EQUIVARIANT GEOMETRY OF SINGULAR CUBIC THREEFOLDS, II

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Abstract. We study linearizability of actions of finite groups on cubic threefolds with nonnodal isolated singularities.

1. Introduction

Among the central problems in birational geometry is the *linearizability* problem, as well as the closely related *rationality problem*. The first is about identifying regular actions of finite groups G on algebraic varieties which are equivariantly birational to actions of G on $\mathbb{P}(V)$, where V is a representation of G. The second could be viewed as a special case, when G is the trivial group, i.e., identifying varieties birational to projective space. These questions can be asked over the complex numbers $\mathbb{C}$, or arbitrary ground fields. One of the distinguishing features of this research is the rich interplay between arithmetic and geometric aspects.

In this paper, we focus on *linearizability* and *stable linerizability* of actions of finite groups on singular cubic threefolds $X \subset \mathbb{P}^4$, over an uncountable algebraically closed field k of characteristic zero; recall that a G-action on X is stably linearizable if the action on $X \times \mathbb{P}^n$ is linearizable, with G acting trivially on the second factor.

We extend our investigations of the *nodal* case in [5], [6] to cover the remaining cases of isolated singularities. We rely on the recent classification of such singularities in [17]. In detail, we only consider situations when the G-action does not fix one of the singular points, since in that case, the G-action is linearizable via projection from this point. Under this assumption, in the nonnodal case, there are at most 6 singular points, all of which are necessarily ADE singularities. The linear position of the singularities affects the possible automorphism

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groups $G \subset \mathrm{Aut}(X)$ - we use the *defect*

$$d = d(X) := \mathrm{rank}\,\mathrm{Cl}(X) - 1,$$

where $\mathrm{Cl}(X)$ is the class group of X, to distinguish some cases. Going through the list of configurations of isolated singularities in [17, Table 7, 8 and 9] we extract all nonnodal cases that are not *a priori* linearizable. We compute, in Section 2, the defect, using [14, Theorem 1.1]:

- $m\mathsf{A}_1$, $m = 2, \ldots, 10$, handled in [5], [6],
- $m\mathsf{A}_2$, $m = 2, 3, 4, 5$, $d = 0$,
- $2\mathsf{A}_2 + m\mathsf{A}_1$, $m = 2, 3, 4$, and $d = \begin{cases} 1 & \text{if } m = 4, \\ 0 & \text{otherwise,} \end{cases}$
- $2\mathsf{A}_3 + m\mathsf{A}_1$, $m = 2, 3, 4$, and $d = \begin{cases} 3 & \text{if } m = 4, \\ 2 & \text{if } m = 3, \\ 1 \text{ or } 2 & \text{if } m = 2, \end{cases}$
- $2\mathsf{A}_3$ and $d = 0$ or 1,
- $2\mathsf{A}_4$ and $d = 0$,
- $2\mathsf{A}_5$ and $d = 1$,
- $3\mathsf{A}_2 + 2\mathsf{A}_1$, $d = 0$,
- $3\mathsf{A}_3$, $d = 1$ or 2,
- $2\mathsf{D}_4$, $d = 2$,
- $2\mathsf{D}_4 + 2\mathsf{A}_1$, $d = 3$,
- $2\mathsf{D}_4 + 3\mathsf{A}_1$, $d = 4$,
- $3\mathsf{D}_4$, $d = 4$, there is a unique such threefold [2, Theorem 3.2].

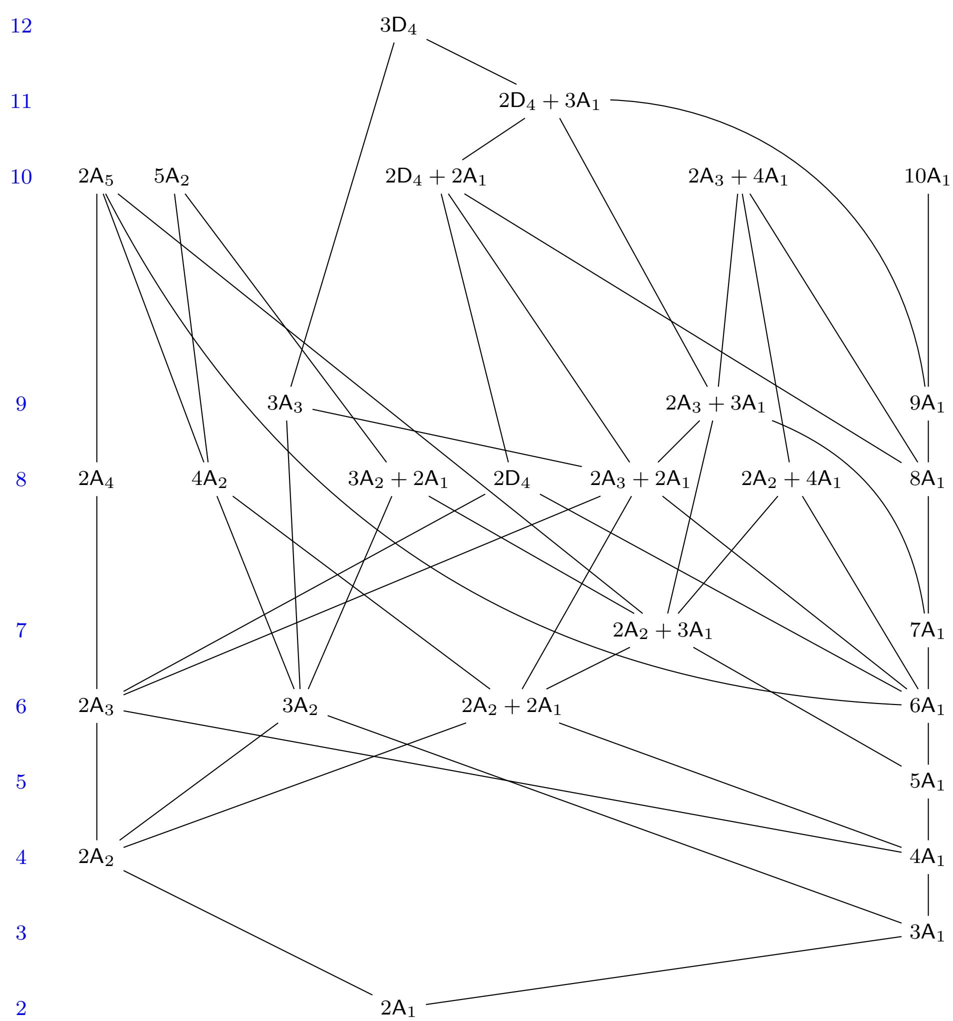
Note that in each of these cases the cubic X is GIT semistable [2].

Starting from here, the strategy is transparent: describe *normal forms* of the cubics for each relevant singularity type, compute the full automorphism group $\mathrm{Aut}(X)$, deploy the known obstructions to linearizability, such as

(H1) cohomology of the G-action on the Picard group $\mathrm{Pic}(\tilde{X})$, for a smooth model $\tilde{X}$ of X,
(IJ) equivariant intermediate Jacobians, see [6, Section 2],
(Burn) Burnside invariants [13],
(Sp) equivariant specialization,

to identify nonlinearizable actions. While the nonvanishing of the **(H1)**, **(IJ)**, or **(Burn)** obstructions exclude linearizability of the given threefold, the specialization technique only yields information for a *very general* member of the corresponding family (which explains our restriction to an uncountable ground field k). In practice, it is very difficult to obtain a result for *every* member; this is well-understood in the study of rationality. In the remaining cases, we look for linearizability constructions. The implementation of this strategy is quite involved, and relies on extensive use of `magma`.

In applications of equivariant specialization **(Sp)**, we need detailed information about degenerations of singularities. Recall that a (combination of) ADE singularities T degenerates to T' if and only if the Dynkin diagram of the root system of T is an induced subgraph of the Dynkin diagram of the root system of T' (see [3, Section 5.9]). We record the possible degenerations of singularities of cubic threefolds:



For a given G-action on a nonnodal cubic X, **(H1)** obstruction does not vanish only in the following cases:

$$2\mathsf{A}_5, \quad 2\mathsf{D}_4 + 2\mathsf{A}_1 \quad \text{and} \quad 3\mathsf{D}_4.$$

In each of these three cases, the full automorphism group $\mathrm{Aut}(X)$ is infinite, and the vanishing of the **(H1)** obstruction is equivalent to the linearization of the G-action, see Proposition 4.1.

We proceed to summarize the results: X is a nonnodal cubic threefold, with singularities as above, and $G \subseteq \mathrm{Aut}(X)$ a finite group.

Two singularities.

- $2\mathsf{A}_2$: the G-action is linearizable if and only if G fixes a singular point.
- $2\mathsf{A}_3$:
  - $d(X) = 0$: if X is very general, the G-action is not linearizable, with a possible exception when $\mathrm{Aut}(X) = \mathfrak{D}_4$ and $G = C_4$, see Corollary 5.5,
  - $d(X) = 1$: X is G-equivariantly birational to a smooth intersection of two quadrics $X_{2,2} \subset \mathbb{P}^5$; the G-action is linearizable if and only if there is a G-stable line on $X_{2,2}$, by [11].
- $2\mathsf{A}_4$: if X is very general, the G-action is not linearizable, with a possible exception when $\mathrm{Aut}(X) = C_6$ and $G = C_2$, see Proposition 5.11.
- $2\mathsf{A}_5$: the G-action is linearizable if and only if the **(H1)** obstruction vanishes, which happens if and only if G acts trivially on the class group $\mathrm{Cl}(X) \simeq \mathbb{Z}^2$, see Corollary 5.14.
- $2\mathsf{D}_4$: **(Burn)** and **(Sp)** settle the linearizability problem for most actions.

Three singularities.

- $3\mathsf{A}_2$ and $3\mathsf{A}_3$: we expect that the G-action is linearizable if and only if G does not fix singular points, and we confirm this in many cases using **(Burn)** and **(Sp)**, to cohomology for a specific C_3-action. In the $3\mathsf{A}_2$ case, the intermediate Jacobian $\mathrm{IJ}(\tilde{X})$ of a minimal resolution of singularities $\tilde{X} \to X$ is the Jacobian of a smooth curve of genus 2, and the intermediate Jacobian obstruction of [7] may be applicable.
- $3\mathsf{D}_4$: the G-action is linearizable if and only if the **(H1)** obstruction vanishes, see Proposition 6.7.

Four singularities. Many G-actions are nonlinearizable, via **(Burn)**, see Proposition 7.11.

- $2\mathsf{A}_2 + 2\mathsf{A}_1$ and $4\mathsf{A}_2$: such X are equivariantly birational to a smooth divisor of degree $(1,1,1,1)$ in $(\mathbb{P}^1)^4$, we expect that the action is linearizable if and only if G fixes a singular point; we prove this for very general X in Proposition 7.2, respectively, in Proposition 7.10.
- $2\mathsf{A}_3 + 2\mathsf{A}_1$:
  - $d(X) = 1$: the G-action is linearizable, by Lemma 7.3.
  - $d(X) = 2$: the G-action on very general X is linearizable if and only if it fixes a singular point, by Proposition 7.5.
- $2\mathsf{D}_4 + 2\mathsf{A}_1$: the G-action is linearizable if and only if the **(H1)** obstruction vanishes, by Corollary 7.8.

Five singularities.

- $2\mathsf{A}_2+3\mathsf{A}_1, 2\mathsf{A}_3+3\mathsf{A}_1, 3\mathsf{A}_2+2\mathsf{A}_1, 5\mathsf{A}_2$: the G-action is linearizable.
- $2\mathsf{D}_4 + 3\mathsf{A}_1$: there is a unique such threefold, with infinite automorphisms, G-equivariantly birational to a smooth quadric without fixed points; **(Burn)** obstructs some of the actions, e.g., for $G = C_2^2 \times \mathfrak{S}_3$. The linearizability problem for smooth quadric threefolds is still open.

Six singularities. All G-actions are linearizable.

Here is the roadmap of the paper: In Section 2, we compute the *defect* $d(X)$, in all cases. In Section 3, we explain how to compute the automorphism group $\mathrm{Aut}(X)$, and implement the algorithm in an example. Section 4 is devoted to computations of the Picard group of a minimal resolution of singularities $\tilde{X}$ of X and of group cohomology $\mathrm{H}^1(G, \mathrm{Pic}(\tilde{X}))$, for subgroups $G \subseteq \mathrm{Aut}(X)$; the nonvanishing of this invariant is an obstruction to linearizability. The subsequent sections are organized by the number of singular points.

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2. The Defect

In this section, we compute the defect

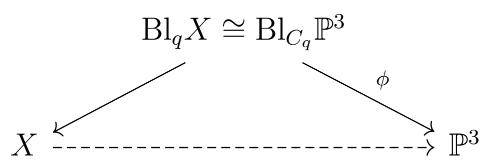
$$d(X) := \mathrm{rk}\,\mathrm{Cl}(X) - 1,$$

where $\mathrm{Cl}(X)$ is the class group, for cubic threefolds X with specified combinations of singularities, using the projection method. The proofs follow closely those in [17, §4], however, we feel that the presentation will be useful for the reader. We always project from the worst singularity $q \in \mathrm{Sing}(X)$, the singular locus of X.

Projection method. We review the projection method outlined in [14] (see also [17]): fix $q \in \mathrm{Sing}(X)$ and choose coordinates so that $q = [1:0:0:0:0]$. Then X is given by

$$x_1 f_2(x_2, \dots, x_5) + f_3(x_2, \dots, x_5) = 0, \tag{2.1}$$

where f_2, f_3 are homogeneous, of degree 2 and 3, respectively. Projection from q gives a birational map $X \dashrightarrow \mathbb{P}^3$, factoring as



This yields

$$\begin{aligned} Q_q &:= \{f_2(x_2, \dots, x_5) = 0\} \subset \mathbb{P}^3, \\ S_q &:= \{f_3(x_2, \dots, x_5) = 0\} \subset \mathbb{P}^3, \\ C_q &:= Q_q \cap S_q \subset \mathbb{P}^3. \end{aligned}$$

The curve C_q parameterizes lines contained in X passing through q.

Lemma 2.1. *Let X be a cubic threefold with singularities as above. Then:*

(1) *If q is a D_4-point, then Q_q is the union of two planes and*

$$d(X) = (\#\ \textit{irreducible components of } C_q) - 2;$$

(2) *If q is an A_n-point with $n \geq 2$, then Q_q is a quadric cone and*

$$d(X) = (\#\ \textit{irreducible components of } C_q) - 1.$$

(3) *If q is an A_n-point with $n \geq 3$, then in addition C_q passes through the cone point of Q_q and has an A_{n-2}-singularity there.*

Proof. The statements about Q_q are a combination of [17, Claim A.10, A.11]. The formula for the defect is [14, Theorem 1.2]. □

Projection from A_2.

Lemma 2.2. *Let X be a cubic threefold. Then $d(X) = 0$, when X has*

- *$m\mathsf{A}_2$-singularities, for $1 \leq m \leq 5$, or*
- *$(2\mathsf{A}_2 + m\mathsf{A}_1)$-singularities, for $1 \leq m \leq 3$, or*
- *$(3\mathsf{A}_2 + 2\mathsf{A}_1)$-singularities.*

Proof. Projecting from $q \in \mathrm{Sing}(X)$, we see that C_q must be irreducible, otherwise X would have at least $4\mathsf{A}_1$-singularities. It follows that C_q is an irreducible $(2, 3)$ complete intersection curve, and so the defect is 0, see [17, Proposition 4.9]. □

Proposition 2.3. *Let X be a cubic with $(4\mathsf{A}_1 + 2\mathsf{A}_2)$-singularities. Then $d(X) = 1$, and*

(1) *X contains exactly one plane Π, containing the 4 nodes,*
(2) *the line containing the $2\mathsf{A}_2$-points is disjoint from Π.*

Proof. Let $q \in \mathrm{Sing}(X)$ be an A_2-point. By [17, Proposition 4.9], we see that $C_q = A \cup B$, where A is a hyperplane section of the quadric cone Q_q not passing through the cone point, and B is an irreducible curve with an A_2-singularity. The computation of $d(X)$ follows from Lemma 2.1.

Let $Z \subset \mathbb{P}^4$ be the cone over A with vertex q; then $Z \subset X$, and Z spans a hyperplane, which intersects X in $Z \cup \Pi$, with $\Pi \cap Z = A$ containing the four nodes of X.

For the second claim, let $L \subset X \subset \mathbb{P}^4$ be the line between the two A_2-points. Then L intersects the hyperplane spanned by Z in exactly one point q, and it follows that $L \cap \Pi = \varnothing$. □

Projection from A_3.

Lemma 2.4. *Let X be a cubic with $2\mathsf{A}_3$-singularities. Then either*

(1) *$d(X) = 0$, or*
(2) *$d(X) = 1$, and there is a unique plane containing the A_3-points.*

Proof. We project from an A_3-point q. The curve $C_q \subset Q_q$ has an A_1-singularity at the cone point of Q_q. Since X has a second A_3-point, so must C_q. This implies that either:

- C_q is irreducible with an A_3-singularity away from the cone point of Q_q, or
- $C_q = A \cup B$, where A is a fiber in the ruling of Q_q, and B is a smooth genus 2 curve tangent to A at a single point, corresponding to the second A_3-point. We see that $d(X) = 1$, and

the plane is given by the cone over A with vertex q - it thus contains the two A_3-points.

□

Lemma 2.5. *Let X be a cubic with $3\mathsf{A}_3$-singularities. Then either*

(1) $d(X) = 1$ *and* $\mathrm{Cl}(X)$ *is freely generated by two classes of cubic scrolls contained in* X*,*
(2) $d(X) = 1$ *and there is a unique plane* Π *containing exactly two* A_3*-points, in particular, there is a distinguished* A_3*-point contained in two planes, or*
(3) $d(X) = 2$ *and there are exactly three planes, each containing exactly two* A_3*-points.*

Proof. We project from an A_3-point q. The curve $C_q \subset Q_q$ must have an A_1-singularity at the cone point of Q_q. According to [17, Proposition 4.8] there are four ways of forcing two additional A_3-singularities:

- $C_q = A \cup B$, where both A and B are twisted cubics passing through the cone point, and tangent in two other points. By [14, Lemma 4.4], X contains two families of cubic scrolls, that freely generate $\mathrm{Cl}(X)$ (see also [9]).
- $C_q = A \cup B$, where A is a hyperplane section of Q_q not passing through the cone point, and B is a smooth curve of genus 1. Further, A is tangent to B at two distinct points with multiplicity 2. In this case, $d(X) = 1$: we see the plane as in Proposition 2.3. It contains the remaining two A_3-points.
- $C_q = A \cup B$, where A is a ruling of Q_q and B a genus 2 curve. Furthermore, A is tangent to B at a point, and B has an additional A_3-singularity. This is the same arrangement as the previous case, where we are instead projecting from an A_3-point that is contained in the unique plane.
- $C_q = A_1 \cup A_2 \cup B$, where B is a smooth curve of genus 1 and each A_i is a distinct line in the ruling of Q_q which is tangent to B. In this case $d(X) = 2$, and the planes are given as the cone over each A_i, along with the residual plane from intersecting the hyperplane spanned by the A_i.

