P}^2 that is tangent to $\pi(C)$ at the point $\pi(P)$, let R_1 be the proper transform on S of the conic in \mathbb{P}^2 that tangents to $\pi(C)$ at the point $\pi(P)$ and passes through the points $\pi(E_2)$, $\pi(E_3)$, and $\pi(E_4)$, and let R_2 be the proper transform on S of the conic in \mathbb{P}^2 that tangents to $\pi(C)$ at the point $\pi(P)$ and passes through the points $\pi(E_1)$, $\pi(E_3)$, and $\pi(E_4)$. Since $P_1 \notin B_3^1 \cup B_4^1$, the curves R_1 and R_2 are irreducible. Hence, $\frac{1}{2}L_{12} + \frac{1}{2}R + \frac{1}{2}R_1 + \frac{1}{2}R_2 \sim_{\mathbb{R}} -K_S$ and $(S, (1 - \beta)C + \lambda\beta(\frac{1}{2}L_{12} + \frac{1}{2}R + \frac{1}{2}R_1 + \frac{1}{2}R_2))$ is log canonical. By Lemma 3.2, we may assume that one curve among R , R_1 , and R_2 is not contained in $\text{Supp}(D)$. Denote this curve by Z , and denote its proper transform on S_1 by Z^1 . Then $P_1 \in Z^1$ and $3 - a - x = Z^1 \cdot \Omega^1 \geq x_1$, which is impossible, since $a + x \leq 2$ and $a + x + 2x_1 > 5$.

We see that $P_1 = B_3^1 \cap F_1$ or $P_1 = B_4^1 \cap F_1$. Without loss of generality, we may assume that $P_1 = B_3^1 \cap F_1$. Then $B_3 \subset \text{Supp}(\Omega)$, since otherwise we would have $2 - a - x = B_3^1 \cdot \Omega^1 \geq x_1$, which is impossible, since $a + x \leq 2$. We write $\Omega = bB_3 + \Delta$, where $b \in \mathbb{R}_{>0}$ and Δ is an effective \mathbb{R} -divisor on S such that $B_3 \not\subset \text{Supp}(\Delta)$. Denote the proper transform of the divisor Δ on S_1 by Δ^1 . Let $y = \text{mult}_P(\Delta)$ and $y_1 = \text{mult}_{P_1}(\Delta^1)$. Then $x = b + y$ and $x_1 = b + y_1$. We have $b - a + y \leq 1$ by Corollary 5.1 and $a + b + y = a + x \leq 2$, which implies a contradiction $a + x + 2x_1 \leq 2 + 2y + 2b \leq 6$. ■

Lemma 5.6. Suppose that $K_S^2 = 4$. Then (4.1) is log canonical at P . □

Proof. Suppose that (4.1) is not log canonical at P . Let us use the notation of Section 2.7. Then $\lambda\beta < \frac{2}{3}$. Without loss of generality, we may assume that $P \in E$. Then $P = E \cap C$. By Lemma 4.7, the point P is not contained in any other (-1) -curve. By Lemma 4.2, we have $E \subset \text{Supp}(D)$.

The log pair $(S, (1 - \beta)C + \lambda\beta(\frac{3}{2}E + \frac{1}{2}(E_1 + E_2 + E_3 + E_4 + E_5)))$ is log canonical and $\frac{3}{2}E + \frac{1}{2}(E_1 + E_2 + E_3 + E_4 + E_5) \sim_{\mathbb{R}} -K_S$. By Lemma 3.2, we may assume that $\text{Supp}(\Omega)$ does not contain one curve among E_1, E_2, E_3, E_4, E_5 . Intersecting this curve with Ω , we get $a \leq 1$. Let L_1, L_2, L_3, L_4, L_5 be the proper transforms on S of the lines in \mathbb{P}^2 that pass through $\pi(P)$ and $\pi(E_1), \pi(E_2), \pi(E_3), \pi(E_4), \pi(E_5)$,

respectively. Then $\frac{2}{3}E + \frac{1}{3}(L_1 + L_2 + L_3 + L_4 + L_5) \sim_{\mathbb{R}} -K_S$, and $(S, (1 - \beta)C + \lambda\beta(\frac{2}{3}E + \frac{1}{3}(L_1 + L_2 + L_3 + L_4 + L_5)))$ is log canonical. By Lemma 3.2, we may assume that $\text{Supp}(\Omega)$ does not contain one curve among L_1, L_2, L_3, L_4, L_5 . Intersecting this curve with Ω , we get $a + x \leq 2$. Recall that $a \leq 1$ by Corollary 5.1. Thus, we have

$$a \leq 1, \quad x - a \leq 1, \quad a + x \leq 2, \tag{5.4}$$

which implies that $x \leq \frac{3}{2}$ and $\lambda\beta(a + x) - \beta \leq 1$. In particular, we have $\lambda\beta x \leq 1$.

