ON FLOPS AND CANONICAL METRICS

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Abstract. This article is concerned with an observation for proving non-existence of canonical Kähler metrics. The idea is to use a rather explicit type of degeneration that applies in many situations. Namely, in a variation on a theme introduced by Ross–Thomas, we consider flops of the deformation to the normal cone. This yields a rather widely applicable notion of stability that is still completely explicit and readily computable, but with wider scope. We describe some applications, among them, a proof of one direction of the Calabi conjecture for asymptotically logarithmic del Pezzo surfaces.

1. Motivation and results

A variety is slope stable in the sense of Ross–Thomas if, roughly, it is K-stable with respect to degenerations to the normal cone of its subvarieties. This notion has been studied extensively by a number of authors and has yielded many non-existence results for canonical metrics on projective Kähler manifolds. Our main purpose in this article is to introduce a slight variation on this theme by considering a somewhat more involved notion of stability that involves additional flops on the degeneration to the normal cone but that is still geometric and computable, and is partly inspired by the work of Li–Xu. This gives many new non-existence results, and most notably allows us to resolve one direction of the Calabi conjecture for asymptotically logarithmic del Pezzo surfaces.

1.1. **Existence theorem for KEE metrics.** Kähler–Einstein edge (KEE) metrics are a natural generalization of Kähler–Einstein metrics: they are smooth metrics on the complement of a divisor, and have a conical singularity of angle $2\pi\beta$ transverse to that 'complex edge'. When $\beta = 1$, of course, this is just an ordinary Kähler–Einstein metric, that extends smoothly across the divisor. One can think of the metric as being 'bent' at an angle $2\pi\beta$ along the divisor. In the case of Riemann surfaces, KEE metrics are just the familiar constant curvature metrics with isolated cone singularities, that have been studied since the late 19th century, e.g., by Picard [25].

A basic question, whose origins trace back to Tian's 1994 lectures in the setting of nonpositive curvature [32], extended in Donaldson's 2009 lectures to the setting of anticanonical divisors on Fano manifolds [10], and further extended in our previous work [5] (see also the survey [27, §8]), is the following:

Problem 1.1. *Under what analytic conditions on the triple (X, D, β) does a KEE metric exist on the Kähler manifold X bent at an angle $2\pi\beta$ along the divisor $D \subset X$?*

This is partly motivated by Troyanov's solution in the Riemann surface case [36], and was settled by Jeffres–Mazzeo–Rubinstein (sufficient condition) and Darvas–Rubinstein (sufficient and necessary conditions) in higher dimensions for smooth D [17, 7]. These results give an analytic criterion characterizing existence, once the cohomological condition

$$-K_X - (1-\beta)D \text{ is } \mu \text{ times an ample class, for some } \mu \in \mathbb{R}, \tag{1.1}$$

is satisfied.

Remark 1.2. The analytic condition of [7, Theorem 9.1] is optimal and in particular improves on that of [17, Theorem 2] in the presence of automorphisms. An alternative proof of the sufficient condition of [17] was later also given by Guenancia–Paun [13], who treated the more general case of a simple normal crossing (snc) D, based on work of Berman et al. [3]; we refer to the survey [27] for a thorough discussion and many more references.

1.2. **Angle increasing to 2π.** The existence theorem of [17] coupled with Berman's work [2] showed that KEE metrics always exist for (X, D, β) when X is a Fano manifold admitting a smooth anticanonical divisor D and β is small [17, Corollary 1].

Following these results, considerable amount of work about KEE metrics in recent years has concerned the behavior of such metrics when the cone angle *increases towards* 2π, the two main issues being to show that when X is Fano admitting a smooth anticanonical divisor D, then:

(a) X admits KEE metrics with angle $2\pi\beta$ along a smooth anticanonical divisor for all angles $\beta < 1$ sufficiently close to 1 iff X is K-semistable;
(b) the limit of these KEE metrics as β tends to 1 is a smooth KE metric iff X is K-stable.