□

Proposition 2.6. *Let X be a cubic with $2\mathsf{A}_3 + 4\mathsf{A}_1$-singularities. Then*

(1) $d(X) = 3$*, and the extra generators of* $\mathrm{Cl}(X)$ *are planes,*
(2) *there is a unique plane* $\Pi \subset X$ *containing all four nodes,*

(3) *each* A_3*-point is contained in two planes, each containing two other nodes,*
(4) *the line containing the* $2\mathsf{A}_3$*-points is disjoint from* Π.

Proof. We project from an A_3-point $q \in X$. By [17, Proposition 4.8, corrected version], we see that $C_q = A_1 \cup A_2 \cup B_1 \cup B_2$, where A_1, A_2 are two distinct lines in the ruling of the quadric cone Q_q, and B_1, B_2 are two hyperplane sections of Q_q not passing through the cone point and tangent to each other at $p \in Q_q$. The computation of $d(X)$ follows from Lemma 2.1.

We see that there are two planes containing q; namely, the cones over A_1, A_2 with vertex q. Each contains two nodes of X: indeed, C_q has a node at each of the points $p_i \in A_i \cap B_i$, and thus by, Lemma 2.1, there is a node on the line $\langle q, p_i\rangle$. Note that the plane Π spanned by A_1, A_2 contains the four nodes of X, and hence is contained in X itself.

Finally, let L denote the line through the two A_3-points of X. We claim $L \cap \Pi = \varnothing$. Indeed, suppose that $L \cap \Pi \neq \varnothing$. Then there exists a hyperplane section $F \subset X$ that contains L and Π. Note that F must split as a union of Π and two other planes such that both of then contains L, and each of them contain two nodes of X. This is impossible: if a plane in X contains four singular points of X, then these singular points must be nodes. □

Lemma 2.7. *Let* X *be a cubic with* $2\mathsf{A}_3 + m\mathsf{A}_1$*-singularities, for* $2 \leq m \leq 3$. *Then*

(1) *When* $m = 2$, *then either*
  - $d(X) = 1$ *and there is a unique plane in* X *that contains the two* A_1*-points and exactly one* A_3*-point,*
  - $d(X) = 1$ *and there is a unique plane in* X *containing the two* A_3*-points and no other singularities, or*
  - $d(X) = 2$ *and there are three planes contained in* X*; one plane contains the two* A_3*-points, and the other planes each contain one of the* A_3*-points and the two* A_1*-points.*

(2) *When* $m = 3$, *then* $d(X) = 2$ *and there are exactly two planes contained in* X*; both planes contain an* A_3*-point and exactly* $2\mathsf{A}_1$*-points.*

Proof. Let $q \in \mathrm{Sing}(X)$ be an A_3-point. By [17, Proposition 4.8, corrected version], C_q is necessarily reducible. If X has $2\mathsf{A}_3 + 2\mathsf{A}_1$-singularities, then there are two possibilities:

- $C_q = A \cup B$, with A and B irreducible and intersecting transversely at the cone point. Thus $d(X) = 1$, and the additional class in $\mathrm{Cl}(X)$ is a plane. There are three options:
  - A is a hyperplane section of the quadric, and B is tangent to A at one point, and intersects transversely in two distinct points. The plane is the residual from the cone over A, and contains the two A_1-points and one A_3-point.
  - A is a ruling of the quadric, intersecting B in two distinct points; B has an additional A_3-singularity. The plane is the cone over A, and contains the two A_1-points and one A_3-point.
  - A is a ruling of the quadric, and is tangent to B at a single point; B has two additional A_1-singularities.
- $C_q = A \cup B_1 \cup B_2$, and each irreducible. Here, B_1 and B_2 are distinct rulings of the quadric cone, and B_1 is tangent to A in a unique point, whereas B_2 intersects A in two distinct points. We see that $d(X) = 2$, and the cone over B_1 is a plane containing the two A_3-points, whereas the cone over B_2 is a plane containing only q and the two A_1-points. Taking the plane spanned by B_1 and B_2 gives a third plane containing one A_3-point and two A_1-points.

Next, suppose that X has $2\mathsf{A}_3 + 3\mathsf{A}_1$-singularities. Then $C_q = A \cup B_1 \cup B_2$, where A is a ruling of the quadric, B_1 is a hyperplane section of the quadric, and B_2 is a twisted cubic. Further, B_1 and B_2 are tangent in one point, and intersect transversely in one other point. We see that $d(X) = 2$, and there are exactly two planes in X: the first is the cone over A, and contains one A_3-point and $2\mathsf{A}_1$-points. The second plane is residual to the cone over B_1, and contains one A_3-point and $2\mathsf{A}_1$-points. Note that there is one A_1-point that belongs to both planes, namely the intersection of A and B_1. □

Projection from A_4.

Lemma 2.8. *Let X be a cubic threefold with $2\mathsf{A}_4$-singularities. Then $d(X) = 0$.*

Proof. This is case (4) of [17, Proposition 4.6]: C_q must be irreducible with one A_4-point (and an A_2-point at the cone point of Q_q); indeed, having multiple components forces additional singularities on X. □

Projection from A_5.

Lemma 2.9. *Let X be a cubic with $2\mathsf{A}_5$-singularities. Then $d(X) = 1$, and $\mathrm{Cl}(X)$ is freely generated by two classes of cubic scrolls contained in X.*

Proof. We project from one of the A_5-points q. The curve C_q necessarily has an A_3-singularity at the cone point of Q_q - this is impossible if C_q is irreducible. The only possibility for an additional A_5-singularity is to have $C_q = A \cup B$, where each component is a smooth twisted cubic passing through the cone point of Q_q, and intersecting each other in one point with multiplicity 4. By [14, Lemma 4.4], X contains two families of cubic scrolls, that freely generate $\mathrm{Cl}(X)$ (see also [9]). $\square$

Projection from D_4. In this case, $Q_q = \Pi_1 \cup \Pi_2$, where $\Pi_i \cong \mathbb{P}^2$, meeting in a line l. Note that $C_q = B_1 \cup B_2$, where $B_i \subset \Pi_i$ is a cubic curve, and C_q intersects l in three simple points, with each B_i smooth at these points.

Proposition 2.10. *Let X be a cubic with a D_4-singularity. Then*

(1) *If X has $2\mathsf{D}_4$-singularities, then $d(X) = 2$ and there are three planes in X, each containing the two singular points.*
(2) *If X has $2\mathsf{D}_4 + 2\mathsf{A}_1$-singularities, then $d(X) = 3$, and there are five planes contained in X. Three planes contain both D_4-points, and each D_4-point is contained in one other plane containing both nodes.*
(3) *If X has $2\mathsf{D}_4 + 3\mathsf{A}_1$-singularities, then $d(X) = 4$, and there are nine planes contained in X. Three planes contain both D_4-points, and each D_4-point is contained in three other planes which contain two of the three nodes.*
(4) *If X has $3\mathsf{D}_4$-singularities, then $d(X) = 4$, and there are nine planes contained in X, each containing exactly two singular points.*

Proof. Since X has at least $2\mathsf{D}_4$-singularities, at least one of the plane cubics, say B_1, is the union of three lines meeting in a point. The cone over each line with vertex q gives a plane in X. Consider $span\langle \Pi_1, q\rangle \subset \mathbb{P}^4$; this hyperplane intersects X in precisely these three planes, giving one relation in $\mathrm{Cl}(X)$ - we see $d(X) = 2$ for the case of $2\mathsf{D}_4$-singularities.

If X has additional A_1-singularities, then B_2 must become singular. When $k = 2$, B_2 must be a conic and a line, for $k = 3$, B_2 becomes three lines in a triangle configuration. Each line gives an additional plane in X, and the case of three lines gives one relation as before. The defect and plane computation follows. We see that q is contained in three planes which contain the other D_4-point (corresponding to the cone over B_1), and in one plane containing the two nodes. This curve configuration is the only possibility; in particular, if we project from the other D_4-point, we have the same configuration. The claim follows.

If X has $3\mathsf{D}_4$-singularities, then both B_1 and B_2 are three lines meeting in a point. The defect is thus $d(X) = 4$. Note that there are six planes that contain q, each containing one other A_3-point. By symmetry, there are nine planes contained in X. □

3. Automorphisms

In this section, we explain how to classify automorphisms $\mathrm{Aut}(X)$ of singular cubics $X \subset \mathbb{P}^4$, with s singular points. By convention, the action on the variety is from the right, and the action on the function field from the left. In coordinates, given $\mathbf{x} = (x_1, \ldots, x_5)$ and $\sigma \in \mathrm{Aut}(X)$, we put

$$\sigma(\mathbf{x}) = \sigma((x_1, \ldots, x_5)) = (x_1, \ldots, x_5) \cdot M_\sigma,$$

where $M_\sigma \in \mathsf{GL}_5$ is the corresponding matrix.

Throughout, we assume that $\mathrm{Aut}(X)$ does not fix any singular points, since otherwise, the action is linearizable. Let H be the maximal subgroup of $\mathrm{Aut}(X)$ which fixes all singular points. The $\mathrm{Aut}(X)$-action preserves the singular locus, yielding an exact sequence

$$0 \to H \to \mathrm{Aut}(X) \xrightarrow{\rho} \mathfrak{S}_s.$$

In particular, $\mathrm{Aut}(X)$ is a semidirect product of H and $\rho(\mathrm{Aut}(X))$. The image of ρ reflects the singularity type, for example, when X has $(2\mathsf{A}_2 + 2\mathsf{A}_1)$-singularities, the image is contained in $\mathfrak{S}_2 \times \mathfrak{S}_2$.

The algorithm for classifying $\mathrm{Aut}(X)$ involves the following steps:

- Find a *normal form* of X, based on the singularity type. This amounts to fixing appropriate coordinates and simplifying the equation.
- Determine possible images of ρ, and find all lifts to $\mathrm{Aut}(X)$, depending on parameters in the equation of X.
- For each lift, determine H.

Here, we explain the process in a simple case, when X is the unique cubic with $3\mathsf{D}_4$-singularities, see [1, Theorem 5.4]:

Proposition 3.1. *Let*

$$X := \{x_1x_2x_3 + x_4^3 + x_5^3 = 0\} \tag{3.1}$$

be the unique cubic threefold with $3\mathsf{D}_4$*-singularities. Then*

$$\mathrm{Aut}(X) = \langle \tau_{a,b}, \eta, \sigma_{(45)}, \sigma_{(123)}, \sigma_{(12)} \rangle \simeq (\mathbb{G}_m^2(k) \times \mathfrak{S}_3) \rtimes \mathfrak{S}_3,$$

where

$$\begin{aligned} \tau_{a,b} &: (\mathbf{x}) \mapsto (ax_1, bx_2, a^{-1}b^{-1}x_3, x_4, x_5), \quad a, b \in k^\times, \\ \eta &: (\mathbf{x}) \mapsto (x_1, x_2, x_3, \zeta_3 x_4, \zeta_3^2 x_5), \\ \sigma_{(45)} &: (\mathbf{x}) \mapsto (x_1, x_2, x_3, x_5, x_4), \\ \sigma_{(123)} &: (\mathbf{x}) \mapsto (x_3, x_1, x_2, x_4, x_5), \\ \sigma_{(12)} &: (\mathbf{x}) \mapsto (x_2, x_1, x_3, x_4, x_5). \end{aligned}$$

In particular, η_1, η_2 *generate the first* $\mathfrak{S}_3$*-factor and* $\sigma_{(12)}, \sigma_{(123)}$ *generate the second* $\mathfrak{S}_3$*-factor in* $\mathrm{Aut}(X)$*.*

Proof. Observe that $\sigma_{(12)}, \sigma_{(123)} \in \mathrm{Aut}(X)$, and that $\mathrm{Aut}(X)$ acts transitively on the three singular points. It remains to find the subgroup $H \subset \mathrm{Aut}(X)$ fixing all three singular points. Based on the form of the equation, we see that an $h \in H$ takes the form

$$(\mathbf{x}) \mapsto (s_1x_1, s_2x_2, s_3x_3, a_1x_4 + a_2x_5, a_3x_4 + a_4x_5),$$

for $s_1, s_2, s_3, a_1, a_2, a_3, a_4 \in k$. There exists an exact sequence

$$0 \to H' \to H \xrightarrow{\psi} \mathsf{PGL}_2,$$

where ψ is the projection of the H-action onto $\mathbb{P}^1_{x_4,x_5}$, given by the coordinates x_4, x_5. The equation of X implies that $\psi(H)$ leaves invariant three points defined by $\{x_4^3 + x_5^3 = 0\} \subset \mathbb{P}^1_{x_4,x_5}$. The maximal subgroup of PGL_2 leaving these three points invariant is $\mathfrak{S}_3$. To show that $\psi(H) = \mathfrak{S}_3$, one can check that $\eta, \sigma_{(45)} \in H$ and their images in $\mathsf{PGL}_2(k)$ generate $\mathfrak{S}_3$. On the other hand, elements in $\tau \in H'$ are diagonal of the form

$$\tau : (\mathbf{x}) \mapsto (s_1x_1, s_2x_2, s_3x_3, x_4, x_5).$$

One can check that $s_1s_2s_3 = 1$ and τ is given by $\tau_{a,b}$, for $a, b \in k^\times$. □

4. Picard groups and cohomology

Let X be a cubic threefold with ADE singularities, and $\tilde{X} \to X$ an $\mathrm{Aut}(X)$-equivariant resolution of singularities; it can be achieved via a sequence of blowups, where at each step we blow up the necessarily $\mathrm{Aut}(X)$-invariant singular locus consisting of all the singular points.

Here we consider the induced G-actions on the Picard group $\mathrm{Pic}(\tilde{X})$ and $\mathrm{Cl}(X)$, for $G \subseteq \mathrm{Aut}(X)$. In particular, if the G-action on X is linearizable, then the G-module $\mathrm{Pic}(\tilde{X})$ is a *stably permutation module*, see [5, Section 2]. If the cohomology groups

$$\mathrm{H}^1(G, \mathrm{Pic}(\tilde{X})), \text{ or } \mathrm{H}^1(G, \mathrm{Pic}(\tilde{X})^\vee)$$

are nonvanishing, then $\mathrm{Pic}(\tilde{X})$ fails to be a stably permutation module. We call this the (**H1**)-obstruction to linearizability. This is also an obstruction to *stable linearizability*, i.e., linearizability of $X \times \mathbb{P}^n$, with trivial action on the second factor. We refer the reader to [5, 6, Section 2] for applications.

The following proposition shows that the only possible combinations of nonnodal singularities with (**H1**)-obstructions are

$$2\mathsf{A}_5, \quad 2\mathsf{D}_4 + 2\mathsf{A}_1 \quad \text{and} \quad 3\mathsf{D}_4.$$

Proposition 4.1. *Let X be a cubic threefold with isolated singularities, and $\tilde{X} \to X$ an* $\mathrm{Aut}(X)$*-equivariant resolution of singularities. Then*

- $\mathrm{Pic}(\tilde{X})$ *is a permutation module for* $\mathrm{Aut}(X)$ *if X is not of one of the following configurations of singularities*

$$6\mathsf{A}_1 \textit{ with defect } 0\ , \quad 8\mathsf{A}_1, \quad 9\mathsf{A}_1, \quad 10\mathsf{A}_1,$$

$$2\mathsf{A}_5, \quad 2\mathsf{D}_4 + 2\mathsf{A}_1 \quad \textit{and} \quad 3\mathsf{D}_4.$$

- *For each of the cubic threefolds X with singularities in the list above, if the* $\mathrm{Aut}(X)$*-action does not fix any singular point then it has an* (**H1**)*-obstruction.*

Proof. If $\mathrm{Aut}(X)$ fixes a singular point, then the $\mathrm{Aut}(X)$-action on X is linearizable and $\mathrm{Pic}(\tilde{X})$ is an $\mathrm{Aut}(X)$-permutation module. So it suffices to consider the singularity types in the diagram in the introduction. The cases of nodal ones are treated in [5]. Here we treat X with a nonnodal singular point via a case-by-case study. Let X be a cubic threefold with singularities not in the list of the first assertion and denote the defect of X by d. Using the analysis of generators of $\mathrm{Cl}(X)$ in Section 2, we find

- When $d = 0$, $\mathrm{Pic}(\tilde{X})$ is freely generated by the classes of the hyperplane section and the exceptional divisors, permuted by the $\mathrm{Aut}(X)$-action;
- When $d = 1$ and the singularity type is $3\mathsf{A}_3$, $\mathrm{Pic}(\tilde{X})$ is freely generated by the class of the hyperplane section, one class of the cubic scrolls in X and the classes of the exceptional divisors, permuted by the $\mathrm{Aut}(X)$-action;

- When $d = 1$ and the singularity type is not $3\mathsf{A}_3$, $\mathrm{Pic}(\tilde{X})$ is freely generated by the classes of the hyperplane section, the unique plane in X and the exceptional divisors, permuted by the $\mathrm{Aut}(X)$-action;
- When $d = 2$ and the singularity type is $2\mathsf{A}_3 + 3\mathsf{A}_1$, $\mathrm{Pic}(\tilde{X})$ is freely generated by the classes of the hyperplane section, two planes in X and the exceptional divisors, permuted by the $\mathrm{Aut}(X)$-action;
- When $d = 2$ and the singularity type is $2\mathsf{A}_3 + 2\mathsf{A}_1$, $3\mathsf{A}_3$ or $2\mathsf{D}_4$, $\mathrm{Pic}(\tilde{X})$ is freely generated by three classes of planes in X and the exceptional divisors, permuted by the $\mathrm{Aut}(X)$-action;
- When $d = 3$ and the singularity type is $2\mathsf{A}_3 + 4\mathsf{A}_1$, the $\mathrm{Aut}(X)$-action on X is linearizable, see Proposition 9.1;
- When $d = 4$ and the singularity type is $2\mathsf{D}_4 + 3\mathsf{A}_1$, then X is $\mathrm{Aut}(X)$-equivariantly birational to a smooth quadric, see Section 8. It follows that $\mathrm{Pic}(\tilde{X})$ is a permutation module.

The proof of the second assertion relies on a detailed analysis on $\mathrm{Aut}(X)$ and the geometry of X, see Propositions 5.12, 6.5 and 7.7. □

The following lemma simplifies computations in subsequent sections.