Denote the proper transform of the curve E on S_1 by E^1 . Then $\lambda\beta(a + x) - \beta \leq 1$, since $a + x \leq 2$. Thus, the log pair $(S_1, (1 - \beta)C^1 + \lambda\beta aE^1 + \lambda\beta\Omega^1 + (\lambda\beta(a + x) - \beta)F_1)$ is not log canonical at the unique point $O_1 \in F_1$ by Lemma 3.3. Note that $\lambda\beta(a + x) - \beta > 0$ by Lemma 3.1. Moreover, either $O_1 = P_1$ or $O_1 = E^1 \cap F_1$, since otherwise we would have $\lambda x = \lambda\beta\Omega^1 \cdot F_1 > 1$ by Theorem 3.4. If $O_1 = E^1 \cap F_1$, then Theorem 3.9 implies

$$\lambda\beta(1 + a - x) = \lambda\beta\Omega^1 \cdot E^1 > 2(1 + \beta - \lambda\beta(x + a))$$

or $\lambda\beta x = \lambda\beta\Omega^1 \cdot F_1 > 2(1 - \lambda\beta a)$. The former inequality gives $\lambda\beta(1 + 3a + x) > 2 + 2\beta$, which is impossible since $1 + 3a + x \leq 5$ by (5.4). The latter inequality gives that $\lambda\beta(x + 2a) > 2$, which is impossible since $x + 2a \leq 3$ by (5.4). Thus, we see that $O_1 = P_1$.

Let R be the proper transform on S of a line in \mathbb{P}^2 that is tangent to $\pi(C)$ at the point $\pi(P)$. Then either $-K_S \cdot R = 3$ or $-K_S \cdot R = 2$. Moreover, $-K_S \cdot R = 3$ if and only if $\pi(R)$ does not contain any of the points $\pi(E_1), \pi(E_2), \pi(E_3), \pi(E_4), \pi(E_5)$.

Suppose that $-K_S \cdot R = 2$. Without loss of generality, we may assume that $R = L_1$. We write $\Omega = bL_1 + \Delta$, where b is a non-negative real number, and Δ is an effective \mathbb{R} -divisor on S whose support does not contain the curve L_1 . Denote the proper transform of the curve L_1 on S_1 by L_1^1 , and denote the proper transform of Δ on S_1 by Δ^1 . Let $y = \text{mult}_P(\Delta)$ and $y_1 = \text{mult}_{P_1}(\Delta^1)$. Then $x = y + b$. Since $(S, (1 - \beta)C + \lambda\beta(E + E_1 + L_1))$ is log canonical and $E + E_1 + L_1 \sim -K_S$, we may assume that $b = 0$ or $\text{Supp}(\Delta)$ does not contain E_1 by Lemma 3.2. Thus, if $b \neq 0$, then $1 - a - b = \Delta \cdot E_1 \geq 0$. With (5.4), this gives $y + 2b \leq 2$ and $2 + a + y + 2b \leq \frac{9}{2}$. On the other hand, we have $2 - a - y = \Delta^1 \cdot L_1^1 \geq y_1$, which implies that $a + 2y_1 \leq 2$, since $y \geq y_1$. Thus, we see that $y_1 \leq 1$. Then $\text{mult}_{P_1}((1 - \beta)C^1 + \lambda\beta\Delta^1) = 1 - \beta + \lambda\beta y_1 \leq 1$. Applying Theorem 3.9, we see that

$$1 - \beta + \lambda\beta(2 - a - y) = ((1 - \beta)C^1 + \lambda\beta\Delta^1) \cdot L_1^1 > 2(1 + \beta - \lambda\beta(a + b + y))$$

or $1 - \beta + \lambda\beta y = ((1 - \beta)C^1 + \lambda\beta\Delta^1) \cdot F_1 > 2(1 - \lambda\beta b)$. This gives $\lambda\beta(2 + a + y + 2b) > 1 + 3\beta$ or $\lambda\beta(y + 2b) > 1 + \beta$. The former inequality is impossible, because $2 + a + y + 2b \leq \frac{9}{2}$. The latter inequality is also impossible, because $y + 2b \leq 2$.