Problems (a)-(b) attracted a good deal of work building on combined efforts of many researchers in the past two decades, culminating in a solution [6, 34].

1.3. **Angle decreasing to 0.** In [5], we initiated a systematic study of the behavior in the other extreme when the *cone angle β goes to zero*. In partial analogy with the previous paragraph, the program initiated in [5] concerns:

(a) Determining all triples (X, D, β) satisfying (1.1) with *sufficiently small β*;
(b) Obtaining a condition equivalent to existence of KEE metrics for such triples;
(c) Understanding the limit, when such exists, of these KEE metrics as β tends to zero.

This program is largely open. In [5] we established (a) in dimension two under a technical assumption that the pair (X, D) is strongly asymptotically log Fano (see Definition 1.3; this is satisfied, e.g., when D is smooth), and made some initial progress towards (b). One of our goals in the present article is to establish one direction of the equivalence in part (b) in dimension two.

To make the notion of 'sufficiently small β' more precise, we introduce some terminology. Consider a pair (X, D) where $D = \sum_{i=1}^m D_i$ is a snc divisor. Denote

$$\mathrm{Amp}(X, D) := \{\beta \in \mathbb{R}_+^m \,:\, -K_X - \sum_{i=1}^m (1-\beta_i) D_i \text{ is ample}\}. \tag{1.2}$$

Definition 1.3. [5, Definition 1.1] *We say (X, D) is asymptotically log Fano (ALF) if* $0 \in \mathrm{Amp}(X, D)$*, and strongly asymptotically log Fano if* $\mathrm{Amp}(X, D)$ *contains a punctured neighborhood of* 0 *in* $\mathbb{R}_+^m \setminus \{0\}$.

When $m = 1$ these two notions coincide. Understanding which pairs (X, D) admit a KEE metric with a small angle along D requires understanding the class of ALF varieties. Recall that by Kawamata–Shokurov's Basepoint-free Theorem if (X, D) is ALF, then $|a(K_X + D)|$ (for some $a \in \mathbb{N}$) is free from base points and gives a morphism

$$\eta\colon X \to Z,$$

so that Z is a point if and only if $D \sim -K_X$ [30, Theorem 2.1] (see also [5, Theorem 1.9]), since $\mathrm{Pic}(X)$ has no torsion [16, Proposition 2.1.2]. The following conjecture, posed in our earlier work, gives a rather complete picture concerning (b)–(c) when D is smooth:

Conjecture 1.4. [5, Conjecture 1.11] *Suppose that (X, D) is asymptotically log Fano manifold with D smooth and irreducible.*

(i) *If η is birational, there exist no KEE metrics for sufficiently small β.*

(ii) *If η is not birational, then there exist KEE metrics ω_β with angle $2\pi\beta$ along D for all sufficiently small $\beta > 0$. Moreover, as β tends to zero (X, D, ω_β) converges in an appropriate sense to a generalized KE metric ω_∞ on $X \setminus D$ that is Calabi–Yau along generic fibers of η.*

This conjecture suggests that the existence problem for KEE metrics in the small angle regime boils down to computing a single intersection number! Namely, checking whether

$$(K_X + D)^n = 0.$$

This is a rather far-reaching simplification as compared to checking the much harder condition of K-stability. Indeed, the easier direction of the Yau–Tian–Donaldson conjecture implies that a KEE metric exists only if the pair (X, D) is log K-stable [2]. However, even in dimension two, it is a very difficult problem to check (log) K-stability as it involves, in theory, computing the Futaki invariant of an infinite number of test configurations.

1.4. **Flop-slope stability and non-existence.** When $n = 2$, Conjecture 1.4 (i) amounts to:

Conjecture 1.5. [5, Conjecture 1.6] *Let S be a smooth surface, and let C be a smooth irreducible curve on S. Suppose that (S, C) is asymptotically log del Pezzo. Then S admits KEE metrics with angle β along C for all sufficiently small β only if $(K_S + C)^2 = 0$.*

Our main result is a verification of Conjecture 1.5.