Lemma 4.2. *Let $G \subseteq \mathrm{Aut}(X)$ be a finite subgroup. Let $\tilde{X} \to X$ be a G-equivariant resolution of singularities and E_i the corresponding exceptional divisors. Then:*

- *If $\mathrm{H}^1(G, \mathrm{Cl}(X)) = 0$, then $\mathrm{H}^1(G, \mathrm{Pic}(\tilde{X})) = 0$.*
- *If $\mathrm{H}^2(G, \oplus_i \mathbb{Z} \cdot E_i) = 0$, then $\mathrm{H}^1(G, \mathrm{Pic}(\tilde{X})) = \mathrm{H}^1(G, \mathrm{Cl}(X))$, where $\oplus_i \mathbb{Z} \cdot E_i$ is the free $\mathbb{Z}$-module generated by E_i.*

Proof. We have a short exact sequence

$$0 \to \oplus_i \mathbb{Z} \cdot E_i \to \mathrm{Cl}(\tilde{X}) \simeq \mathrm{Pic}(\tilde{X}) \to \mathrm{Cl}(X) \to 0, \tag{4.1}$$

giving rise to the long exact sequence

$$\begin{aligned} &\ldots \to \\ &\mathrm{H}^1(G, \oplus_i \mathbb{Z} \cdot E_i) \to \mathrm{H}^1(G, \mathrm{Pic}(\tilde{X})) \to \mathrm{H}^1(G, \mathrm{Cl}(X)) \to \mathrm{H}^2(G, \oplus_i \mathbb{Z} \cdot E_i) \\ &\to \ldots \end{aligned} \tag{4.2}$$

Moreover, $\oplus_i \mathbb{Z} \cdot E_i$ is naturally a G-permutation module, induced by the permutation action on the singular points and the exceptional divisors over those points. Therefore, $\mathrm{H}^1(G, \oplus_i \mathbb{Z} \cdot E_i) = 0$ and the assertions follow from (4.2). □

5. Two singular points

Assume that the cubic threefold $X \subset \mathbb{P}^4$ is singular at

$$p_1 = [1:0:0:0:0], \quad p_2 = [0:1:0:0:0].$$

We are interested in the following combinations of singularity types

$$2\mathsf{A}_n, n = 2, 3, 4, 5, \quad 2\mathsf{D}_4.$$

Up to a change of coordinates, X is given by

$$x_1x_2x_3 + x_1q_1 + x_2q_2 + f_3 = 0, \tag{5.1}$$

for some quadratic forms $q_1, q_2 \in k[x_4, x_5]$ and a cubic $f_3 \in k[x_3, x_4, x_5]$. As in [6], we see that X is $\mathrm{Aut}(X)$-birational to the hypersurface V_4

$$z_1z_2 = q_1q_2 - x_3f_3 \subset \mathbb{P}(2,2,1,1,1),$$

where $z_1 = x_1x_3 + q_2$ and $z_2 = x_2x_3 + q_1$. This V_4 has 2 singular points of type $\frac{1}{2}(1,1,1)$ at $[1:0:0:0:0]$ and $[0:1:0:0:0]$. The blowup $\tilde{V}_4$ of these points yields an $\mathrm{Aut}(X)$-equivariant commutative diagram:

$$\begin{array}{ccc} & \tilde{V}_4 & \\ \swarrow & & \searrow \\ V_4 & \dashrightarrow & \mathbb{P}^2 \end{array}$$

where $V_4 \dashrightarrow \mathbb{P}^2$ is the map induced by the projection to the last three coordinates of $\mathbb{P}(2,2,1,1,1)$, and $\tilde{V}_4 \to \mathbb{P}^2$ is a conic bundle. The discriminant curve of the conic bundle is a plane quartic curve

$$D = \{q_1q_2 - x_3f_3 = 0\} \subset \mathbb{P}^2_{x_3,x_4,x_5}.$$

Singularity type $2\mathsf{A}_2$. Up to isomorphism, X is given by (5.1) where:

$$q_1 = x_4^2, \quad q_2 = x_4^2 \text{ or } x_5^2, \text{ and } f_3 \text{ a generic cubic form.}$$

The discriminant curve $D \subset \mathbb{P}^2$ of the conic bundle is smooth in either case, and we obtain a natural homomorphism

$$\gamma : \mathrm{Aut}(X) \to \mathrm{Aut}(D).$$

Proposition 5.1. *Let X be a cubic threefold with $2\mathsf{A}_2$-singularities. Let $G \subset \mathrm{Aut}(X)$ be a subgroup not fixing any singular point of X. Then the G-action on X is not linearizable.*

Proof. The proof is essentially the same as the proof of [6, Theorem 3.3], where the claim was proved for a cubic threefold with two nodes. Namely, the group G contains an element ι switching the singular points of X such that its actions on $\mathrm{IJ}(\tilde{X})$ and $\mathrm{IJ}(D)$ differ by multiplication by -1, which implies that the G-action on X is not linearizable. We refer to [6, Theorem 3.3] for the details. □

Remark 5.2. The analysis of the induced actions on intermediate Jacobians does not help to settle the linearizability problem when the singularities are worse than those considered above; in particular, when $\mathrm{IJ}(\tilde{X}) \cong \mathrm{J}(C)$, for a curve C which is either reducible with rational components, or has $g(C) \leq 2$.

Singularity type $2\mathsf{A}_3$ **with no plane.** Up to isomorphism, X is given by

$$x_1x_2x_3 + x_1x_4^2 + x_2q_2 + f_3 = 0,$$

where

$$f_3 = t_1x_3^3 + x_3^2(t_2x_4 + t_3x_5) + x_3(t_4x_4^2 + t_5x_5^2 + t_6x_4x_5) + t_7x_4^2x_5 + t_8x_4x_5^2 + t_9x_4^3.$$

Since X contains no planes and has singular points of type A_3, we have

$$q_2 = x_5^2 \quad \text{and} \quad t_9 = 0.$$

The change of variables

$$x_1 \mapsto x_1 - \frac{t_7^2x_3}{4} - t_7x_5, \quad x_2 \mapsto x_2 - \frac{t_8^2x_3}{4} - t_8x_5$$
$$x_3 \mapsto x_3, \quad x_4 \mapsto x_4 + \frac{t_8x_3}{2}, \quad x_5 \mapsto x_5 + \frac{t_7x_3}{2} \tag{5.2}$$

eliminates the terms $x_4^2x_5$, $x_4x_5^2$, and we may assume that $t_7 = t_8 = 0$.

Proposition 5.3. *Let X be a cubic threefold with $2\mathsf{A}_3$-singularities and $d(X) = 0$, i.e., not containing a plane. Assume that $\mathrm{Aut}(X)$ does not fix any singular point of X. Then, up to isomorphism, X is given by*

$$x_1x_2x_3 + x_1x_4^2 + x_2x_5^2 + t_1x_3^3 + x_3^2(t_2x_4 + t_2x_5) + x_3(t_4x_4^2 + t_4x_5^2 + t_6x_4x_5) = 0, \tag{5.3}$$

where $t_1, t_2, t_4, t_6 \in k$ and $(\mathrm{Aut}(X), X)$ is one of the following:

- $\mathrm{Aut}(X) = \langle \sigma_{(12)(45)}, \eta_1, \eta_2 \rangle \simeq \mathfrak{D}_4$, *for general* $t_1, t_4 \in k$ *and* $t_2 = t_6 = 0$, *generated by*

$$\sigma_{(12)(45)} : (\mathbf{x}) \mapsto (x_2, x_1, x_3, x_5, x_4),$$
$$\eta_1 : (\mathbf{x}) \mapsto (x_1, x_2, x_3, -x_4, -x_5),$$
$$\eta_2 : (\mathbf{x}) \mapsto (x_1, x_2, x_3, x_4, -x_5).$$

- $\mathrm{Aut}(X) = \langle \sigma_{(12)(45)}, \eta_1 \rangle \simeq C_2^2$, *for general* $t_1, t_4, t_6 \in k$ *and* $t_2 = 0$.
- $\mathrm{Aut}(X) = \langle \sigma_{(12)(45)} \rangle \simeq C_2$, *for general* $t_1, t_2, t_4, t_6 \in k$.

Proof. We follow the algorithm from Section 3. Let f be the defining equation of X, i.e.,

$$f = x_1x_2x_3 + x_1x_4^2 + x_2x_5^2 + t_1x_3^3 + x_3^2(t_2x_4 + t_3x_5) + x_3(t_4x_4^2 + t_5x_5^2 + t_6x_4x_5),$$

and $\iota \in \mathrm{Aut}(X)$ an element switching the two singular points. Based on the form of f, one observes that ι takes the form

$$\iota = \begin{pmatrix} 0 & s_1 & 0 & 0 & 0 \\ s_2 & 0 & 0 & 0 & 0 \\ a_1 & a_4 & 1 & a_7 & a_{10} \\ a_2 & a_5 & 0 & a_8 & a_{11} \\ a_3 & a_6 & 0 & a_9 & a_{12} \end{pmatrix}, \quad s_1, s_2 \in k^\times,\ a_1, \ldots, a_{12} \in k,$$

and $\iota^*(f) = s_1s_2f$. This leads to a system of 24 equations in 20 variables, starting with:

$$s_1a_{12}^2 = 0, \quad s_2a_8^2 = 0, \quad s_2a_5 + 2s_2a_7a_8 = 0, \quad a_3a_9^2 + a_6a_{12}^2 = 0,$$
$$2a_2a_8a_9 + a_3a_8^2 + 2a_5a_{11}a_{12} + a_6a_{11}^2 = 0,$$
$$a_2a_9^2 + 2a_3a_8a_9 + a_5a_{12}^2 + 2a_6a_{11}a_{12} = 0,$$
$$s_1a_2 + 2s_1a_{10}a_{11} = 0, \quad s_2a_6 + 2s_2a_7a_9 = 0,$$
$$s_1a_1 + s_1a_{10}^2 = 0, \quad s_2a_4 + s_2a_7^2 = 0, \quad \ldots$$

These quickly imply (in order)

$$a_{12} = a_8 = a_5 = a_3 = a_6 = a_2 = a_{10} = a_7 = a_1 = a_4 = 0;$$

it remains to solve the system of equations given by the vanishing of:

$$s_1s_2t_1 - t_1, \quad s_1s_2t_2 - t_3a_{11}, \quad s_1s_2 - s_1a_{11}^2, \quad s_1s_2t_4 - t_5a_{11}^2,$$
$$s_1s_2t_3 - t_2a_9, \quad s_1s_2t_6 - t_6a_9a_{11}, \quad s_1s_2 - s_2a_9^2, \quad s_1s_2t_5 - t_4a_9^2.$$

We do this using the `magma` function `ProbableRadicalDecomposition`. Excluding solutions giving rise to cubics with other singularity types, we found that ι exists if and only if

$$s_1 - a_9^2 = s_2 - a_{11}^2 = t_2 - t_3 a_{11} = t_4 - t_5 a_{11}^2 = a_9 a_{11} - 1 = 0.$$

Up to a scaling of $x_1, \dots, x_5$, we may assume that $t_2 = t_3$ and $t_4 = t_5$. Under these conditions, we find all possibilities for the subgroup H not fixing singular points. In particular, any element $\eta \in H$ takes the form

$$\eta = \begin{pmatrix} s_1 & 0 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 & 0 \\ a_1 & a_4 & 1 & a_7 & a_{10} \\ a_2 & a_5 & 0 & a_8 & a_{11} \\ a_3 & a_6 & 0 & a_9 & a_{12} \end{pmatrix}, \quad s_1, s_2 \in k^\times, \ a_1, \dots, a_{12} \in k.$$

The equality $\eta^*(f) = s_1 s_2 f$ gives another system of equations. The same method as above yields:

- $t_2 = t_3 = 0, s_1 = s_2 = a_{12} = 1, a_8 = -1$, or
- $t_2 = t_3 = t_6 = 0, s_1 = s_2 = a_8 = 1, a_{12} = -1$,

and all remaining a_j vanish. □

The following proposition relies on notation from Proposition 5.3.

Proposition 5.4. *The $\langle \sigma_{(12)(45)} \rangle$-action from Proposition 5.3 on a very general cubic threefold X with $2\mathsf{A}_3$-singularities and defect $d(X) = 0$ is not stably linearizable.*

Proof. We use specialization, as in [6, Proposition 2.9], applied to a higher-dimensional family. Fixing $t_1 \in k^\times$ and $t_6 \in k$, we consider the family of cubic threefolds

$$\mathcal{X} \to \mathbb{A}^2_{t_2, t_4},$$

where the fiber $X_{t_2,t_4} \subset \mathbb{P}^4$ is given by

$$(5.4) \quad x_1x_2x_3 + x_1x_4^2 + x_2x_5^2 + t_1x_3^3 + x_3^2(t_2x_4 + t_2x_5) + \\ + x_3(t_4x_4^2 + t_4x_5^2 + t_6x_4x_5) = 0.$$

The $\sigma_{(12)(45)}$-action naturally extends to $\mathcal{X}$. For very general $t_2, t_4 \in k$, the fiber X_{t_2,t_4} is a cubic threefold with $2\mathsf{A}_3$-singularities. The special fiber $X_{0,0}$, at $t_2 = t_4 = 0$, has $2\mathsf{A}_5$-singularities. The $\sigma_{(12)(45)}$-action on $X_{0,0}$ is not stably linearizable, by Proposition 5.12.

To apply specialization, we resolve, equivariantly, the singularities of the generic fiber of the family $\mathcal{X}$ via blowing up the $2\mathsf{A}_3$-points twice. This brings us into the situation of a smooth generic fiber and BG-rational singularities in the special fiber: the special fiber has $2\mathsf{A}_1$-singularities in the same $\langle \sigma_{(12)(45)} \rangle$-orbit. The argument works for any fixed $t_1 \in k^\times$ and $t_6 \in k$, thus, applying specialization, we conclude the $\langle \sigma_{(12)(45)} \rangle$-action on a very general cubic given by (5.3) is not stably linearizable.

□

Corollary 5.5. *A G-action on a very general cubic threefold in each of the three cases in Proposition 5.3 is not stably linearizable if and only if it does not fix two singular points, except possibly one case:* $\mathrm{Aut}(X) = \mathfrak{D}_4$ *and* $G = \langle \sigma_{(12)(45)} \eta_2 \rangle \simeq C_4$.

Proof. Any action switching two singular points, except the one specified in the assertion, specializes to an action on a cubic with $2\mathsf{A}_5$-singularities such that there are **(H1)** obstructions, as in the proof of Proposition 5.4. □

Remark 5.6. The exceptional case described in Corollary 5.5 also specializes to a cubic with $2\mathsf{A}_5$-singularities, but to the group satisfying **(H1)**, and is linearizable, see Proposition (5.13).

Singularity type $2\mathsf{A}_3$ containing a plane. Similar to the case with no plane, X is given by

$$x_1x_2x_3 + x_1x_4^2 + x_2q_2 + f_3 = 0,$$

where

$$f_3 = t_1x_3^3 + x_3^2(t_2x_4 + t_3x_5) + x_3(t_4x_4^2 + t_5x_5^2 + t_6x_4x_5) + t_7x_4^2x_5 + t_8x_4x_5^2 + t_9x_4^3.$$

Since X contains a plane, we see that

$$q_2 = x_4^2, \quad t_8 \neq 0.$$

Up to a change of variables, one may assume that $t_6 = t_7 = t_9 = 0$.

Proposition 5.7. *Let X be a cubic threefold with $2\mathsf{A}_3$-singularities and $d(X) = 1$, i.e., containing a plane. Then, up to isomorphism, X is given by*

$$x_1x_2x_3 + (x_1 + x_2)x_4^2 + t_1x_3^3 + x_3^2(t_2x_4 + t_3x_5) + \\ + x_3(t_4x_4^2 + t_5x_5^2) + t_8x_4x_5^2 = 0. \tag{5.5}$$

Assume that $\mathrm{Aut}(X)$ does not fix any singular point. Then, up to isomorphism, $(\mathrm{Aut}(X), X)$ is one of the following:

- $\mathrm{Aut}(X) = \langle \sigma_{(12)}, \eta_3 \rangle \simeq C_2 \times C_8$, *for general $t_1, t_8 \in k^\times$, and $t_2 = t_3 = t_4 = t_5 = 0$, generated by*

$$\sigma_{(12)} : (\mathbf{x}) \mapsto (x_2, x_1, x_3, x_4, x_5),$$
$$\eta_3 : (\mathbf{x}) \mapsto (-x_1, -x_2, x_3, -\zeta_8^2 x_4, \zeta_8 x_5).$$

- $\mathrm{Aut}(X) = \langle \sigma_{(12)}, \eta_3^2 \rangle \simeq C_2 \times C_4$, *for general* $t_4 \in k$, $t_1, t_8 \in k^\times$, *and* $t_2 = t_3 = t_5 = 0$.
- $\mathrm{Aut}(X) = \langle \sigma_{(12)}, \eta_3^4 \rangle \simeq C_2 \times C_2$, *for general* $t_1, t_2, t_4, t_5 \in k$, $t_8 \in k^\times$ *and* $t_3 = 0$.
- $\mathrm{Aut}(X) = \langle \sigma_{(12)} \rangle \simeq C_2$, *for general* $t_1, t_2, t_4, t_5 \in k$, $t_3, t_8 \in k^\times$.

Proof. We apply the algorithm of Section 3, as in Proposition 5.3. □

Proposition 5.8. *Let X be a cubic threefold with* $2\mathsf{A}_3$*-singularities and* $d(X) = 1$*. Let* $G \subseteq \mathrm{Aut}(X)$ *be a finite subgroup. Then the G-action on X is not linearizable if and only if no singular points are fixed by G and X does not contain a G-invariant line disjoint from the unique plane* $\Pi \subset X$*.*

Proof. Unprojection from the plane Π produces, equivariantly, a smooth intersection of two quadrics $X_{2,2} \subset \mathbb{P}^5$. By [10], it is linearizable if and only if $X_{2,2}$ contains G-invariant lines. This is equivalent to G fixing a singular point or leaving invariant a line disjoint from Π, see [6, Proposition 5.6]. □

Corollary 5.9. *Let X be a cubic threefold with* $2\mathsf{A}_3$*-singularities and* $d(X) = 1$*. Then the G-action on X is linearizable if and only if G fixes a singular point or* $G = \langle \sigma_{(12)} \rangle$*.*

Proof. For $G = \langle \sigma_{(12)} \rangle$, the G-fixed locus on X is a smooth cubic surface. Its image under the unprojection to $X_{2,2}$ is a smooth del Pezzo surface of degree 4, with 16 lines. Then G is linearizable. All the other possible subgroups G in Proposition 5.7 not fixing any singular points contain an element of the form $\sigma_{(12)}\eta_3^r$. One can check that for all $r = 1, \dots, 7$, $\sigma_{(12)}\eta_3^r$ does not leave invariant any line in X disjoint from Π. Therefore, the corresponding G-action is not linearizable by Proposition 5.8. □

Singularity type $2\mathsf{A}_4$. Up to isomorphism, X is given by

$$q_1 = x_4^2, \quad q_2 = x_5^2, \quad f_3 = t_1 x_3^3 + x_3^2(t_2 x_4 + t_3 x_5) + \\ + x_3\left(-\frac{t_7^2}{4}x_4^2 - \frac{t_8^2}{4}x_5^2 + t_6 x_4 x_5\right) + t_7 x_4^2 x_5 + t_8 x_5^2 x_4$$

for general parameters $t_1, t_2, t_3, t_6, t_7, t_8 \in k$. As above, we may assume that $t_7 = t_8 = 0$ and $t_2 = t_3$, up to a change of variables.