We have $-K_S \cdot R = 3$. Then R is irreducible and $R + E \sim -K_S$. Since $(S, (1 - \beta)C + \lambda\beta(R + E))$ is log canonical, we may assume that $R \not\subset \text{Supp}(\Omega)$ by Lemma 3.2. Denote the proper transform of the curve R on the surface S_1 by R^1 . Then $3 - 2a - x = \Omega^1 \cdot R^1 \geq x_1$, which gives $x + x_1 + 2a \leq 3$. Then $\lambda\beta(a + x + x_1) - 2\beta \leq 1$ by (5.4). Thus, the log pair

$$(S_2, (1 - \beta)C^2 + \lambda\beta\Omega^2 + (\lambda\beta(a + x) - \beta)F_1^2 + (\lambda\beta(a + x + x_1) - 2\beta)F_2)$$

is not log canonical at a unique point $O_2 \in F_2$ by Lemma 3.3. Note that $\lambda\beta(a + x + x_1) - 2\beta > 0$ by Lemma 3.1. If $O_2 \neq P_2$ and $O_2 \neq F_1^2 \cap F_2$, then Theorem 3.4 gives $\lambda\beta x_1 = \lambda\beta\Omega^2 \cdot F_2 > 1$, which is impossible, since $\lambda\beta x_1 \leq \lambda\beta x \leq 1$ by (5.4). If $O_2 = F_1^2 \cap F_2$, then Theorem 3.4 gives

$$\lambda\beta(a + 2x) - 2\beta \geq (\lambda\beta\Omega^2 + (\lambda\beta(a + x + x_1) - 2\beta)F_2) \cdot F_1^2 > 1,$$

which is impossible, since $a + 2x \leq \frac{7}{2}$, by (5.4). Hence, we see that $O_2 = P_2$.

One has $\lambda\beta(a + x + x_1 + x_2) - 3\beta \leq 1$ by (5.4), since $x + x_1 + 2a \leq 3$ and $x_2 \leq x_1 \leq x$. Thus, it follows from Lemma 3.3 that

$$(S_3, (1 - \beta)C^3 + \lambda\beta\Omega^3 + (\lambda\beta(a + x + x_1) - 2\beta)F_2^3 + (\lambda\beta(a + x + x_1 + x_2) - 3\beta)F_3)$$

is not log canonical at a unique point $O_3 \in F_3$. Note that $\lambda\beta(a + x + x_1 + x_2) - 3\beta > 0$ by Lemma 3.1. If $O_3 \neq P_3$ and $O_3 \neq F_2^3 \cap F_3$, then Theorem 3.4 gives $\lambda\beta x_2 = \lambda\beta\Omega^3 \cdot F_3 > 1$, which is impossible, since $\lambda\beta x_2 \leq \lambda\beta x \leq 1$ by (5.4). If $O_3 = F_2^3 \cap F_3$, then Theorem 3.4 gives

$$\lambda\beta(a + x + 2x_1) - 3\beta = (\lambda\beta\Omega^3 + (\lambda\beta(a + x + x_1 + x_2) - 3\beta)F_3) \cdot F_2^3 > 1,$$

which contradicts (5.4), since $x + x_1 + 2a \leq 3$. Thus, we have $O_3 = P_3$. Then Theorem 3.4 gives

$$\beta \geq 4\lambda\beta - 3\beta = C^3 \cdot (\lambda\beta\Omega^3 + (\lambda\beta(a + x + x_1 + x_2) - 3\beta)F_3) > 1,$$

which is impossible, since $\beta \in (0, 1]$. ■

This completes the proof of Lemma 4.9.

Acknowledgements

We thank Yanir Rubinstein and Gabor Székelyhidi for useful discussions. Most of the results in this article were obtained in the PhD Thesis of the second author (see [13]). This article contains simplified proofs and, in some cases, completely new proofs. A part of this work was done during a visit of both authors to the National Center for Theoretical Sciences in Taipei. We are grateful to

Jungkai Chen for making this visit possible. The first author thanks the Max Planck Institute for Mathematics in Bonn, where part of this work was completed.

Funding

The work was supported by a grant of Russian Science Foundation (project No. 15-11-20036).

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