Theorem 1.6. *Let S be a smooth surface, and let C be a smooth irreducible curve on S. Suppose that (S, C) is asymptotically log del Pezzo and $(K_S + C)^2 \neq 0$. Then S does not admit KEE metrics with angle β along C for all sufficiently small β. Moreover, this statement holds for all $\beta \in \mathrm{Amp}(M, D)$ for which* (5.3) *is negative.*

In other words, we give a completely elementary and verifiable criterion that is equivalent to log K-unstability in the small angle regime. The proof involves a modification of the notion of slope stability due to Ross and Thomas [26], where we additionally perform flops on the deformation to the normal cone. In Li–Xu [21] it was shown that the generalized Futaki invariant decreases under certain modifications and our construction is partly inspired by those general results, although we do not make use of them. This construction using flops occupies most of this article, and we believe it is of independent interest.

This *flop-slope* construction is essential to the proof of Theorem 1.6 since for asymptotically logarithmic del Pezzo surfaces the more traditional obstructions of Matsushima, Futaki, and Ross–Thomas [22, 11, 26] are not sufficient, as examples in this article and in [5] show. We expect the method developed in this article to yield many more new examples of non-existence in different settings and in higher dimensions.

We remark that the converse to Conjecture 1.5 is open: we refer to [27, §9] for a discussion of partial results.

1.5. **Organization.** In §2 we review some preliminaries: the intersection-theoretic formula for the generalized Futaki invariant, (log) slope stability, and also derive some related useful formulas for asymptotically log del Pezzo surfaces. In §3 we apply these formulas to prove Theorem 1.6 for the simplest subclass of asymptotically log del Pezzo surfaces: the Maeda class for which $0 \in \mathrm{Amp}(M, D)$. Section 4 is the heart of the article, and contains our modification of slope stability, which we call flop-slope stability. The main result here is Proposition 4.9 that gives a formula for the Futaki invariant for the flopped test configuration. Some technical intersection-theoretic result needed here is proved in Appendix A. The proof of Theorem 1.6 is then carried out in §5. In §6 we collect some further examples.

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2. Preliminaries

2.1. **Generalized Futaki invariant.** Let $\beta = (\beta_1, \ldots, \beta_m) \in (0,1]^m$ be a vector. Let X be a normal $\mathbb{Q}$-factorial variety (of complex dimension n), let $D = \sum_{i=1}^m D_i$ be a divisor (where the D_i are distinct $\mathbb{Q}$-Cartier prime Weil divisors) on X, and let L_β be an ample $\mathbb{R}$-divisor on X (that a priori may depend on β). Put

$$D_\beta = \sum_{i=1}^m (1-\beta_i) D_i.$$

Let $(\mathcal{X}, \mathcal{L}_\beta, \mathcal{D}, \beta)$ be a quadruple consisting of a normal $\mathbb{Q}$-factorial variety $\mathcal{X}$ of dimension $n+1$, equipped with a flat surjective map $p\colon \mathcal{X} \to \mathbb{P}^1$, $\mathbb{R}$-divisor $\mathcal{L}_\beta$ [1]; a divisor $\mathcal{D} = \sum_{i=1}^m \mathcal{D}_i$ (where the $\mathcal{D}_i$ are distinct $\mathbb{Q}$-Cartier prime Weil divisors) on $\mathcal{X}$. Suppose that all fibers of p except the fiber over $[0:1]$ (which we call the central fiber) are isomorphic to X, and the divisors $\mathcal{L}_\beta$ and $\mathcal{D}_i$ restricted to these fibers are L_β and D_i, respectively. Thus, $\mathrm{Supp}(\mathcal{D})$ does not contain components of the fibers of p (if it did, $\mathcal{D}_i$ restricted to different fibers would be different, but we assume the restriction is always the same, namely, D_i), and so in particular it does not contain components of the central fiber. The generalized Futaki invariant is

$$\mathrm{F}(\mathcal{X}, \mathcal{L}_\beta, \mathcal{D}, \beta) := n \frac{-(K_X + \sum_{i=1}^m (1-\beta_i) D_i).L_\beta^{n-1}}{L_\beta^n} \mathcal{L}_\beta^{n+1} + (n+1)\Big(K_{\mathcal{X}} - p^\star(K_{\mathbb{P}^1}) + \sum_{i=1}^m (1-\beta_i)\mathcal{D}_i\Big).\mathcal{L}_\beta^n. \tag{2.1}$$