Proposition 5.10. *Let X be a cubic threefold with $2\mathsf{A}_4$-singularities. Assume that $\mathrm{Aut}(X)$ does not fix a singular point. Then, up to isomorphism, X is given by*

$$x_1x_2x_3 + x_1x_4^2 + x_2x_5^2 + t_1x_3^3 + x_3^2(t_2x_4 + t_2x_5) + t_6x_3x_4x_5 = 0, \tag{5.6}$$

with general parameters $t_1, t_2, t_6 \in k$ and $(\mathrm{Aut}(X), X)$ is one of the following:

- $\mathrm{Aut}(X) = \langle \sigma_{(12)(45)}, \eta_4 \rangle \simeq C_6$, *for general $t_2 \in k^\times$ and $t_1 = t_6 = 0$, generated by*

$$\sigma_{(12)(45)} : (\mathbf{x}) \mapsto (x_2, x_1, x_3, x_5, x_4),$$
$$\eta_4 : (\mathbf{x}) \mapsto (\zeta_3^2x_1, \zeta_3^2x_2, x_3, \zeta_3x_4, \zeta_3x_5).$$

- $\mathrm{Aut}(X) = \langle \sigma_{(12)(45)} \rangle \simeq C_2$ *for general parameters $t_1, t_2, t_6 \in k$.*

Proof. Similar to the proof of Proposition 5.3. □

Proposition 5.11. *Let X be a very general cubic with $2\mathsf{A}_4$-singularities such that $\mathrm{Aut}(X)$ switches two singular points. Then the $\langle \sigma_{(12)(45)} \rangle$-action on X is not stably linearizable.*

Proof. By the classification in Proposition 5.10, it suffices to show the $\langle \sigma_{(12)(45)} \rangle$-action on a very general $2\mathsf{A}_4$ cubic threefold X is not stably linearizable. We use specialization, as in 5.4.

Fix $t_6 \in k$ and $t_1 \in k^\times$, and consider the family of cubic threefolds

$$\pi : \mathcal{X} \to \mathbb{A}^1_{t_2}$$

whose generic fiber $X_{t_2} := \mathcal{X}_{t_2}$ is the cubic threefold given by

$$x_1x_2x_3 + x_1x_4^2 + x_2x_5^2 + t_1x_3^3 + x_3^2(t_2x_4 + t_2x_5) + t_6x_3x_4x_5 = 0.$$

The $\sigma_{(12)(45)}$-action extends to $\mathcal{X}$. For very general $t_2 \in k$, the fiber X_{t_2} is a cubic threefold with $2\mathsf{A}_4$-singularities. The special fiber X_0 at $t_2 = 0$ has $2\mathsf{A}_5$-singularities. Moreover, by Proposition 5.12, the $\langle \sigma_{(12)(45)} \rangle$-action on X_0 is not stably linearizable. As in Proposition 5.4, applying specialization to a resolution of singularities of the generic fiber of the family $\mathcal{X}$ completes the proof.

□

Singularity type $2\mathsf{A}_5$. According to [1], see also [2, Theorem 3.2(iii)], any cubic threefold X_b with $2\mathsf{A}_5$-singularities is given by

$$X_b = \{x_1x_2x_3 + x_1x_4^2 + x_2x_5^2 + x_3^3 + bx_3x_4x_5 = 0\}, \quad b^2 \neq -4. \tag{5.7}$$

One has

$$\mathrm{Aut}(X_b) = \begin{cases} \langle \tau_a, \sigma_{(12)(45)} \rangle \simeq \mathbb{G}_m(k) \rtimes C_2, & b^2 \neq 0, -4, \\ \langle \tau_a, \sigma_{(12)(45)}, \eta_2 \rangle \simeq (C_2 \times \mathbb{G}_m(k)) \rtimes C_2, & b = 0, \end{cases}$$

where

$$\begin{aligned} \tau_a &: (\mathbf{x}) \mapsto (a^2 x_1, a^{-2} x_2, x_3, a^{-1} x_4, a x_5), \quad a \in k^\times, \\ \sigma_{(12)(45)} &: (\mathbf{x}) \mapsto (x_2, x_1, x_3, x_5, x_4), \\ \eta_2 &: (\mathbf{x}) \mapsto (x_1, x_2, x_3, x_4, -x_5). \end{aligned} \tag{5.8}$$

Cohomology. By results in Section 2, the defect of X_b with $2\mathsf{A}_5$-singularities equals 1, and $\mathrm{Cl}(X)$ is generated by two classes of rational normal cubic scrolls in X. Projecting from $q = [1:0:0:0:0]$, we see that the associated $(2,3)$-curve

$$R_q = \{x_2x_3 + x_4^2 = x_2x_5^2 + bx_3x_4x_5 + x_3^3 = 0\} \subset \mathbb{P}^3_{x_2,x_3,x_4,x_5}$$

is the union of two twisted cubic curves, given by

$$\begin{aligned} R_1 = \{x_2x_3 + x_4^2 = x_3^2 - \frac{-b+\sqrt{b^2+4}}{2} x_4x_5 = \\ = x_3x_4 + \frac{-b+\sqrt{b^2+4}}{2} x_2x_5 = 0\} \subset \mathbb{P}^3_{x_2,x_3,x_4,x_5} \end{aligned}$$

and

$$\begin{aligned} R_2 = \{x_2x_3 + x_4^2 = x_3^2 - \frac{-b-\sqrt{b^2+4}}{2} x_4x_5 = \\ = x_3x_4 + \frac{-b-\sqrt{b^2+4}}{2} x_2x_5 = 0\} \subset \mathbb{P}^3_{x_2,x_3,x_4,x_5}. \end{aligned}$$

Let $\widehat{R}_1$, respectively $\widehat{R}_2$, be the cones over R_1, respectively R_2. The classes of $\widehat{R}_1$ and $\widehat{R}_2$ in $\mathrm{Cl}(X)$ give another set of generators of $\mathrm{Cl}(X)$, equivalent to the classes of two cubic scrolls.

Proposition 5.12. *Let X be a cubic threefold of singularity type $2\mathsf{A}_5$, and $G = \langle \sigma_{(12)(45)} \rangle$ given by (5.8). Then*

$$\mathrm{H}^1(G, \mathrm{Pic}(\tilde{X})) = \mathbb{Z}/2.$$

Proof. First, one checks that $\widehat{R}_1 \cup \sigma_{(12)(45)}(\widehat{R}_1)$ is cut out by the quadric hypersurface section of X given by

$$\widehat{R}_1 \cup \sigma_{(12)(45)}(\widehat{R}_1) = \{x_3^2 + \frac{b-\sqrt{b^2+4}}{2} x_4x_5 = 0\} \cap X.$$

This implies that $\sigma_{(12)(45)}$ switches the two generators of $\mathrm{Cl}(X)$. As in [6, Proposition 7.5], we compute

$$\mathrm{H}^1(G, \mathrm{Pic}(\tilde{X})) = \mathbb{Z}/2.$$

□

Linearizability. When $b = 0$, the action of $\eta_2 \cdot \sigma_{(12)(45)}$ switches two nodes and has vanishing cohomology. This action is linearizable:

Proposition 5.13. *Let X be the cubic threefold given by* (5.7) *with $b = 0$ and $G \simeq \mathbb{G}_m(k) \rtimes C_2$, generated by $\eta_2\sigma_{(12)(45)}$ and $\tau_a, a \in k^\times$. Then the G-action on X is linearizable.*

Proof. Recall that $X = \{x_1x_2x_3 + x_1x_4^2 + x_2x_5^2 + x_3^3 = 0\}$ and

$$\begin{aligned} \eta_2\sigma_{(12)(45)} &: (\mathbf{x}) \mapsto (x_2, x_1, x_3, x_5, -x_4), \\ \tau_a &: (\mathbf{x}) \mapsto (a^2x_1, a^{-2}x_2, x_3, a^{-1}x_4, ax_5), \quad a \in k^\times. \end{aligned}$$

in particular, it leaves the affine chart $\{x_3 \neq 0\}$ invariant. Thus we can assume that $x_3 = 1$, and consider the G-equivariant change of coordinates

$$y_1 = x_1 + x_5^2, \quad y_2 = x_2 + x_4^2,$$

yielding the equation

$$y_1y_2 + (1 - x_4x_5)(1 + x_4x_5) = 0.$$

Let

$$z_1 = \frac{y_1}{(1 + x_4x_5)}, \quad z_2 = \frac{y_2}{(1 - x_4x_5)};$$

this G-equivariant birational change of coordinates gives a G-birational map $X \dashrightarrow Y$, where

$$Y = \{z_1z_2 + t^2 = 0\} \subset \mathbb{P}^4_{z_1,z_2,x_4,x_5,t}.$$

Thus Y is a cone over a smooth conic, with G-action generated by

$$\begin{aligned} \eta_2\sigma_{(12)(45)} &:(z_1, z_2, x_4, x_5, t) \mapsto (z_2, z_1, x_5, -x_4, t), \\ \tau_a &:(z_1, z_2, x_4, x_5, t) \mapsto (a^2z_1, a^{-2}z_2, x_4, x_5, t). \end{aligned}$$

Projecting from the G-fixed point $[0 : 0 : 1 : \zeta_4 : 0]$, we obtain linearization. □

Combining Proposition 5.12 and 5.13, we settle the linearizability problem of cubic threefolds with $2\mathsf{A}_5$-singularities:

Corollary 5.14. *Let X be a cubic threefold with $2\mathsf{A}_5$-singularities given by* (5.7) *and $G \subseteq \mathrm{Aut}(X)$. Then the G-action on X is not (stably) linearizable if and only if G contains an element conjugate to $\sigma_{(12)(45)}$ given by* (5.8).

Proof. By Proposition 5.12, the G-action is not stably linearizable if $\sigma_{(12)(45)} \in G$. When G switches two singular points but does not contain any element conjugate to $\sigma_{(12)(45)}$, we are in the situation where $b = 0$ in (5.7) and G is a subgroup of the group generated by $\eta_2\sigma_{(12)(45)}$ and $\tau_a, a \in k^\times$. Such G-actions are linearizable, by Proposition 5.13. □

Singularity type $2\mathsf{D}_4$.

Proposition 5.15. *Let X be a cubic threefold with $2\mathsf{D}_4$-singularities. Up to isomorphism, X is given by*

$$x_1x_2x_3 + f_3(x_3, x_4, x_5) = 0, \tag{5.9}$$

where f_3 is a generic cubic form in x_3, x_4, x_5, i.e.,

$$\begin{aligned} f_3 &= t_1x_3^3 + x_3^2h_1 + x_3h_2 + h_3,\\ h_1 &= t_2x_4 + t_3x_5,\\ h_2 &= t_4x_4^2 + t_5x_5^2 + t_6x_4x_5,\\ h_3 &= t_7x_4^2x_5 + t_8x_4x_5^2 + t_9x_4^3 + t_{10}x_5^3 \end{aligned} \tag{5.10}$$

for general $t_1, \dots, t_{10} \in k$, with $\mathrm{Aut}(X)$ one of the following:

Case (1): $\mathrm{Aut}(X) = \langle \sigma_{(12)}, \tau_a, \eta_1, \eta_3, \sigma_{(45)} \rangle \simeq (\mathbb{G}_m(k) \rtimes C_2) \times \mathfrak{S}_3 \times C_3$, *$X$ is given by*

$$x_1x_2x_3 + x_3^3 + x_4^3 + x_5^3 = 0.$$

Case (2): $\mathrm{Aut}(X) = \langle \sigma_{(12)}, \tau_a, \eta_1, \sigma_{(45)} \rangle \simeq (\mathbb{G}_m(k) \rtimes C_2) \times \mathfrak{S}_3$, *$X$ is given by*

$$x_1x_2x_3 + x_3^3 + t_6x_3x_4x_5 + x_4^3 + x_5^3 = 0,$$

for general $t_6 \in k$.

Case (3): $\mathrm{Aut}(X) = \langle \sigma_{(12)}, \tau_a, \eta_3, \sigma_{(45)} \rangle \simeq (\mathbb{G}_m(k) \rtimes C_2) \times C_6$, *$X$ is given by*

$$x_1x_2x_3 + x_3^3 + (x_4 + x_5)(r_4x_4 + r_5x_5)(r_5x_4 + r_4x_5) = 0,$$

for general $r_4, r_5 \in k$.

Case (4): $\mathrm{Aut}(X) = \langle \sigma_{(12)}, \tau_a, \sigma_{(45)} \rangle \simeq (\mathbb{G}_m(k) \rtimes C_2) \times C_2$, *$X$ is given by*

$$\begin{aligned} x_1x_2x_3 + t_1x_3^3 + r_1x_3^2(x_4 + x_5) + x_3(r_2x_4 + r_3x_5)(r_3x_4 + r_2x_5) +&\\ + (x_4 + x_5)(r_4x_4 + r_5x_5)(r_5x_4 + r_4x_5) = 0,&\end{aligned}$$

for general $t_1, r_1, \dots, r_5 \in k$.

Case (5): $\mathrm{Aut}(X) = \langle \sigma_{(12)}, \tau_a, \eta_2 \rangle \simeq (\mathbb{G}_m(k) \rtimes C_2) \rtimes C_2$, *$X$ is given by*

$$x_1x_2x_3 + x_3^2(t_2x_4 + t_3x_5) + t_7x_4^2x_5 + t_8x_4x_5^2 + t_9x_4^3 + t_{10}x_5^3 = 0,$$

for general $t_2, t_3, t_7, t_8, t_9, t_{10} \in k$.

Case (6): $\mathrm{Aut}(X) = \langle \sigma_{(12)}, \tau_a \rangle \simeq \mathbb{G}_m(k) \rtimes C_2$, *$X$ is given by*

vanishing of (5.9) *where f_3 is a generic cubic form*

such that $h_2 \not\equiv 0$,

where

$$\begin{aligned}
\sigma_{(12)} &: (\mathbf{x}) \mapsto (x_2, x_1, x_3, x_4, x_5),\\
\tau_a &: (\mathbf{x}) \mapsto (ax_1, a^{-1}x_2, x_3, x_4, x_5), \quad a \in k^\times,\\
\eta_1 &: (\mathbf{x}) \mapsto (x_1, x_2, x_3, \zeta_3x_4, \zeta_3^2x_5),\\
\eta_2 &: (\mathbf{x}) \mapsto (-x_1, x_2, x_3, -x_4, -x_5),\\
\eta_3 &: (\mathbf{x}) \mapsto (x_1, x_2, x_3, \zeta_3x_4, \zeta_3x_5),\\
\sigma_{(45)} &: (\mathbf{x}) \mapsto (x_1, x_2, x_3, x_5, x_4).
\end{aligned}$$

Proof. For any such cubic X, $\mathrm{Aut}(X)$ contains a subgroup isomorphic to the infinite dihedral group generated by

$$\begin{aligned}
\sigma_{(12)} &: (\mathbf{x}) \mapsto (x_2, x_1, x_3, x_4, x_5),\\
\tau_a &: (\mathbf{x}) \mapsto (ax_1, a^{-1}x_2, x_3, x_4, x_5), \quad a \in k^\times.
\end{aligned} \tag{5.11}$$

To find possibilities of $\mathrm{Aut}(X)$, it suffices to find $H \subset \mathrm{Aut}(X)$, the subgroup fixing both singular points. Based on the form of (5.9), one sees that any element in H takes the form

$$\begin{pmatrix} s_1 & 0 & 0 & 0 & 0\\ 0 & s_2 & 0 & 0 & 0\\ 0 & 0 & 1 & b_1 & b_4\\ 0 & 0 & 0 & b_2 & b_5\\ 0 & 0 & 0 & b_3 & b_6, \end{pmatrix}, \quad s_1, s_2, b_1, \dots, b_6 \in k,$$

Then up to a change of variables only in coordinates x_4 and x_5, we may assume $b_1 = b_4 = 0$ without changing the form of (5.9). Namely, H preserves (5.9) and acts on the ambient $\mathbb{P}^4$ via $\mathbb{P}(I_1 \oplus I_2 \oplus I_3 \oplus V)$, where I_1, I_2 and I_3 are 1-dimensional representations of H, acting respectively on coordinates x_1, x_2 and x_3, and V is a 2-dimensional representation of H acting on x_4, x_5. In the plane $\mathbb{P}^2_{x_3,x_4,x_5}$, the group H leaves both the line $l = \{x_3 = 0\}$ and the cubic curve $C = \{f_3 = 0\}$ invariant. Since X is a cubic with $2\mathsf{D}_4$-singularities, by Proposition 2.10, X contains three distinct planes, corresponding to the points defined by $l \cap C$. This implies $l \cap C$ defines three distinct points, in the same H-orbit. Consider the exact sequence

$$0 \to H' \to H \to \bar{H} \to 0$$

where H' contains elements in H acting via scalars in V, and $\bar{H}$ acts faithfully on $\mathbb{P}(V) = \mathbb{P}^1_{x_4,x_5}$. Since H leaves invariant three points, the possibilities of $\bar{H}$ are

$$\bar{H} = C_1, C_2, C_3, \text{ or } \mathfrak{S}_3,$$

where $\mathfrak{S}_3$ is generated by

$$\sigma = \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^2 \end{pmatrix} \quad \text{and} \quad \iota = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and the other possibilities are the corresponding subgroups of $\mathfrak{S}_3$. Moreover, H leaves invariant each of the following subsets of $\mathbb{P}^1_{x_4,x_5}$, defined by

$$Q_1 = \{h_1 = 0\}, \quad Q_2 = \{h_2 = 0\}, \quad Q_3 = \{h_3 = 0\}.$$

Using this, we classify the possibilities of H and $\bar{H}$.

When $\bar{H} = C_1$: In this case $H = H'$. We find below all possibilities of H'. By definition, any element in $\eta \in H'$ takes the form

$$\eta : (\mathbf{x}) \mapsto (s_1x_1, s_2x_2, x_3, s_3x_4, s_3x_5), \quad s_1, s_2, s_3 \in k^\times.$$

The weights of the η-action on h_1, h_2, h_3 are respectively s_3, s_3^2, s_3^3. Since $h_3 \not\equiv 0$, there are the following cases:

- When $h_2 \not\equiv 0$: we have $s_3 = 1$, η is the toric action (5.11) and $H' \simeq k^\times$.
- When $h_2 \equiv 0$, $h_1 \not\equiv 0$ and $t_1 = 0$: we have $s_3 = -1$, $s_1s_2 = -1$, and $H' \simeq C_2 \times k^\times$.
- When $h_2 \equiv 0$, $h_1 \equiv 0$ and $t_1 \neq 0$: we have $s_3^3 = 1$ and $H' \simeq C_3 \times k^\times$.
- When $h_2 \equiv 0$, $h_1 \equiv 0$ and $t_1 = 0$: X has $3\mathsf{D}_4$-singularities.

When $\bar{H} = \mathfrak{S}_3$: Since $\mathfrak{S}_3$ has no fixed points in $\mathbb{P}^1$, one has $h_1 \equiv 0$, $h_2 = t_6x_4x_5$ and $h_3 = x_4^3 + x_5^3$.

When $\bar{H} = \langle\sigma\rangle \simeq C_3$: We know that Q_1 is a fixed point of $\bar{H}$, i.e., $h_3 = t_2x_4$ or t_3x_5. Similarly, Q_2 can also only contain fixed points of $\bar{H}$, i.e., $h_3 = t_4x_4^2$, $t_5x_5^2$, or $t_6x_4x_5$, and Q_3 contains three distinct points in one $\bar{H}$-orbit, thus, up to scaling, $h_3 = x_4^3 + x_5^3$. Matching the weights of the σ-actions on each of the monomials appearing in f_3, one sees that the only choice is $h_1 \equiv 0$, $h_2 = t_6 x_4 x_5$ and $h_3 = x_4^3 + x_5^3$. Then we go back to the situation above. Thus, $\bar{H} \not\simeq C_3$, i.e., $\sigma \in \bar{H}$ implies that $\bar{H} \simeq \mathfrak{S}_3$.

When $\bar{H} = \langle \iota \rangle \simeq C_2$: As above, using symmetries on $\mathbb{P}^1_{x_4,x_5}$, and matching the weights on the monomials, we find two cases:

- $h_1 = r_1(x_4 + x_5)$, for some $r_1 \in k$,
- $h_2 = (r_2 x_4 + r_3 x_5)(r_3 x_4 + r_2 x_5)$, for some $r_2, r_3 \in k$,
- $h_3 = (x_4 + x_5)(r_4 x_4 + r_5 x_5)(r_5 x_4 + r_4 x_5)$, for some $r_4 \neq r_5 \in k^\times$;

or

- $h_1 = r_1(x_4 - x_5)$, for some $r_1 \in k$,
- $h_2 = r_2(x_4^2 - x_5^2)$, for some $r_2, r_3 \in k$,
- $h_3 = (x_4 - x_5)(r_4 x_4 + r_5 x_5)(r_5 x_4 + r_4 x_5)$, for some $r_4 \neq r_5 \in k^\times$,
- $t_1 = 0$.

Combining all possibilities of $\bar{H}$ and of H', and checking the singularity types, we obtain the assertion. □

The following applies in Cases(1), (2), (3), and (4) of Proposition 5.15.

Proposition 5.16. *Let X be a cubic threefold with $2\mathsf{D}_4$-singularities admitting the action of $H := \langle \sigma_{(12)}, \sigma_{(45)} \rangle$. Then, for any $G \subseteq \mathrm{Aut}(X)$ containing H, the G-action on X is not linearizable.*

Proof. The $\sigma_{(12)}$-action fixes a smooth cubic surface $S \subset X$ and the residual $\sigma_{(45)}$-action fixes a genus 1 curve on S, producing an incompressible symbol, in the terminology of, e.g., [15, Section 3]. We conclude as in [6, Proposition 2.6]. □

Proposition 5.17. *Let X be a cubic threefold with $2\mathsf{D}_4$-singularities and $G = \langle \sigma_{(12)}, \tau_a \rangle \simeq \mathfrak{D}_{2n}$, $n \geq 2$, where $a = \zeta_{2n}$ is a primitive $2n$-th root of unity and $\sigma_{(12)}, \tau_a$ are described in Proposition 5.15. Then the G-action on X is not linearizable.*

Proof. Recall that $G = \mathfrak{D}_{2n}$ is the dihedral group of order $4n$. Observe that G pointwise fixes a smooth elliptic curve $E = \{x_1 = x_2 = 0\} \subset X$. To apply the Burnside formalism, one has to pass to a standard model, and, in particular, blow up strata with nonabelian generic stabilizers. Thus, one needs to blow up E in X, see [8, Section 7.2] for definitions. The exceptional divisor has generic stabilizer C_2. It follows that on a standard form for the action $X \circlearrowleft G$, we find the symbol

$$(C_2, \mathfrak{D}_n \circlearrowright k(S), (1)), \tag{5.12}$$

where $S = \mathbb{P}(\mathcal{N}_{E/X})$, the projectivization of the normal bundle of E in X, and in particular, S is a $\mathbb{P}^1$-bundle over E. This symbol is incompressible: the $\mathfrak{D}_n$-action on S is not birational to any actions on the blowup of a genus 1 curve with abelian stabilizers. To see this, one can apply the Burnside formalism in dimension 2. Notice that the $\mathfrak{D}_n$-action on S is trivial on the base of the fibration $S \to E$. So we find an *incompressible* symbol in the class $[S \circlearrowleft G]$:

$$(C_n, \mathrm{triv} \circlearrowright k(E), (\beta)), \tag{5.13}$$

for some character β of C_n. On the other hand, such symbols do not arise from any $\mathfrak{D}_n$-action on the blowup of a genus 1 curve on any standard model – on a standard model, the curve has abelian stabilizer and receives a nontrivial action from $\mathfrak{D}_n$. It can never produce a divisorial symbol with a trivial residual action as in (5.13). Equivariant nonbirationality of S with a blowup of a genus 1 curve with abelian stabilizers also follows from the the functoriality of passage to MRC quotients, see [12, Theorem IV.5.5]. Therefore, we conclude that symbols (5.12) are incompressible. Such symbols do not appear from linear actions on $\mathbb{P}^3$, which implies that the G-action on X is not linearizable. □

Proposition 5.18. *The $\langle \sigma_{(12)}\sigma_{(45)} \rangle$-action on a very general cubic threefold with $2\mathsf{D}_4$-singularities described in Case (4) in Proposition 5.15 is not stably linearizable.*

Proof. Recall that such X with $2\mathsf{D}_4$-singularities are given by

$$\begin{aligned} x_1x_2x_3 + t_1x_3^3 + r_1x_3^2(x_4+x_5) + x_3(r_2x_4+r_3x_5)(r_3x_4+r_2x_5) + \\ + (x_4+x_5)(r_4x_4+r_5x_5)(r_5x_4+r_4x_5) = 0, \end{aligned} \tag{5.14}$$

for general parameters $t_1, r_1, r_2, r_3, r_4, r_5 \in k$, and $\sigma_{(12)}\sigma_{(45)}$ takes the form

$$\sigma_{(12)}\sigma_{(45)} : (\mathbf{x}) \to (x_2, x_1, x_3, x_5, x_4).$$

Now we view (5.14) as a family $\mathcal{X} \to \mathbb{A}^6$ of cubic threefolds with $2\mathsf{D}_4$-singularities parameterized by $t_1, r_1, r_2, r_3, r_4, r_5 \in k$. The general fibers above $r_2 - r_3 = t_1 = 0$ are cubic threefolds with $2\mathsf{D}_4 + 2\mathsf{A}_1$-singularities. In particular, under the change of variables

$$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = \frac{1}{4}(2x_3 - \frac{r_4 - r_5}{\sqrt{r_1}}x_4 + \frac{r_4 - r_5}{\sqrt{r_1}}x_5)$$

$$y_4 = \frac{1}{4}(2x_3 + \frac{r_4 - r_5}{\sqrt{r_1}}x_4 - \frac{r_4 - r_5}{\sqrt{r_1}}x_5), \quad y_5 = x_4 + x_5,$$

the fibers above $r_2 - r_3 = t_1 = 0$ are given by

(5.15)

$$y_1y_2y_3 + y_1y_2y_4 + 4r_1y_3y_4y_5 + r_2^2y_5^2(y_3 + y_4) + \frac{1}{4}(r_4 + r_5)^2y_5^3 = 0,$$

and $\sigma_{(12)}\sigma_{(45)}$ under the new basis is

$$\sigma_{(12)}\sigma_{(45)} : (\mathbf{y}) \mapsto (y_2, y_1, y_4, y_3, y_5).$$

From Proposition 7.7, we see that the $\langle\sigma_{(12)}\sigma_{(45)}\rangle$-action on the cubic given by (5.15) is not stably linearizable. Applying specialization to the resolution of the $2\mathsf{D}_4$-singularities in the generic fiber of the family $\mathcal{X}$, we conclude that the $\langle\sigma_{(12)}\sigma_{(45)}\rangle$-action on a very general member in $\mathcal{X}$ is not stably linearizable. □

Proposition 5.19. *Let X be a cubic threefold with $2\mathsf{D}_4$-singularities. Then the $\langle\sigma_{(12)}\tau_a\rangle$-action on X from Proposition 5.15 is linearizable for any $a \in k^\times$.*

Proof. The $\sigma_{(12)}\tau_a$-action preserves each of the three planes and pointwise fixes a smooth cubic surface in X for any $a \in k^\times$. Unprojection from one plane birationally transforms X to an intersection of two quadrics $X_{2,2}$ in $\mathbb{P}^5$, with $2\mathsf{A}_1$-singularities. The cubic surface becomes a smooth del Pezzo surface of degree 4 in $X_{2,2}$, and it contains 16 lines, fixed by the action. Projection from any of the lines yields a linearization of the $\sigma_{(12)}\tau_a$-action on X. □

6. Three singular points

Let X have three singular points. We may assume that they are at

$$p_1 = [1 : 0 : 0 : 0 : 0], \quad p_2 := [0 : 1 : 0 : 0 : 0], \quad p_3 := [0 : 0 : 1 : 0 : 0],$$

so that X is given by

$$x_1x_2x_3 + x_1q_1 + x_2q_2 + x_3q_3 + f_3 = 0, \tag{6.1}$$

where $q_1, q_2, q_3, f_3 \in k[x_4, x_5]$. There are three possibilities:

$$3\mathsf{A}_2, \quad 3\mathsf{A}_3, \quad 3\mathsf{D}_4.$$

All of these are specializations of the $3\mathsf{A}_1$ case, studied in [6, Section 4]. Here, we use similar arguments.

Singularity Types $3\mathsf{A}_2$ and $3\mathsf{A}_3$. Since p_1, p_2 and p_3 are A_n-points with $n = 2, 3$, the rank of q_1, q_2, q_3 is 1, i.e., $q_i = l_i^2$ for some linear forms $l_i \in k[x_4, x_5]$, $i = 1, 2, 3$. Observe that if the singularity type is $3\mathsf{A}_2$, then X contains no plane. It follows that q_1, q_2, q_3 and f_3 do not share a common factor.

Proposition 6.1. *Let X be a cubic threefold with singularity types $3\mathsf{A}_2$ or $3\mathsf{A}_3$. Assume that $\mathrm{Aut}(X)$ does not fix any singular points.*

If X has $3\mathsf{A}_2$-points then either:

(1) $\mathrm{Aut}(X) = \langle \sigma_{(123)}, \sigma_{(23)}, \eta_1 \rangle \simeq C_3 \times \mathfrak{S}_3$, *where*

$$\sigma_{(123)} : (\mathbf{x}) \mapsto (x_3, x_1, x_2, x_4, x_5),$$
$$\sigma_{(23)} : (\mathbf{x}) \mapsto (x_1, x_3, x_2, x_4, x_5),$$
$$\eta_1 : (\mathbf{x}) \mapsto (x_1, x_2, x_3, x_4, \zeta_3 x_5),$$

and X is given by

$$x_1x_2x_3 + x_4^2(x_1 + x_2 + x_3) + ax_4^3 + x_5^3 = 0,$$

for general $a \in k^\times$;

(2) $\mathrm{Aut}(X) = \langle \sigma_{(123)}, \sigma_{(23)}, \eta_2 \rangle \simeq C_6 \times \mathfrak{S}_3$, *where*

$$\eta_2 : (\mathbf{x}) \mapsto (x_1, x_2, x_3, -x_4, \zeta_3 x_5),$$

and X is given by

$$x_1x_2x_3 + x_4^2(x_1 + x_2 + x_3) + x_5^3 = 0.$$

(3) $\mathrm{Aut}(X) = \langle \sigma_{(123)}, \sigma_{(23)} \rangle \simeq \mathfrak{S}_3$, *and X is given by*

$$x_1x_2x_3 + x_4^2(x_1 + x_2 + x_3) + f_3 = 0,$$

with $f_3 \in k[x_4, x_5]$ a general cubic form;

(4) $\mathrm{Aut}(X) = \langle \sigma_{(123)}\eta_1 \rangle \simeq C_3$, *and X is given by*

$$x_1x_2x_3 + x_1(x_4 + x_5)^2 + x_2(x_4 + \zeta_3 x_5)^2 + x_3(x_4 + \zeta_3^2 x_5)^2 + f_3 = 0,$$

where $f_3 = ax_4^3 + bx_5^3$, for $a \neq b \in k$, or $f_3 = cx_4^3$, for $c \in k^\times$;

If X has $3\mathsf{A}_3$-points then:

(1) $\mathrm{Aut}(X) = \langle \sigma_{(123)}, \sigma_{(23)}, \eta_3 \rangle \simeq C_4 \times \mathfrak{S}_3$, *where*

$$\eta_3 : (\mathbf{x}) \mapsto (x_1, x_2, x_3, -x_4, \zeta_4 x_5),$$

and X is given by

$$x_1x_2x_3 + x_4^2(x_1 + x_2 + x_3) + ax_4x_5^2 = 0,$$

for general $a \in k^\times$.

(2) $\mathrm{Aut}(X) = \langle \sigma_{(123)}\eta_1, \sigma\rangle \simeq \mathfrak{S}_3$, *where*

$$\sigma : (\mathbf{x}) \mapsto (x_1, \zeta_3 x_3, \zeta_3^2 x_2, x_5, x_4),$$

and X *is given by*

$$x_1x_2x_3+x_1(x_4+x_5)^2+x_2(x_4+\zeta_3x_5)^2+x_3(x_4+\zeta_3^2x_5)^2+a(x_4^3+x_5^3) = 0,$$

for general $a \in k^\times$.

(3) $\mathrm{Aut}(X) = \langle \sigma_{(123)}, \sigma_{(23)}\rangle \simeq \mathfrak{S}_3$, *and* X *is given by*

$$x_1x_2x_3 + x_4^2(x_1 + x_2 + x_3) + x_4 f_2 = 0,$$

with $f_2 \in k[x_4, x_5]$ *a general quadratic form*;

Proof. We know that $\mathrm{Aut}(X)$ fits into the exact sequence

$$0 \to H \to \mathrm{Aut}(X) \xrightarrow{\rho} \mathfrak{S}_3, \tag{6.2}$$

where H is the subgroup of $\mathrm{Aut}(X)$ fixing three singular points. Assume $\mathrm{Aut}(X)$ does not fix any singular point, i.e., there exists an element $\sigma_{123} \in \mathrm{Aut}(X)$ with $\rho(\sigma_{123}) = (1, 2, 3) \in \mathfrak{S}_3$. Since σ_{123} preserves the form (6.1) and q_1, q_2, q_3 define at most 3 points in $\mathbb{P}^1_{x_4,x_5}$, we may assume that σ_{123} takes the form

$$\sigma_{123} : (\mathbf{x}) \mapsto (s_1x_2, s_2x_3, s_3x_1, x_4, \zeta_3^r x_5),$$

for $r = 0$ or 1, and $s_1, s_2, s_3 \in k^\times$. The cyclic action, together with the torus action on x_1, x_2, x_3, imply that

$$q_2 = \sigma_{123}^*(q_1), \quad \text{and} \quad q_3 = \sigma_{123}^*(q_2).$$

It follows that $s_1 = s_2 = s_3 = \pm 1$. Now we discuss two cases of r:

If $r = 0$, we may assume that $q_1 = q_2 = q_3 = x_4^2$. Then $\mathrm{Aut}(X)$ contains a natural $\mathfrak{S}_3$-action, permuting the coordinates x_1, x_2, x_3. It remains to classify possibilities of H. Assume that H is nontrivial. A $\tau \in H$ acts diagonally on $x_1, \ldots, x_4$, since it preserves (6.1), and one can diagonalize τ without changing the form of (6.1). Thus, we may assume that τ takes the form

$$\tau : (\mathbf{x}) \mapsto (x_1, x_2, x_3, a_1x_4, a_2x_5), \quad a_1 = \pm 1, \quad a_2 \in k^\times.$$

Recall that f_3 defines at most three points on $\mathbb{P}^1_{x_4,x_5}$, preserved by τ. We have the following cases:

- f_3 defines three distinct points. It follows that $a_1 = 1$, $a_2 = \zeta_3$, and $f_3 = ax_4^3+x_5^3$, for some $a \in k$. In this case, X has singularity type $3\mathsf{A}_2$ and $\mathrm{Aut}(X) = C_3 \times \mathfrak{S}_3$.

- f_3 defines two distinct points, necessarily fixed by τ. It follows that $f_3 = ax_4x_5^2$, for some $a \in k^\times$. Note that $f_3 = ax_4^2x_5$ would give nonisolated singularities on X, and we exclude this. Thus, we have $a_1 = -1, a_2 = \zeta_4$, X has $3\mathsf{A}_3$-singularities and $\mathrm{Aut}(X) = C_4 \times \mathfrak{S}_3$.
- f_3 defines one point, necessarily fixed by τ. We have $f_3 = x_5^3$, X has $3\mathsf{A}_2$-singularities and $\mathrm{Aut}(X) = C_6 \times \mathfrak{S}_3$.

If f_3 is a general cubic form, then X has $3\mathsf{A}_2$-singularities and $\mathrm{Aut}(X) = \mathfrak{S}_3$. If $f_3 = x_4f_2$, where f_2 is a general quadratic form, then X has $3\mathsf{A}_3$-singularities and $\mathrm{Aut}(X) = \mathfrak{S}_3$.

Now we consider the case $r = 1$. Up to a change of variables, we may assume that

$$q_1 = (x_4 + x_5)^2, \quad q_2 = (x_4 + \zeta_3x_5)^2, \quad q_3 = (x_4 + \zeta_3^2x_5)^2.$$

Let $\sigma_{23} \in \mathrm{Aut}(X)$ be an element fixing p_1 and switching p_2 and p_3. Then σ_{23} also fixes the point in $\mathbb{P}^1_{x_4,x_5}$ defined by q_1 and switches the points defined by q_2, q_3. The only possible such action on $\mathbb{P}^1$ is switching the coordinates x_4 and x_5. But the points defined by f_3 need to be preserved by both σ_{123} and σ_{12}. The only possibility is $f_3 = a(x_4^3 + x_5^3)$, for some $a \in k^\times$, and X has $3\mathsf{A}_3$-singularities. In particular, σ_{23} takes the form

$$\sigma_{23} : (\mathbf{x}) \mapsto (x_1, \zeta_3x_3, \zeta_3^2x_2, x_5, x_4).$$

In the case of $3\mathsf{A}_2$-singularities, σ_{12} does not exists, i.e., $\rho(\mathrm{Aut}(X)) = C_3$. We then classify the possibilities of H. For any $\eta \in H$, η fixes three singularities of X in $\mathbb{P}^4$ and three points in $\mathbb{P}^1$ defined by q_1, q_2, q_3. One sees that η acts on $\mathbb{P}^4$ diagonally, with weights $(a_1, a_2, a_3, a_4, a_4)$. As above, we see that f_3 takes the following forms:

- f_3 defines three distinct points and $f_3 = a(x_4^3 + x_5^3)$, for some $a \in k^\times$. In this case X has $3\mathsf{A}_3$-singularities.
- f_3 defines three distinct points and $f_3 = ax_4^3 + bx_5^3$ for some $a \neq b \in k^\times$. In this case X has $3\mathsf{A}_2$-singularities.
- f_3 defines two distinct points, necessarily fixed by σ_{123}, i.e., $f_3 = ax_4^2x_5$ or $ax_4x_5^2$, for some $a \in k^\times$. But in this case, X is not $\langle\sigma_{123}\rangle$-invariant.
- f_3 defines one point, necessarily fixed by σ_{123}, i.e., $f_3 = ax_4^3$ or ax_5^3, for some $a \in k^\times$. In this case, X has $3\mathsf{A}_2$-singularities.