Whenever the triple $(\mathcal{X}, \mathcal{L}_\beta, \mathcal{D})$ is a test configuration in the sense of Tian [33] and Donaldson [9], then $F(\mathcal{X}, \mathcal{L}_\beta, \mathcal{D})$ equals its Futaki invariant in the sense of Ding–Tian or Donaldson [8, 9, 37, 23, 2, 21, 35]. If

$$L_\beta \sim_{\mathbb{R}} -(K_X + \sum_{i=1}^m (1-\beta_i) D_i) \tag{2.2}$$

(so that $(X, \sum_{i=1}^m (1-\beta_i) D_i)$ is a log Fano variety), the formula for $\mathrm{F}(\mathcal{X}, \mathcal{L}_\beta, \mathcal{D}, \beta)$ simplifies to

$$\mathrm{F}(\mathcal{X}, \mathcal{L}_\beta, \mathcal{D}, \beta) = n\mathcal{L}_\beta^{n+1} + (n+1)\Big(K_{\mathcal{X}} - p^\star K_{\mathbb{P}^1} + \sum_{i=1}^m (1-\beta_i)\mathcal{D}_i\Big).\mathcal{L}_\beta^n. \tag{2.3}$$

We recall the following result:

Theorem 2.1. [2, Theorem 4.8] *Suppose that X is smooth, $D = \sum_{i=1}^m D_i$ is a simple normal crossing divisor, and* (2.2) *holds. Let $(\mathcal{X}, \mathcal{L}_\beta, \mathcal{D}, \beta)$ be a test configuration for (X, L_β, D_β). Assume that $\mathcal{L}_\beta$ is p-ample. If $\mathrm{F}(\mathcal{X}, \mathcal{L}_\beta, \mathcal{D}, \beta) < 0$, then (X, D, β) does not admit a KEE metric.*

The simplest possible case (beyond a product configuration) when we can effectively apply this theorem is when X is smooth and the triple $(\mathcal{X}, \mathcal{L}_\beta, \mathcal{D})$ is a very particular test configuration obtained via deformation to the normal cone of a smooth subvariety in X. This construction is originally due to Ross–Thomas [26]. We now turn to describe it.

[1] We do *not* assume $\mathcal{L}_\beta$ is p-ample in the definition of $\mathrm{F}(\mathcal{X}, \mathcal{L}_\beta, \mathcal{D}, \beta)$. An $\mathbb{R}$-Cartier divisor A on $\mathcal{X}$ is p-ample (resp., p-big) if $A + p^\star B$ is ample (resp., big) for some divisor B on $\mathbb{P}^1$.

2.2. **Slope stability.** Let X be a smooth variety, and let Z be a smooth subvariety in X. Consider the blow-up of $Z \times \{[0:1]\}$ in $X \times \mathbb{P}^1$. We denote the resulting space (of complex dimension $n+1$) by $\mathcal{X}$ and denote the blow-down map by π_Z. Denote the π_Z-exceptional divisor by E_Z. Let $p_{\mathbb{P}^1}\colon X \times \mathbb{P}^1 \to \mathbb{P}^1$ and $p_X\colon X \times \mathbb{P}^1 \to X$ denote the natural projections.