It is not hard to check that in all cases above, H is trivial, and $\mathrm{Aut}(X) = C_3$ when X has $3\mathsf{A}_2$-singularities; $\mathrm{Aut}(X) = \mathfrak{S}_3$ when X has $3\mathsf{A}_3$-singularities. □

Proposition 6.2. *Let X be a cubic with $3\mathsf{A}_2$ or $3\mathsf{A}_3$-singularities. Then the following G-actions are not stably linearizable, for very general X in the corresponding families in Proposition 6.1:*

(1) *X has $3\mathsf{A}_2$-singularities and $\mathrm{Aut}(X) = \mathfrak{S}_3$: $G = \langle\sigma_{(123)}\rangle$,*
(2) *X has $3\mathsf{A}_2$-singularities and $\mathrm{Aut}(X) = C_3 \times \mathfrak{S}_3$: $G = \langle\sigma_{(123)}\rangle$,*
(3) *X has $3\mathsf{A}_3$-singularities and $\mathrm{Aut}(X) = \mathfrak{S}_3$: $G = \langle\sigma_{(123)}\rangle$.*

Proof. We use specialization, as in Proposition 5.4. By Proposition 6.1, cubic threefolds in Case (1) are given by

$$x_1x_2x_3 + x_4^2(x_1 + x_2 + x_3) + f_3 = 0,$$

for a general cubic form $f_3 \in k[x_4, x_5]$ or $f_3 = x_4f_2$. We may assume that f_3 defines three distinct points in $\mathbb{P}^1_{x_4,x_5}$, and is isomorphic to the cubic form $x_4^3 + x_5^3$. Up to a change of variables, a very general cubic in Case (1) is a fiber of the family

$$\mathcal{X} \to \mathbb{A}^2_{s,t}, \tag{6.3}$$

whose generic fiber is given by

$$x_1x_2x_3 + (sx_4 + tx_5)^2(x_1 + x_2 + x_3) + x_4^3 + x_5^3 = 0.$$

The $\sigma_{(123)}$-action extends to the family $\mathcal{X}$ and remains unchanged under the change of variables since it acts trivially on x_4 and x_5. The generic fiber of $\mathcal{X}$ is a cubic with $3\mathsf{A}_2$-singularities at

$$p_1 = [1 : 0 : 0 : 0 : 0], \quad p_2 = [0 : 1 : 0 : 0 : 0], \quad p_3 = [0 : 0 : 1 : 0 : 0].$$

The special fiber $X_{0,0}$ above $s = t = 0$ is a cubic with $3\mathsf{D}_4$-singularities. The $\sigma_{(123)}$-action on $X_{0,0}$ is not stably linearizable, by Proposition 6.7. As in Proposition 5.4, we apply specialization to a resolution of singularities of the generic fiber. This can be achieved by blowing up three sections of $\mathcal{X} \to \mathbb{A}^2_{s,t}$ corresponding to p_1, p_2 and p_3. After the blowup, the new family has smooth generic fiber, and the special fiber above $s = t = 0$ has BG-rational singularities: it has $9\mathsf{A}_1$-singularities forming three $\sigma_{(123)}$-orbits. As in [6, Proposition 2.9], we conclude that a very general member in the family is not stably linearizable.

The same argument applies to cubic threefolds in Case (2) and (3) as they form a subfamily of Case (1), with the same $\sigma_{(123)}$-action. □

Example 6.3. Let X be the cubic with $3\mathsf{A}_2$-singularities and $\mathrm{Aut}(X) = C_6 \times \mathfrak{S}_3$. The element η_2^2 fixes a cubic surface S with $3\mathsf{A}_1$-singularities in X, given by

$$x_1x_2x_3 + (x_1 + x_2 + x_3)x_4^2 = 0,$$

contributing to a symbol

$$(C_3, \mathfrak{D}_6 \circlearrowright k(S), (1)). \tag{6.4}$$

The residual $\mathfrak{D}_6$-action on S is realized as permutations of the coordinates x_1, x_2, x_3 and the -1 sign change on x_4. The standard Cremona transformation on $\mathbb{P}^3$

$$(x_1, x_2, x_3, x_4) \mapsto (\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \frac{1}{x_4})$$

birationally transforms S into the smooth quadric surface

$$Q = \left\{x_1x_2 + x_2x_3 + x_3x_1 + x_4^2 = 0\right\} \subset \mathbb{P}^3$$

with the same $\mathfrak{D}_6$-action on the ambient $\mathbb{P}^3$. This $\mathfrak{D}_6$-action on Q is not birational to an action on a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, see [16, Example 9.1]. Using the same argument as there, one sees that (6.4) is an incompressible symbol. This symbol cannot appear in classes of linear actions. It follows that the $C_6 \times \mathfrak{S}_3$-action on X is not linearizable.

The same argument applies to the cubic X with $3\mathsf{A}_3$-singularities and $\mathrm{Aut}(X) = C_4 \times \mathfrak{S}_3$, given by

$$x_1x_2x_3 + x_4^2(x_1 + x_2 + x_3) + ax_4x_5^2 = 0, \quad a \in k^\times.$$

For $G = \mathrm{Aut}(X)$-action, the element η_3^2 contributes to a symbol

$$(\langle \eta_3^2 \rangle, \mathfrak{D}_6 \circlearrowright k(S), (1)), \tag{6.5}$$

where S is the same cubic surface carrying the same $\mathfrak{D}_6$-action as in the symbol (6.4). As above, we see that the symbol (6.5) is also incompressible and the $\mathrm{Aut}(X)$-action on X is not linearizable.

Singularity Type $3\mathsf{D}_4$. There is a unique such cubic threefold, see [1, Theorem 5.4], given by

$$X = \{x_1x_2x_3 + x_4^3 + x_5^3\}.$$

By Proposition 3.1, we have

$$\mathrm{Aut}(X) = \langle \tau_{a,b}, \eta, \sigma_{(45)}, \sigma_{(123)}, \sigma_{(12)} \rangle \simeq (\mathbb{G}_m^2(k) \times \mathfrak{S}_3) \rtimes \mathfrak{S}_3,$$

with generators described in that proposition.

Example 6.4. Let $G = \langle \sigma_{(123)}, \sigma_{(45)}\sigma_{(12)}, \eta, \tau_{1,-1}, \tau_{-1,1} \rangle \simeq C_3 \rtimes \mathfrak{S}_4$. The action is not linearizable, as G cannot act linearly and generically freely on $\mathbb{P}^3$.

Cohomology. For the $3\mathsf{D}_4$ case, we compute $\mathrm{H}^1(G, \mathrm{Pic}(\tilde{X}))$ for finite subgroups $G \subset \mathrm{Aut}(X)$. The analysis is similar to the $9\mathsf{A}_1$-cubic in [6]. Recall that the defect $\sigma(X) = 4$ and $\mathrm{rk}\,\mathrm{Cl}(X) = 5$. In particular, $\mathrm{Cl}(X)$ is generated by the nine planes in X:

$$\Pi_{i,j} = \{x_i = x_4 + \zeta_3^j x_5 = 0\}, \quad i = 1,2,3, \quad j = 1,2,3 \quad \zeta_3 = e^{\frac{2\pi i}{3}},$$

subject to relations

$$\sum_{i=1}^{3} \Pi_{i,j} = F, \quad \text{for } j = 1,2,3, \quad \sum_{j=1}^{3} \Pi_{i,j} = F, \quad \text{for } i = 1,2,3, \tag{6.6}$$

where F denotes the class of the hyperplane section on X.

Proposition 6.5. *Let X be the cubic of singularity type $3\mathsf{D}_4$ and*

$$\sigma_{(123)} : (\mathbf{x}) \mapsto (x_3, x_1, x_2, x_4, x_5).$$

Then

$$\mathrm{H}^1(\langle \sigma_{(123)} \rangle, \mathrm{Pic}(\tilde{X})) = \mathbb{Z}/3.$$

Proof. Using the generators and relations of $\mathrm{Cl}(X)$ described above, one can compute

$$\mathrm{H}^1(\langle \sigma_{(123)} \rangle, \mathrm{Cl}(X)) = \mathbb{Z}/3.$$

Let $\tilde{X} \to X$ be an $\mathrm{Aut}(X)$-equivariant resolution of singularities via successive blowups of the singular points, and $E_i, i = 1, \ldots, r$ the corresponding exceptional divisors. Since $\sigma_{(123)}$ acts transitively on the $3\mathsf{D}_4$-points, it permutes the exceptional divisors without leaving any one fixed. Then

$$\mathrm{H}^2(\langle \sigma_{(123)} \rangle, \oplus_{i=1}^r \mathbb{Z} \cdot E_i) = 0.$$

Using Lemma 4.2, we conclude

$$\mathrm{H}^1(\langle \sigma_{(123)} \rangle, \mathrm{Pic}(\tilde{X})) = \mathrm{H}^1(\langle \sigma_{(123)} \rangle, \mathrm{Cl}(X)) = \mathbb{Z}/3.$$

□

Linearizability.

Proposition 6.6. *Let X be the cubic threefold with $3\mathsf{D}_4$-singularities and $H = \langle \tau_{ab}, \eta\sigma_{(123)}, \sigma_{(45)}\sigma_{(12)} \rangle$, from Proposition 3.1. Then the H-action on X is linearizable.*

Proof. Under the rational map $\rho : X \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$(\mathbf{x}) \mapsto (-x_3, x_4 + x_5) \times (-x_1, \zeta_3^2 x_4 + \zeta_3 x_5) \times (-x_2, \zeta_3 x_4 + \zeta_3^2 x_5) \times (x_4, x_5),$$

X is birationally transformed to $S \times \mathbb{P}^1$ where S is a smooth del Pezzo surface of degree 6, realized as

$$\{u_1v_1w_1 = u_2v_2w_2\} \subset \mathbb{P}^1_{u_1,u_2} \times \mathbb{P}^1_{v_1,v_2} \times \mathbb{P}^1_{w_1,w_2}.$$

The map ρ is H-equivariant. The H-action on $S \times \mathbb{P}^1$ is faithful on the factor S: H acts on S via the $\mathfrak{S}_3$-permutation of three copies of $\mathbb{P}^1$ and the 2-dimensional torus action. The H-action on the $\mathbb{P}^1$ in $S \times \mathbb{P}^1$ factors through $\mathfrak{S}_3$. Observe that the H-action on S is also birational to the H-action on $\mathbb{P}^2$ via permutation of coordinates and the standard $\mathbb{G}_m^2$ torus action on $\mathbb{P}^2$. Thus, the H-action on $S \times \mathbb{P}^1$ is birational to the corresponding action on $\mathbb{P}^2 \times \mathbb{P}^1$. This action is birational to an action on a rank-1 vector bundle over $\mathbb{P}^2$. By the no-name lemma, this is birational to an action on $\mathbb{A}^1 \times \mathbb{P}^2$, with trivial action on the first factor and generically free action on the second factor, which is linearizable. ◻

Proposition 6.7. *Let X be the cubic threefold with $3\mathsf{D}_4$-singularities and $G \subseteq \mathrm{Aut}(X)$. Then the G-action on X is not (stably) linearizable if and only if G contains an element conjugate to $\sigma_{(123)}$.*

Proof. When G contains an element conjugate to $\sigma_{(123)}$, the G-action on X has an **(H1)**-obstruction and is not stably linearizable, by Proposition 6.5. When G does not contain an element conjugate to $\sigma_{(123)}$ and G does not fix any singular point, then

$$G \subset H = \langle \tau_{ab}, \eta\sigma_{(123)}, \sigma_{(45)}\sigma_{(12)} \rangle \simeq \mathbb{G}_m^2(k) \rtimes \mathfrak{S}_3.$$

From Proposition 6.6, we see that the G-action on X is linearizable. ◻

7. Four singular points

With our assumptions, the possible combinations of singularities, with specializations, are:

$$\begin{array}{ccccc} 2\mathsf{A}_2 + 2\mathsf{A}_1 & \longrightarrow & 2\mathsf{A}_3 + 2\mathsf{A}_1 & \longrightarrow & 2\mathsf{D}_4 + 2\mathsf{A}_1 \\ \big\downarrow & & & & \\ 4\mathsf{A}_2 & & & & \end{array}$$

In all cases, the singularities are in linear general position - indeed, if four singularities are contained in a plane, they must be $4\mathsf{A}_1$, treated in [6]. We can thus assume that the singularities are at

$$p_1 = [1:0:0:0:0], \quad p_2 = [0:1:0:0:0],$$
$$p_3 = [0:0:0:1:0], \quad p_4 = [0:0:0:1:0].$$

Then X is given by

$$
\begin{aligned}
(7.1)\quad & t_1x_1x_2x_3+t_2x_1x_2x_4+t_4x_1x_3x_4+t_8x_2x_3x_4+\\
&+t_{15}x_5^3+x_5^2(t_7x_1+t_{11}x_2+t_{13}x_3+t_{14}x_4)+\\
&+x_5(t_3x_1x_2+t_5x_1x_3+t_6x_1x_4+t_9x_2x_3+t_{10}x_2x_4+t_{12}x_3x_4)=0,
\end{aligned}
$$

for some $t_1,\dots,t_{15}\in k$. Up to a change of variables by torus actions, we may assume that $t_i=0$ or 1, for $i=1,2,4,8$. In the remaining of this section, we assume p_1 and p_2 are of the same singularity type and p_3 and p_4 are of the same type. Then there is an exact sequence

$$0\to H\to \mathrm{Aut}(X)\xrightarrow{\rho}\mathfrak{S}_4,$$

and, except for $4\mathsf{A}_2$, the image of ρ is at most $C_2^2=\langle(1,2),(3,4)\rangle$.

Singularity Type $2\mathsf{A}_2+2\mathsf{A}_1$. Assume that p_1, p_2 are A_2-points.

Proposition 7.1. *If X is a cubic threefold with $2\mathsf{A}_2+2\mathsf{A}_1$-singularities then:*

- *Up to isomorphism, X can be given by*

$$
\begin{aligned}
(7.2)\quad & x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4+x_5^3+\\
&+x_5^2\Big(a_1x_3+a_2x_4-\frac{1}{4}a_3^2(x_1+x_2)\Big)+\\
&+x_5\big(a_3(x_1x_3+x_2x_4)+a_4x_3x_4\big)=0,
\end{aligned}
$$

  for general $a_1,a_2,a_3,a_4\in k$.
- *If $\mathrm{Aut}(X)$ does not fix any singular point, then one of the following holds*

$$\mathrm{Aut}(X)=\begin{cases}\langle\sigma_{(12)(34)},\sigma_{(12)}\rangle\simeq C_2^2, & \text{when } a_1=a_2,\ a_3=0,\\ \langle\sigma_{(12)(34)}\rangle\simeq C_2, & \text{when } a_1=a_2,\ a_3\neq 0,\end{cases}$$

  where

$$
\begin{aligned}
\sigma_{(12)(34)}&:(\mathbf{x})\mapsto(x_2,x_1,x_4,x_3,x_5),\\
\sigma_{(12)}&:(\mathbf{x})\mapsto(x_2,x_1,x_3,x_4,x_5).
\end{aligned}
$$

Proof. Since p_1 and p_2 are A_2-singularities, the quadratic terms after x_1, respectively x_2, define two quaternary quadratic forms of rank 3. This gives two nonlinear conditions on the parameters in (7.1):

$$
\begin{aligned}
(7.3)\quad & t_1^2t_6^2+4t_1t_2t_4t_7-2t_1t_2t_5t_6-2t_1t_3t_4t_6+t_2^2t_5^2-2t_2t_3t_4t_5+t_3^2t_4^2=0,\\
& t_1^2t_{10}^2+4t_1t_2t_8t_{11}-2t_1t_2t_9t_{10}-2t_1t_3t_8t_{10}+t_2^2t_9^2-2t_2t_3t_8t_9+t_3^2t_8^2=0.
\end{aligned}
$$

When $t_1 = t_2 = 0$, X has nonisolated singularities. Thus, up to a change of variables, one may assume that

$$t_1 = 1, \quad t_3 = t_5 = t_9 = 0,$$

which reduces (7.3) to

$$4t_2t_4t_7 + t_6^2 = 4t_2t_8t_{11} + t_{10}^2 = 0. \tag{7.4}$$

It follows that $t_2 \neq 0$ since otherwise $t_2 = t_6 = t_{10} = 0$ and X has nonisolated singularities. Similarly, one may check $t_4, t_8 \neq 0$, since otherwise it introduces A_3-singularities. Hence, $t_1 = t_2 = t_4 = t_8 = 1$. Up to a change of variables, we may assume $t_6 = 0$, simplifying (7.4) as $t_7 = 4t_{11} + t_{10}^2 = 0$. One can also check $t_{15} \neq 0$, since otherwise X has $2\mathsf{A}_1 + 2\mathsf{A}_3$-singularities. Thus, we may put $t_{15} = 1$, by scaling x_5, and the equation of X is of the form (7.2).

Now assume that $\mathrm{Aut}(X)$ does not fix any singular point, i.e., there exists $\sigma \in \mathrm{Aut}(X)$ such that $\rho(\sigma) = (1,2)(3,4)$ and σ takes the form

$$\sigma : (\mathbf{x}) \mapsto (s_2x_2 + r_2x_5, s_1x_1 + r_1x_5, s_4x_4 + r_4x_5, s_3x_3 + r_3x_5, x_5)$$

for $s_1, \dots, s_4 \in k^\times$ and $r_1, \dots, r_4 \in k$. The fact that σ leaves X invariant leads to a system of equations in the parameters $s_1, \dots, s_4, r_1, \dots, r_4$. Solving the system, we find that such an element σ exists if and only if

$$a_1 = a_2, \quad s_1 = s_2 = s_3 = s_4 = 1, \quad r_1 = r_2 = r_3 = r_4 = 0.$$

Using the same method, we find that when $a_1 = a_2$, an element $\tau \in \mathrm{Aut}(X)$ with $\rho(\tau) = (1,2)$ exists if and only if $a_3 = 0$, and

$$\tau : (\mathbf{x}) \mapsto (x_2, x_1, x_3, x_4, x_5).$$

Moreover, any $h \in H$ fixing four singular points is trivial. $\square$

Proposition 7.2. *Let X be a very general cubic threefold with $2\mathsf{A}_2 + 2\mathsf{A}_1$-singularities, given by (7.2) with $a_1 = a_2$. Then the $\langle \sigma_{(12)(34)} \rangle$-action on X, specified in Proposition 7.1, is not stably linearizable.*

Proof. We use the notation from Proposition 7.1. Let $a = a_4^{-1/4}$. Under the change of variables

$$y_1 = ax_1, \quad y_2 = ax_2, \quad y_3 = \frac{1}{a^2}x_3, \quad x_4 = \frac{1}{a^2}x_4, \quad y_5 = x_5,$$

the equation (7.2) becomes

$$x_1x_2x_3 + x_1x_2x_4 + a^3x_1x_3x_4 + a^3x_2x_3x_4 + a_3ax_5(x_1x_3 + x_2x_4) + \\ + x_3x_4x_5 + x_5^2(a_2a^2(x_3 + x_4) - \frac{a_3^2}{4a}(x_1 + x_2)) + x_5^3. \tag{7.5}$$

For fixed $a_2, \lambda \in k$, we can consider the 1-parameter family of cubic $\mathcal{X} \to \mathbb{A}_a$ parameterized by a given by (7.5) with a_2 and $a_3 = \lambda a$. In particular, generic fiber of $\mathcal{X}$ is given by

$$x_1x_2x_3 + x_1x_2x_4 + a^3x_1x_3x_4 + a^3x_2x_3x_4 + a^2x_5(x_1x_3 + x_2x_4)+ \\ + x_3x_4x_5 + x_5^2(a_2a^2(x_3 + x_4) - \frac{a}{4}(x_1 + x_2)) + x_5^3.$$

The $\langle\sigma_{(12)(34)}\rangle$-action naturally extends to $\mathcal{X}$ and is not stably linearizable on the special fiber X_0 above $a = 0$: X_0 has $2\mathsf{D}_4+2\mathsf{A}_1$-singularities and has an **(H1)**-obstruction by Proposition 7.7.

Similarly as before, e.g., in Proposition 5.11, we apply specialization to the resolution of singularities of the generic fiber in the family $\mathcal{X}$ to conclude the $\langle\sigma_{(12)(34)}\rangle$-action on a very general member in the family $\mathcal{X}$, i.e., a very general cubic with $2\mathsf{A}_2 + 2\mathsf{A}_1$-singularities, is not stably linearizable.

□

Singularity Type $2\mathsf{A}_3 + 2\mathsf{A}_1$. Assume p_1, p_2 are A_3-points. Recall from Lemma 2.7 that a cubic threefold X with $2\mathsf{A}_3 + 2\mathsf{A}_1$-singularities can have $d(X) = 1$ or 2.

Lemma 7.3. *Let X be a cubic threefold with $2\mathsf{A}_3 + 2\mathsf{A}_1$-singularities and $d(X) = 1$. Then the $\mathrm{Aut}(X)$-action on X is linearizable.*

Proof. Let $G = \mathrm{Aut}(X)$. If $d(X) = 1$ there is a unique, necessarily G-invariant plane Π contained in X. There are two possibilities: either Π contains both A_3-points, or only one. In the first case, X contains an G-invariant line that is disjoint from the plane, namely the line between the two A_1-points. In the second case, the A_3-point contained in Π is fixed. In both cases, the $\mathrm{Aut}(X)$-action on X is linearizable. □

Thus we focus on the $d(X) = 2$ case.

Proposition 7.4. *Let X be a cubic with $2\mathsf{A}_3 + 2\mathsf{A}_1$-singularities and defect $d(X) = 2$. Then, up to isomorphism, X is given by*

(7.6)
$$x_1x_2(x_3 + x_4) + x_5^2(x_1 + x_2 + a_1x_3 + a_1x_4) + x_3x_4x_5 + a_2x_5^3 = 0,$$

for general $a_1, a_2 \in k$,

$$\mathrm{Aut}(X) = \begin{cases} \langle\sigma_{(12)}, \sigma_{(12)(34)}, \eta_1\rangle \simeq C_2 \times C_6, & \text{when } a_1 = a_2 = 0, \\ \langle\sigma_{(12)}, \sigma_{(12)(34)}\rangle \simeq C_2^2, & \text{otherwise,} \end{cases}$$

where

$$\sigma_{(12)}: (\mathbf{x}) \mapsto (x_2, x_1, x_3, x_4, x_5),$$
$$\sigma_{(12)(34)}: (\mathbf{x}) \mapsto (x_2, x_1, x_4, x_3, x_5),$$
$$\eta_1: (\mathbf{x}) \mapsto (\zeta_3 x_1, \zeta_3 x_2, \zeta_3^2 x_3, \zeta_3^2 x_4, x_5).$$

Proof. Following the proof of Proposition 7.1, we know that up to change of variables, the parameters in (7.1) satisfy

$$t_1 = t_2 = 1, \quad t_3 = t_5 = t_9 = 4t_4t_7 + t_6^2 = 4t_8t_{11} + t_{10}^2 = 0.$$

When the defect $d(X) = 2$, by Proposition 2.7, we know that X contains three planes, and two of them are spanned by

$$\Pi_1 = \langle p_1, p_3, p_4 \rangle = \{x_2 = x_5 = 0\}, \quad \Pi_2 = \langle p_2, p_3, p_4 \rangle = \{x_1 = x_5 = 0\}.$$

This implies $t_4 = t_8 = 0$, and thus $t_6 = t_{10} = 0$. Then up to a change of variables, we obtain the desired form (7.6).

Using the same method as in the proof of Proposition 7.1, one can find all possibilities of $\mathrm{Aut}(X)$ as stated in both cases. □

Proposition 7.5. *Let X be a very general cubic with* $2\mathsf{A}_3 + 2\mathsf{A}_1$-*singularities and $d(X) = 2$. Then the $\langle \sigma_{(12)(34)} \rangle$-action on X from Proposition 7.4, is not stably linearizable.*

Proof. By Proposition 7.4, we know that all such cubics are given by (7.6). Let $a = a_2^{-\frac{1}{12}}$. Under the change of variables

$$y_1 = \frac{1}{a}x_1, \quad y_2 = \frac{1}{a}x_2, \quad y_3 = a^2 x_3, \quad y_4 = a^2 x_4, \quad y_5 = a^4 x_5,$$

the equation (7.6) becomes

$$y_1y_2(y_3 + y_4) + y_3y_4y_5 + x_5^2(a^9(x_1 + x_2) + a_1a^6(x_3 + x_4)). \tag{7.7}$$

For any fixed $\lambda \in k$, we may consider all the cubic threefolds given by (7.7) with $a_1a^6 = \lambda$ as a 1-parameter family parameterized by a:

$$\mathcal{X} \to \mathbb{A}_a^1,$$

where the general fiber of $\mathcal{X}$ is a cubic threefold with $2\mathsf{A}_3 + 2\mathsf{A}_1$-singularities given by

$$y_1y_2(y_3 + y_4) + y_3y_4y_5 + x_5^2(a^9(y_1 + y_2) + \lambda(y_3 + y_4)) = 0.$$

The special fiber $X_0 = \mathcal{X}_0$ above $a = 0$ has $2\mathsf{D}_4 + 2\mathsf{A}_1$-singularities. The $\sigma_{(12)(34)}$-action on X_0 is not stably linearizable, see Proposition 7.7.

Applying specialization to a resolution of singularities of the generic fiber, we conclude that the $\langle \sigma_{(12)(34)} \rangle$-action on a very general member in the family $\mathcal{X}$ is not stably linearizable. □

Singularity Type $2\mathsf{D}_4 + 2\mathsf{A}_1$**.** Assume that p_1, p_2 are D_4-points.

Proposition 7.6. *Let X be a cubic with $2\mathsf{D}_4+2\mathsf{A}_1$-singularities. Then, up to isomorphism, X is given by*

$$x_1x_2x_3 + x_1x_2x_4 + a_1x_3x_4x_5 + x_5^2(a_2x_3 + a_2x_4) + x_5^3 = 0, \tag{7.8}$$

for general $a_1, a_2 \in k$, and

$$\mathrm{Aut}(X) = \begin{cases} \langle \tau_a, \eta_2, \sigma_{(12)}, \sigma_{(12)(34)} \rangle \simeq (\mathbb{G}_m(k) \times C_2) \rtimes C_2^2, & \textit{when } a_2 = 0, \\ \langle \tau_a, \sigma_{(12)}, \sigma_{(12)(34)} \rangle \simeq \mathbb{G}_m(k) \rtimes C_2^2, & \textit{otherwise,} \end{cases}$$

where

$$\begin{aligned} \tau_a :&(\mathbf{x}) \mapsto (ax_1, a^{-1}x_2, x_3, x_4, x_5), \quad a \in k^\times, \\ \eta_2 :&(\mathbf{x}) \mapsto (x_1, -x_2, -x_3, -x_4, x_5), \\ \sigma_{(12)} :&(\mathbf{x}) \mapsto (x_2, x_1, x_3, x_4, x_5), \\ \sigma_{(12)(34)} :&(\mathbf{x}) \mapsto (x_2, x_1, x_4, x_3, x_5). \end{aligned}$$

Proof. Existence of D_4-points p_1 and p_2 implies that the quadratic terms after x_1, respectively x_2, define two quadratic forms in 4 variables of rank 2. This imposes a system of nonlinear conditions on the parameters $t_1, \dots, t_{15}$. Solving the system via `magma`, and excluding the components of the solutions whose general members define a cubic with nonisolated singularities, we find the conditions on parameters:

$$\begin{aligned} &t_4 = t_8 = 0, \;\; t_1t_6 = t_2t_5, \;\; t_1t_7 = t_3t_5, \;\; t_1t_{10} = t_2t_9, \;\; t_1t_{11} = t_3t_9, \\ &t_2t_7 = t_3t_6, \;\; t_2t_{11} = t_3t_{10}, \;\; t_5t_{10} = t_6t_9, \;\; t_5t_{11} = t_7t_9, \;\; t_6t_{11} = t_7t_{10}. \end{aligned} \tag{7.9}$$

Up to a change of variables, we may assume $t_1 = 1, t_3 = t_5 = t_9 = 0$, reducing (7.9) to $t_6 = t_7 = t_{10} = t_{11} = 0$. One may check $t_2 \neq 0$, since otherwise X has nonisolated singularities, and X is of the form

$$x_1x_2x_3 + x_1x_2x_4 + t_{15}x_5^3 + x_5^2(t_{13}x_3 + t_{14}x_4) + t_{12}x_3x_4x_5 = 0.$$

Up to a change of variables, we may assume that $t_{13} = t_{14}$ and $t_{15} = 1$.

To find $\mathrm{Aut}(X)$, we first observe that $\langle \sigma_{(12)(34)}, \sigma_{(12)} \rangle \subset \mathrm{Aut}(X)$ as specified in the assertion. So it suffices to classify $g \in \mathrm{Aut}(X)$ fixing all four singular points. Such elements take the form

$$g : (\mathbf{x}) \mapsto (s_1x_1 + r_1x_5, s_2x_2 + r_2x_5, s_3x_3 + r_3x_5, s_4x_4 + r_4x_5x_5),$$

for some $s_1, \dots, s_4 \in k^\times$ and $r_1, \dots, r_4 \in k$. Let f be the equation (7.8). As before, $g^*(f) = s_1s_2s_3f$ imposes a system of equations on the parameters. Solving this system leads to the assertions about $\mathrm{Aut}(X)$. $\square$

Cohomology. From Proposition 2.10, we know that $\mathrm{Cl}(X)$ is generated by the five planes in X. Using (7.8), one finds their equations:

$$\Pi_1 = \{x_1 = x_5 = 0\}, \quad \Pi_2 = \{x_2 = x_5 = 0\},$$
$$\Pi_3 = \{x_3 + x_4 = x_5 + \sqrt{a_1}x_3 = 0\},$$
$$\Pi_4 = \{x_3 + x_4 = x_5 = 0\}, \quad \Pi_5 = \{x_3 + x_4 = x_5 - \sqrt{a_1}x_3 = 0\}.$$

The class group $\mathrm{Cl}(X)$ is generated by $\Pi_1, \ldots, \Pi_5$, with relation

$$\Pi_1 + \Pi_2 = \Pi_3 + \Pi_5.$$

The involution $\sigma_{(12)(34)}\sigma_{(12)}$ and η_2 both switch Π_3 and Π_5 and leave other planes invariant, while $\sigma_{(12)}$ switches Π_1 and Π_2 and leaves other planes invariant.

Proposition 7.7. *With notation as above, one has*

$$\mathrm{H}^1(\langle \sigma_{(12)(34)} \rangle, \mathrm{Pic}(\tilde{X})) = \mathbb{Z}/2.$$

Proof. Choose a basis of $\mathrm{Cl}(X)$ consisting of the classes

$$\Pi_3, \quad \Pi_1 + \Pi_2 - \Pi_3, \quad \Pi_4, \quad \Pi_2 - \Pi_3.$$

The involution $\sigma_{(12)(34)}$ switches the first two elements, fixes the third one, and acts on $\Pi_2 - \Pi_3$ via (-1) multiplication. This implies

$$\mathrm{H}^1(\langle \sigma_{(12)(34)} \rangle, \mathrm{Cl}(X)) = \mathbb{Z}/2.$$

Since $\sigma_{(12)(34)}$ does not fix any singular points, it does not fix any class of exceptional divisors E_i of the resolution of singularities. In particular,

$$\mathrm{H}^2(\langle \sigma_{(12)(34)} \rangle, \oplus_i E_i) = 0.$$

By Lemma 4.2, we conclude

$$\mathrm{H}^1(\langle \sigma_{(12)(34)} \rangle, \mathrm{Pic}(\tilde{X})) = \mathrm{H}^1(\langle \sigma_{(12)(34)} \rangle, \mathrm{Cl}(X)) = \mathbb{Z}/2.$$

□

Linearizability.

Corollary 7.8. *Let X be a cubic threefold with $2\mathsf{D}_4 + 2\mathsf{A}_1$-singularities and $G \subseteq \mathrm{Aut}(X)$. The G-action on X is not (stably) linearizable if and only if G contains an element conjugate to $\sigma_{(12)(34)}$.*

Proof. If G contains an element conjugate to $\sigma_{(12)(34)}$. By Proposition 7.7, we know the G-action on X has an **(H1)**-obstruction and is not stably linearizable. Conversely, assume G does not contain such an element and G does not fix any singular point. From the classification in Proposition 7.6, we are in the case when $a_2 = 0$ and up to conjugation, $G \subseteq G'$ where G' is the group generated by $\tau_a, \sigma_{(12)}$ and

$\eta_2\sigma_{(12)(34)}$. One can then check that G' leaves invariant the plane Π_3 and the line $\{x_1 = x_2 = x_5 = 0\} \subset X$ disjoint from Π_3. It follows that the G-action on X is linearizable. □

Singularity Type $4\mathsf{A}_2$. Assume that p_1, p_2, p_3, p_4 are A_2-points on X. We start with a classification of actions and normal forms.

Proposition 7.9. *Let X be a cubic threefold with $4\mathsf{A}_2$-singularities. Then, up to isomorphism, X is given by*

(7.10)
$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + x_5^3 + ax_5^2(x_1 + x_2 + x_3 + x_4) + \\ + x_5(r_1(x_1x_2 + x_3x_4) + r_2(x_1x_3 + x_2x_4) + r_3(x_1x_4 + x_2x_3)) = 0,$$

for general $r_1, r_2, r_3 \in k$, and

$$a = -(\frac{1}{4}r_1^2 - \frac{1}{2}r_1r_2 - \frac{1}{2}r_1r_3 + \frac{1}{4}r_2^2 - \frac{1}{2}r_2r_3 + \frac{1}{4}r_3^2),$$

with

$$\mathrm{Aut}(X) = \begin{cases} \langle \eta_3, \sigma_{(12)}, \sigma_{(1234)} \rangle \simeq C_3 \times \mathfrak{S}_4, & \textit{when } r_1 = r_2 = r_3 = 0, \\ \langle \sigma_{(12)}, \sigma_{(1234)} \rangle \simeq \mathfrak{S}_4, & \textit{when } r_1 = r_2 = r_3 \neq 0, \\ \langle \eta_3^2\sigma_{(234)}, \sigma_{(12)(34)} \rangle \simeq \mathfrak{A}_4, & \textit{when } r_1 = \zeta_3 r_2 = \zeta_3^2 r_3 \neq 0, \\ \langle \sigma_{(12)}, \sigma_{(12)(34)} \rangle \simeq \mathfrak{D}_4, & \textit{when } r_2 = r_3, \\ \langle \sigma_{(13)(24)}, \sigma_{(12)(34)} \rangle \simeq C_2^2, & \textit{otherwise,} \end{cases}$$

where

$$\begin{aligned} \eta_3 &: (\mathbf{x}) \mapsto (x_1, x_2, x_3, x_4, \zeta_3 x_5), \\ \sigma_{(12)} &: (\mathbf{x}) \mapsto (x_2, x_1, x_3, x_4, x_5), \\ \sigma_{(1234)} &: (\mathbf{x}) \mapsto (x_2, x_3, x_4, x_1, x_5), \\ \sigma_{(234)} &: (\mathbf{x}) \mapsto (x_1, x_3, x_4, x_2, x_5), \\ \sigma_{(12)(34)} &: (\mathbf{x}) \mapsto (x_2, x_1, x_4, x_3, x_5), \\ \sigma_{(13)(24)} &: (\mathbf{x}) \mapsto (x_3, x_4, x_1, x_2, x_5). \end{aligned}$$

Proof. Four A_2-points impose the system of equations

$$
\begin{aligned}
(7.11)\quad & t_1^2t_{10}^2+4t_1t_2t_8t_{11}-2t_1t_2t_9t_{10}-2t_1t_3t_8t_{10}+t_2^2t_9^2-2t_2t_3t_8t_9+t_3^2t_8^2\\
&=t_1^2t_{12}^2+4t_1t_4t_8t_{13}-2t_1t_4t_9t_{12}-2t_1t_5t_8t_{12}+t_4^2t_9^2-2t_4t_5t_8t_9+t_5^2t_8^2\\
&=t_2^2t_{12}^2+4t_2t_4t_8t_{14}-2t_2t_4t_{10}t_{12}-2t_2t_6t_8t_{12}+t_4^2t_{10}^2-2t_4t_6t_8t_{10}+t_6^2t_8^2\\
&=t_1^2t_6^2+4t_1t_2t_4t_7-2t_1t_2t_5t_6-2t_1t_3t_4t_6+t_2^2t_5^2-2t_2t_3t_4t_5+t_3^2t_4^2=0
\end{aligned}
$$

on parameters $t_1,\dots,t_{15}$ in (7.1). At least two of t_1,t_2,t_4,t_8 are nonzero, since otherwise X has $3\mathsf{A}_1+\mathsf{D}_4$-singularities. Up to a change of variables, we may assume $t_1=t_2=1$, $t_3=t_5=t_9=0$. If $t_4=0$, (7.11) implies $t_6=t_{12}=0$ and X has nonisolated singularities. Hence $t_4=1$ and the same argument shows $t_8=1$. Then we may assume $t_6=0$ reducing the system (7.11) to

$$t_7=t_{10}^2+4t_{11}=t_{12}^2+4t_{13}=(t_{10}-t_{12})^2+4t_{14}=0.$$

One may check that a general solution defines a cubic with $4\mathsf{A}_2$-points. After a change of variables we obtain (7.10); the automorphisms $\mathrm{Aut}(X)$ can be classified using an argument similar to that in Proposition 7.1. □

Proposition 7.10. *Let X be a very general cubic with $4\mathsf{A}_2$-singularities given by (7.10), with $r_2=r_3$. Then X is not $\langle\sigma_{(13)(24)}\rangle$-stably linearizable.*

Proof. When $r_2=r_3$, one may assume that $r_1=0$ up to isomorphisms. Under the change of variables

$$y_1=-r_2^{1/2}(x_1+x_2),\quad y_2=-r_2^{1/2}(x_3+x_4),\quad y_3=x_5,$$
$$y_4=-r_2^{-1/4}x_4,\quad y_5=-r_2^{-1/4}x_2,$$

equation (7.10) becomes

$$y_1y_2y_3+y_2y_5^2+y_1y_4^2+y_3^3-r_2^{-3/4}y_1y_2y_4-r_2^{-3/4}y_1y_2y_5=0. \tag{7.12}$$

Let $r=-r_2^{-3/4}$, one may view (7.12) as a family of cubic threefolds $\mathcal{X}\to\mathbb{A}^1_r$ parameterized by $r\in k$. The $\sigma_{(13)(24)}$-action extends to the family $\mathcal{X}$ and takes the form

$$\iota:(\mathbf{y})\mapsto(y_2,y_1,y_3,y_5,y_4).$$

The generic fiber of the family has $4\mathsf{A}_1$-singularities at

$$p_1=[1:0:0:0:0],\quad p_2=[1:0:0:0:-r],$$
$$p_3=[0:1:0:0:0],\quad p_4=[0:1:0:-r:0],$$

and the special fiber X_0 when $r = 0$ has $2\mathsf{A}_5$-singularities. The action of $\langle \sigma_{(13)(24)} \rangle$ on X_0 is not stably linearizable, by Proposition 5.12. To resolve the singularities in the generic fiber, one can first equivariantly blow up two disjoint sections passing through the singular points p_1 and p_3 respectively, and then blow up those passing through p_2 and p_3, respectively. The resulting family has smooth generic fiber and the special fiber above $a = 0$ has BG-rational singularities: it has $2\mathsf{A}_1$-singularities in the same $\langle \sigma_{(13)(24)} \rangle$-orbit. Applying specialization, we obtain the desired assertion. □

Burnside obstructions.