Put

$$p := p_{\mathbb{P}^1} \circ \pi_Z.$$

The morphism $p\colon \mathcal{X} \to \mathbb{P}^1$ is flat [14, Proposition 9.7]. Its fibers over every point that is different from $[0:1]$ are isomorphic to X. The fiber $\mathcal{X}_0$ over $[0:1] \in \mathbb{P}^1$ is the union $E_Z \cup X_0$, where X_0 is the proper transform of $X \times \{[0:1]\}$, and

$$E_Z = \mathbb{P}(\nu_Z \oplus \mathcal{O}_Z) \tag{2.4}$$

is a smooth ruled variety. Here ν_Z denotes the normal bundle of Z in X, and $\mathcal{O}_Z$ denotes the trivial line bundle over Z. Of course, $\nu_Z \oplus \mathcal{O}_Z$ is the normal bundle of $Z \times \{[0:1]\}$ in $X \times \mathbb{P}^1$. Note that X_0 is the blow-up of X at Z. Thus, if Z is a divisor in X, then X_0 is simply a copy of X.

Denote by π_0 the morphism $p_X \circ \pi_Z|_{X_0} : X_0 \to X$, which is just the blow-down map of Z in X. In fact, E_Z intersect X_0 exactly at the exceptional locus of π_0 (here we slightly abuse language, since when Z is a divisor, this locus is not exceptional, but is just a copy of Z, the proper transform of Z).

Let $\beta = (\beta_1, \ldots, \beta_m) \in (0,1]^m$ be a vector, and let L_β be an ample $\mathbb{R}$-divisor on X that may depend on the vector β. Put

$$\mathcal{L}_{\beta,c} := (p_X \circ \pi_Z)^\star L_\beta - cE_Z \tag{2.5}$$

for some $c > 0$. Recall the definition of the Seshadri constant of (X, Z) with respect to L_β,

$$\epsilon(X, Z, L_\beta) = \sup\big\{c > 0 \,:\, \pi_0^\star(L_\beta) - cE_Z|_{X_0} \text{ is ample}\big\}. \tag{2.6}$$

Thus, if $c \geq \epsilon(X, Z, L_\beta)$, then $\mathcal{L}_\beta$ is not p-ample. The following is a special case of [26, Lemma 4.1]. We give a simple direct proof for the reader's convenience. We make use of the following simple fact more than once in this article, so we record it here:

$$\text{if } C \text{ is a curve contained in the central fiber then } C.S_0 = -C.E_Z. \tag{2.7}$$

Indeed, since C is contained in a fiber of $p : \mathcal{X} \to \mathbb{P}^1$ and $S_0 \cup E_Z$ is such a fiber (the central fiber) then $C.(S_0 + E_Z) = 0$.

Lemma 2.2. *Suppose that $c \in (0, \epsilon(X, Z, L_\beta))$. Then $\mathcal{L}_{\beta,c}$ is p-ample.*

Proof. Since L_β is ample, by Kleiman's criterion there is a positive constant γ_0 depending only on L_β such that

$$L_\beta.C \geq \gamma_0$$

for every curve C in X. Similarly, there is a positive constant γ_1 depending on L_β and c alone such that

$$\big(\pi_0^\star(L_\beta) - cE_Z|_{X_0}\big).C \geq \gamma_1$$

for every curve $C \subset X_0$, because $c < \epsilon(X, Z, L_\beta)$.

Put

$$\gamma := \min\{c, \gamma_0, \gamma_1\}.$$

We claim that $\mathcal{L}_{\beta,c}.C \geq \gamma$ for every curve $C \subset \mathcal{X}$ such that $p(C)$ is a point. The latter implies p-ampleness of the divisor $\mathcal{L}_{\beta,c}$.

Let C be a curve in $\mathcal{X}$ such that $p(C)$ is a point (so that C lies in some fiber). If C is not in the central fiber $E_Z \cup X_0$, then

$$\mathcal{L}_{\beta,c}.C = L_\beta.p_X \circ \pi_Z(C) \geq \gamma_0 \geq \gamma.$$

If C is in the central fiber and is contracted by π_Z to a point, i.e., C is contained in a fiber of $E_Z \mapsto Z$ (a $\mathbb{P}^{n-1}$ bundle), then $\mathcal{L}_{\beta,c}.C = -cE_Z.C \geq c \geq \gamma$, since $-E_Z.C \geq 1$ in this case as $-E_Z$ restricts to the hyperplane bundle on each fiber. If $C \subset E_Z$, $C \not\subset X_0$ and C is not contracted by π_Z to a point, then using (2.7),

$$\mathcal{L}_{\beta,c}.C = L_\beta.p_X \circ \pi_Z(C) - cE_Z.C = L_\beta.p_X \circ \pi_Z(C) + cX_0.C \geq L_\beta.p_X \circ \pi_Z(C) \geq \gamma_0 \geq \gamma.$$

If $C \subset X_0$, then

$$\mathcal{L}_{\beta,c}.C = \big((p_X \circ \pi_Z)^\star(L_\beta) - cE_Z\big)|_{X_0}.C = \big(\pi_0^\star(L_\beta) - cE_Z|_{X_0}\big).\pi_Z(C) \geq \gamma_1 \geq \gamma,$$

concluding the proof. □

Let $D = \sum_{i=1}^m D_i$ be a simple normal crossing divisor on X, where the D_i are distinct smooth prime divisors on X. For every $c \in (0, \epsilon(X, Z, L_\beta))$, $(\mathcal{X}, \mathcal{L}_{\beta,c}, \mathcal{D}_\beta)$ is a test configuration for (X, L_β, D_β). We assume that $\mathcal{D}_i$ is the proper transform of $D_i \times \mathbb{P}^1$ in $\mathcal{X}$. Recall the following definition due to Ross–Thomas and Li–Sun.

Definition 2.3. *The triple (X, L_β, D_β) is slope unstable with respect to Z if $F(\mathcal{X}, \mathcal{L}_{\beta,c}, \mathcal{D}, \beta) < 0$ for some $c \in (0, \epsilon(X, Z, L_\beta))$.*

Note that according to (2.6) and Lemma 2.2, the assumption on c in Definition 2.3 guarantees that Theorem 2.1 is applicable.

Corollary 2.4. *If (X, L_β, D_β) is slope unstable with respect to Z, then (X, D, β) does not admit a KEE metric.*

The importance of this corollary is that the number $\mathrm{F}(\mathcal{X}, \mathcal{L}_{\beta,c}, \mathcal{D}, \beta)$ is readily computable for the test configuration described in this subsection (compared to a general test configuration).

Remark 2.5. In all cases we considered so far, if $F(\mathcal{X}, \mathcal{L}_{\beta,c}, \mathcal{D}, \beta) < 0$ for some $c \in (0, \epsilon(X, Z, L_\beta))$, then $F(\mathcal{X}, \mathcal{L}_{\beta,c}, \mathcal{D}, \beta) < 0$ for $c = \epsilon(X, Z, L_\beta)$.

In the next section, we compute $F(\mathcal{X}, \mathcal{L}_{\beta,c}, \mathcal{D}, \beta)$ in a particular situation.

2.3. **Slope stability for logarithmic surfaces.** Let us use the notation and assumptions of §2.2. Suppose, in addition, that D is a smooth curve in a smooth surface X, i.e., $m = 1, n = 2$, and $D = D_1, \mathcal{D} = \mathcal{D}_1$, and Z is a smooth curve in X. For transparency, we put $S = X$, $S_0 = X_0$, $p_X = p_S$, $C = D = D_1$, $\beta = \beta_1$, and $D_\beta = (1-\beta)C$. Then $\mathcal{X}$ is a threefold, and the fiber over $[0:1] \in \mathbb{P}^1$ is the union of two surfaces $E_Z \cup S_0$, where S_0 is the proper transform of the fiber of $p_{\mathbb{P}^1}$ over $[0:1]$. Since C is a curve, we have $S_0 \cong S$. Note that the exceptional divisor $E_Z \cong \mathbb{P}(\nu_Z \oplus \mathcal{O}_Z)$ is a smooth ruled surface, where ν_Z denotes the normal bundle of Z in S, and $\mathcal{O}_Z$ denotes the trivial line bundle over Z.