Proposition 7.11. *The following G-actions on the following cubic threefolds are nonlinearizable, for general values of parameters of the corresponding families:*

(1) $2\mathsf{A}_2 + 2\mathsf{A}_1$, *and* $G = \langle \sigma_{(12)}, \sigma_{(12)(34)} \rangle$, *from Proposition 7.1,*
(2) $2\mathsf{A}_3 + 2\mathsf{A}_1$, $d(X) = 2$, *and* $G = \langle \sigma_{(12)}, \sigma_{(12)(34)} \rangle$, *from Proposition 7.4,*
(3) $2\mathsf{D}_4 + 2\mathsf{A}_1$, *and* $G = \langle \sigma_{(12)}, \sigma_{(12)(34)} \rangle$, *from Proposition 7.6,*
(4) $4\mathsf{A}_2$, *and* $G = \langle \sigma_{(12)}, \sigma_{(12)(34)} \rangle$ *in the cases when* $r_2 = r_3$ *from Proposition 7.9,*
(5) $4\mathsf{A}_2$, *and* $G = \langle \eta, \sigma_{(234)}, \sigma_{(12)(34)} \rangle \simeq C_3 \times \mathfrak{A}_4$, *from Proposition 7.9.*

Proof. In Cases (1)–(4), we are in the situation of Proposition 5.16: the involution $\sigma_{(12)}$ gives rise to a Burnside symbol of the form

$$(\langle \sigma_{(12)} \rangle, Y \circlearrowright k(S), (1)) \in \mathrm{Burn}_3(G), \tag{7.13}$$

where $S \subset X$ is a cubic surface. The residual Y-action on S fixes a smooth cubic curve, for general values of parameters, so that

$$\mathrm{H}^1(Y, \mathrm{Pic}(\tilde{S})) = (\mathbb{Z}/2)^2,$$

by [4], i.e., the symbol is incompressible. Moreover, linear actions do not contribute such symbols.

In Case (5), we have an incompressible symbol

$$(C_3, \mathfrak{A}_4 \circlearrowright k(S'), (\zeta_3)),$$

where S' is the Cayley cubic surface (unique cubic surface with 4 nodes). The $\mathfrak{A}_4$-action on S' is birational to the linear $\mathfrak{A}_4$-action on $\mathbb{P}^2$. This symbol is incompressible, appears in the class $[X \circlearrowleft G]$ with multiplicity one, and distinguishes the given G-action on X from a linear action, as in [5, Remark 6.4]. □

8. Five Singular points

Let X be a cubic threefold with five singular points. Under our assumptions, we need to consider the following combinations of singularities:

$$2\mathsf{A}_2 + 3\mathsf{A}_1, \quad 2\mathsf{A}_3 + 3\mathsf{A}_1, \quad 3\mathsf{A}_2 + 2\mathsf{A}_1, \quad 5\mathsf{A}_2, \quad 2\mathsf{D}_4 + 3\mathsf{A}_1.$$

We adapt the argument in [6, §6], which handles $5\mathsf{A}_1$-singularities. First note that if the singularities are not in linearly general position, then there is a distinguished G-fixed singular point, and the G-action on X is linearizable. Thus we can assume that the singular points of X are

$$p_1 = [1:0:0:0:0], \quad p_2 = [0:1:0:0:0], \quad p_3 = [0:0:1:0:0],$$
$$p_4 = [0:0:0:1:0], \quad p_5 = [0:0:0:0:1].$$

Automorphisms $\mathrm{Aut}(\mathbb{P}^4, 5)$ of $\mathbb{P}^4$ respecting these points fit into the exact sequence:

$$0 \to \mathbb{G}_m^4(k) \to \mathrm{Aut}(\mathbb{P}^4, 5) \xrightarrow{\rho} \mathfrak{S}_5 \to 0.$$

Lemma 8.1. *Let X be a cubic threefold with at most A_n-singularities and $\mathrm{Sing}(X) = \{p_1, \dots, p_5\}$. Let $G \subseteq \mathrm{Aut}(X)$ be a finite subgroup acting intransitively on $\mathrm{Sing}(X)$. Then the G-action on X is linearizable.*

Proof. If G fixes a singular points, it is linearizable via projection. It suffices to show linearizability when $\rho(G) = C_2 \times C_3$ or $C_2 \times \mathfrak{S}_3$, i.e., when G preserves the set $\{p_1, p_2\}$ and $\{p_3, p_4, p_5\}$. In these cases, we can find an element $\sigma \in G$ such that $\rho(\sigma) = (1,2)(3,4,5)$. By conjugation under the torus action, one may assume that σ permutes the coordinates $x_1, \dots, x_5$ as the cycle $(1,2)(3,4,5)$. The $\langle\sigma\rangle$-invariant cubic threefolds with only A_n-singularities are given by

$$(8.1) \quad x_3x_4x_5 + a(x_1x_2x_3 + x_1x_2x_4 + x_1x_2x_5) + b(x_1x_3x_4 + x_2x_3x_4 + x_1x_3x_5 + x_2x_3x_5 + x_1x_4x_5 + x_2x_4x_5) = 0,$$

for some $a, b \in k^\times$ (via `Magma`). Notice that if $a = 0$, the cubic has nonisolated singularities, and if $b = 0$, p_1 and p_2 are D_4-points. Based on the form of (8.1), one can see that the embedding $\mathbb{G}_m^4(k) \subset G$ is trivial, i.e., G does not contain diagonal elements.

If $\rho(G) = C_2 \times \mathfrak{S}_3$, then there exists $\tau \in G$ such that $\rho(\tau) = (3,4)$ and τ had order 2. Thus τ takes the form

$$\rho((x_1, x_2, x_3, x_4, x_5)) = (a_1x_1, a_2x_2, x_4, x_3, a_3x_5), \quad a_1, a_2, a_3 = \pm 1.$$

The only possibility for leaving (8.1) invariant is $a_1 = a_2 = a_3 = 1$.

Therefore, we conclude that $G = C_2 \times C_3$ or $C_2 \times \mathfrak{S}_3$, acting via corresponding permutations on the coordinates. Then G pointwise fixes the line $l \subset \mathbb{P}^4$ through $[1:1:0:0:0]$ and $[0:0:1:1:1]$. Let ι be the standard Cremona transformation on $\mathbb{P}^4$. Observe that ι birationally transforms X to a smooth quadric threefold Q, and $\iota(l) = l$. The intersection $l \cap Q$ contains G-fixed points, which implies the assertion. $\square$

Now we consider the cases when $G \subseteq \mathrm{Aut}(X)$ acts transitively on $\mathrm{Sing}(X)$: it follows that the only possible singularity type is $5\mathsf{A}_2$. There exists an element $(1,2,3,4,5) \in \rho(G)$. Up to conjugation by the torus action, we may assume that G contains an element permuting the coordinates $x_1, \ldots, x_5$ via the 5-cycle $(1,2,3,4,5) \in \mathfrak{S}_5$. Then X is given by

$$\begin{aligned} &x_1x_2x_3 + x_2x_3x_4 + x_1x_2x_5 + x_1x_4x_5 + x_3x_4x_5 \\ &\quad + a(x_1x_2x_4 + x_1x_3x_4 + x_1x_3x_5 + x_2x_3x_5 + x_2x_4x_5) = 0, \end{aligned} \tag{8.2}$$

for some $a \in k$. To force five A_2-singularities on X, one needs $a = \zeta_3$ or ζ_3^2. Then $\mathrm{Aut}(X) = \mathfrak{D}_5$, generated by permutations $(1,2,3,4,5)$ and $(4,3,2,1,5)$ on the coordinates. The image $\iota(X)$ under the standard Cremona transformation is a smooth quadric threefold. This $\mathfrak{D}_5$-action on a smooth quadric is linearizable, see [6, Lemma 6.2].

We summarize:

Corollary 8.2. *Let X be a cubic threefold with singularities of type*

$$2\mathsf{A}_2 + 3\mathsf{A}_1, \quad 2\mathsf{A}_3 + 3\mathsf{A}_1, \quad 3\mathsf{A}_2 + 2\mathsf{A}_1, \ \textit{or} \ \ 5\mathsf{A}_2.$$

Then the $\mathrm{Aut}(X)$*-action on X is linearizable.*

Singularity type $2\mathsf{D}_4 + 3\mathsf{A}_1$. Assume that p_1, p_2 are the D_4-points and p_3, p_4, p_5 the A_1-points. Following the proof of Lemma 8.1, if G does not fix a singular point, then X is given by

$$x_3x_4x_5 = x_1x_2(x_3 + x_4 + x_5),$$

with the $\mathfrak{S}_3$-action permuting x_3, x_4, x_5 and the infinite dihedral group $\mathfrak{D}_\infty = \mathbb{G}_m(k) \rtimes C_2$ acting on x_1, x_2. Applying the Cremona transformation ι, based at the 5 singular points, we obtain the smooth quadric

$$x_1x_2 = x_3x_4 + x_4x_5 + x_3x_5.$$

The G-action does not have fixed points, by our assumptions.

We can apply the Burnside formalism. Consider

$$G \simeq C_2^2 \times \mathfrak{S}_3 \subset \mathfrak{D}_\infty \times \mathfrak{S}_3,$$

where one generator of C_2^2 switches x_1, x_2 and the other multiplies x_1, x_2 by -1. The first gives rise to the symbol

$$(C_2, C_2 \times \mathfrak{S}_3 \circlearrowright k(S), (1)) \in \mathrm{Burn}_3(C_2^2),$$

with residual action on the quadric S, given by $x_1^2 = x_3x_4 + x_4x_5 + x_3x_5$. By [15, Example 9.2], this is an incompressible symbol, and the G-action on the quadric threefold is not linearizable.

9. Six singularities

The relevant cases are

$$2\mathsf{A}_2 + 4\mathsf{A}_1, \quad 2\mathsf{A}_3 + 4\mathsf{A}_1. \tag{9.1}$$

By Propositions 2.6 and 2.3, we may assume that the $4\mathsf{A}_1$-points are

$$p_1 = [1 : 1 : 1 : 0 : 0], \qquad p_2 = [-1 : 1 : 1 : 0 : 0],$$
$$p_3 = [1 : -1 : 1 : 0 : 0], \qquad p_4 = [1 : 1 : -1 : 0 : 0],$$

and the two A_2 or A_3-points are

$$p_5 = [0 : 0 : 0 : 1 : 0], \qquad p_6 := [0 : 0 : 0 : 0 : 1].$$

Proposition 9.1. *Let X be a cubic threefold with six singularities which are not all A_1-points. Then the $\mathrm{Aut}(X)$-action on X is linearizable.*

Proof. In both cases of (9.1), the four A_1-singularities are necessarily in a G-stable plane $\Pi \subset X$, and the two points with worse singularities define a G-stable line $l \subset X$, which is disjoint from Π. Arguing as in [5, Lemma 1.1], we obtain linearization. $\square$

Normal forms in these two cases in (9.1) are not needed for the study of linearizability. Nevertheless, we present them, for completeness.

Proposition 9.2. *Let X be a cubic threefold singular at $p_1, \dots, p_6$.*

- *If X has singularity type $2\mathsf{A}_2 + 4\mathsf{A}_1$, then X is given by*

$$\begin{aligned}(a_1x_4 + a_2x_5)(x_1^2 - x_3^2) + (a_3x_4 + a_4x_5)(x_2^2 - x_3^2) +& \\ + (a_5x_1 + a_6x_2 + a_7x_3)x_4x_5 = 0,&\end{aligned} \tag{9.2}$$

  for general $a_1, \dots, a_7 \in k$ satisfying

$$\begin{aligned}a_1^2a_6^2 + a_1a_3a_5^2 + a_1a_3a_6^2 - a_1a_3a_7^2 + a_3^2a_5^2 = 0,\\ a_2^2a_6^2 + a_2a_4a_5^2 + a_2a_4a_6^2 - a_2a_4a_7^2 + a_4^2a_5^2 = 0.\end{aligned} \tag{9.3}$$

- *If X has singularity type $2\mathsf{A}_3 + 4\mathsf{A}_1$, then X is given by*

$$(x_1^2 - x_3^2)x_4 + (x_2^2 - x_3^2)x_5 + x_3x_4x_5 = 0. \tag{9.4}$$

Proof. Singularities at $p_1, \dots, p_6$ impose linear conditions on the vector space $\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}^4_{\mathbb{P}}(3))$. In particular, every cubic threefold singular at $p_1, \dots, p_6$ is of the form (9.2), with general parameters $a_1, \dots, a_7$. Assume that p_5, p_6 are A_2-points. This implies that the quadratic terms locally at $x_4 = 1$ and $x_5 = 1$ define a degenerate quadratic form in four variables. This gives the nonlinear conditions (9.3). A general solution to this system of equations in $a_1, \dots, a_7$ defines a cubic with $2\mathsf{A}_2 + 4\mathsf{A}_1$-singularities via (9.2).

If X has $2\mathsf{A}_3 + 4\mathsf{A}_1$-singularities, from Proposition 2.6, we know that it contains the five planes spanned by points

$$\Pi_1 \supset \{p_2, p_3, p_5\}, \quad \Pi_2 \supset \{p_1, p_4, p_5\}, \quad \Pi_3 \supset \{p_1, p_2, p_3, p_4\},$$

$$\Pi_4 \supset \{p_1, p_2, p_6\}, \quad \Pi_5 \supset \{p_3, p_4, p_6\}.$$

This imposes further linear conditions $a_2 = a_3 = 0$. Substituting into (9.3), we also have $a_1 a_6 = a_4 a_5 = 0$. When a_1 or $a_4 = 0$, the cubic will be reducible, thus $a_5 = a_6 = 0$. Moreover, $a_7 \neq 0$, since otherwise X has nonisolated singularities. By scaling x_4 and x_5, we obtain the form (9.4). □

References

[1] D. Allcock. The moduli space of cubic threefolds. *J. Algebr. Geom.*, 12(2):201–223, 2003.

[2] D. Allcock, J. A. Carlson, and D. Toledo. *The moduli space of cubic threefolds as a ball quotient*, volume 985 of *Mem. Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 2011.

[3] V. I. Arnold, V. V. Goryunov, O. V. Lyashko, and V. A. Vasil'ev. *Singularity theory. I.* Springer-Verlag, Berlin, english edition, 1998. Translated from the 1988 Russian original by A. Iacob.

[4] F. Bogomolov and Yu. Prokhorov. On stable conjugacy of finite subgroups of the plane Cremona group, I. *Cent. Eur. J. Math.*, 11(12):2099–2105, 2013.

[5] I. Cheltsov, Yu. Tschinkel, and Zh. Zhang. Equivariant geometry of the Segre cubic and the Burkhardt quartic, 2023. `arXiv:2308.15271`.

[6] I. Cheltsov, Yu. Tschinkel, and Zh. Zhang. Equivariant geometry of singular cubic threefolds, 2024. `arXiv:2401.10974`.

[7] T. Ciurca, Sh. Tanimoto, and Yu. Tschinkel. Intermediate Jacobians and linearizability, 2024. `arXiv:2403.06047`.

[8] B. Hassett, A. Kresch, and Yu. Tschinkel. Symbols and equivariant birational geometry in small dimensions. In *Rationality of varieties*, volume 342 of *Progr. Math.*, pages 201–236. Birkhäuser/Springer, Cham, [2021] ©2021.

[9] B. Hassett and Yu. Tschinkel. Flops on holomorphic symplectic fourfolds and determinantal cubic hypersurfaces. *J. Inst. Math. Jussieu*, 9(1):125–153, 2010.

[10] B. Hassett and Yu. Tschinkel. Rationality of complete intersections of two quadrics over nonclosed fields. *Enseign. Math.*, 67(1-2):1–44, 2021. With an appendix by Jean-Louis Colliot-Thélène.
[11] B. Hassett and Yu. Tschinkel. Equivariant geometry of odd-dimensional complete intersections of two quadrics. *Pure Appl. Math. Q.*, 18(4):1555–1597, 2022.
[12] J. Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 1996.
[13] A. Kresch and Yu. Tschinkel. Equivariant birational types and Burnside volume. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 23(2):1013–1052, 2022.
[14] L. Marquand and S. Viktorova. The defect of a cubic threefold, 2023. `arXiv:2312.05118`.
[15] Yu. Tschinkel, Kaiqi Yang, and Zhijia Zhang. Combinatorial Burnside groups. *Res. Number Theory*, 8(2):Paper No. 33, 2022.
[16] Yu. Tschinkel, Kaiqi Yang, and Zhijia Zhang. Equivariant birational geometry of linear actions, 2023. `arXiv:2302.02296`.
[17] S. Viktorova. On the classification of singular cubic threefolds, 2023. `arXiv:2304.10452`.

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