In the case when $L_\beta \sim_{\mathbb{R}} -K_S - (1-\beta)C$, there is an explicit formula for $F(\mathcal{X}, \mathcal{D}, \mathcal{L}_\beta, \beta)$. First, recall some intersection formulas.

Lemma 2.6. *One has,*

$$E_Z^3 = -\mathrm{deg}(N_{Z/\mathcal{X}}) = -Z^2, \tag{2.8}$$

and

$$(p_S^\star L_\beta)^3 = 0, \qquad ((p_S \circ \pi_Z)^\star L_\beta).E_Z^2 = -(p_S^\star L_\beta).Z = -L_\beta.Z. \tag{2.9}$$

Proof. The first equality in (2.8) follows from [12, p. 608] while the second equality follows from the fact that $N_{Z/\mathcal{X}}$ decomposes as $\mathcal{O}_Z(-1) \oplus \mathcal{O}_Z$. Since E_Z is the projectivization of $N_{Z/\mathcal{X}}$, the previous decomposition also implies (2.9) as $\pi_Z(E_Z) = Z$. □

Proposition 2.7. *Suppose that* (2.2) *holds. Then,*

$$F(\mathcal{X},\mathcal{L}_{\beta,c},\mathcal{D},\beta)=\begin{cases}\Big(6\beta c-3c^2\Big)L_\beta.Z+\Big(2c^3-3c^2\beta\Big)Z^2 & \text{if } Z=C,\\ \Big(6c-3c^2\Big)L_\beta.Z+\Big(2c^3-3c^2\Big)Z^2 & \text{if } Z\neq C.\end{cases}$$

Proof. First, using Lemma 2.6,

$$\begin{aligned}\mathcal{L}^3_{\beta,c}&=\Big((p_S\circ\pi_Z)^\star L_\beta-cE_Z\Big)^3\\ &=((p_S\circ\pi_Z)^\star L_\beta)^3-3c\Big((p_S\circ\pi_Z)^\star L_\beta\Big)^2.E_Z+3c^2\Big((p_S\circ\pi_Z)^\star L_\beta\Big).E_Z^2-c^3E_Z^3\\ &=(p_S^\star L_\beta)^3+3c^2\Big((p_S\circ\pi_Z)^\star L_\beta\Big).E_Z^2-c^3E_Z^3=3c^2\Big((p_S\circ\pi_Z)^\star L_\beta\Big).E_Z^2-c^3E_Z^3\\ &=-3c^2(p_S^\star L_\beta).Z-c^3E_Z^3=-3c^2L_\beta.Z+c^3Z^2.\end{aligned}$$

For the second term in (2.3), suppose first that $Z=C$ (this is only used in the second line in computing $\mathcal{D}$). Using Lemma 2.6 and the formula for the canonical bundle and a general divisor under a blow-up [12, p. 187, 476],

$$\begin{aligned}&\Big(K_\mathcal{X}-p^\star K_{\mathbb{P}^1}+(1-\beta)\mathcal{D}\Big).\mathcal{L}^2_{\beta,c}\\ &\quad=\Big((p_S\circ\pi_Z)^\star K_S+E_Z+(1-\beta)\pi_Z^\star(C\times\mathbb{P}^1)-(1-\beta)E_Z\Big).\Big((p_S\circ\pi_Z)^\star L_\beta-cE_Z\Big)^2\\ &\quad=\Big((p_S\circ\pi_Z)^\star K_S+(1-\beta)(p_S\circ\pi_Z)^\star C+\beta E_Z\Big).\Big(((p_S\circ\pi_Z)^\star L_\beta)^2-2cE_Z.(p_S\circ\pi_Z)^\star L_\beta+c^2E_Z^2\Big)\\ &\quad=c^2(p_S\circ\pi_Z)^\star K_S.E_Z^2+(1-\beta)c^2(p_S\circ\pi_Z)^\star C.E_Z^2-2\beta c(p_S\circ\pi_Z)^\star L_\beta.E_Z^2+\beta c^2E_Z^3\\ &\quad=-c^2p_S^\star K_S.Z-(1-\beta)c^2\pi_Z^\star C.Z+2\beta cL_\beta.Z-\beta c^2Z^2\\ &\quad=-c^2K_S.Z-(1-\beta)c^2C.Z+2\beta cL_\beta.Z-\beta c^2Z^2.\end{aligned}$$

If $Z\neq C$, then $\mathcal{D}=(1-\beta)\pi_Z^\star(C\times\mathbb{P}^1)$, so the previous calculation gives

$$\big(K_\mathcal{X}-p^\star K_{\mathbb{P}^1}+(1-\beta)\mathcal{D}\big).\mathcal{L}^2_{\beta,c}=-c^2K_S.Z-(1-\beta)c^2C.Z+2cL_\beta.Z-c^2Z^2.$$

Thus, if $Z=C$, we have

$$F(\mathcal{X},\mathcal{L}_{\beta,c},\mathcal{D},\beta)=2[-3c^2L_\beta.Z+c^3Z^2]+3[-c^2K_S.Z+2\beta cL_\beta.Z-c^2Z^2],$$

while if $Z\neq C$, we have

$$F(\mathcal{X},\mathcal{L}_{\beta,c},\mathcal{D},\beta)=2[-3c^2L_\beta.Z+c^3Z^2]+3[-c^2K_S.Z-(1-\beta)c^2C.Z+2cL_\beta.Z-c^2Z^2].$$

Plugging in (2.2) now yields the desired formulas. □

In the next section, we will show how to apply Proposition 2.7 to compute $F(\mathcal{X},\mathcal{D},\mathcal{L}_\beta,\beta)$ in some cases (cf. Li–Sun [20, Proposition 3.15, Example 3.16]). Before doing so, we illustrate with a simple example.

Example 2.8. Suppose that $S=\mathbb{F}_1$ and C is a smooth rational curve in $|E+F|$, where F is a fiber of the natural projection $S\to\mathbb{P}^1$, and E is the unique -1-curve in S. Then L_β is ample for every $\beta\in(0,1]$. The automorphism group of the pair (S,C) is reductive [5, Proposition 7.1] so the edge version of Matsushima's obstruction [5, Theorem 1.12] is not applicable. If $Z=C$ or $Z=E$, then $\epsilon(S,L_\beta,Z)=1+\beta$. In addition, if $Z=C$, Proposition 2.7 gives

$$F(\mathcal{X},\mathcal{L}_{\beta,c},\mathcal{D},\beta)=2(1+\beta)(\beta^2+2\beta-2)$$

for $c=1+\beta$, so $F(\mathcal{X},\mathcal{L}_{\beta,c},\mathcal{D},\beta)<0$ for $\beta<\sqrt{3}-1$. Similarly, if $Z=E$, Proposition 2.7 gives

$$F(\mathcal{X},\mathcal{L}_{\beta,c},\mathcal{D},\beta)=(1+\beta)(2-\beta^2-2\beta).$$

for $c = 1+\beta$, so $F(\mathcal{X}, \mathcal{L}_{\beta,c}, \mathcal{D}, \beta) < 0$ for all $\beta > \sqrt{3}-1$.

3. Maeda's class

Let C be a smooth curve on a smooth surface S. Suppose that (S, C) is asymptotically log del Pezzo. Put

$$L_\beta \sim_{\mathbb{R}} -K_S - (1-\beta)C,$$

where $\beta \in (0,1]$ is such that L_β is ample.

The following result proves in a unified manner than whenever $-K_S - C$ is ample, Conjecture 1.5 holds. Alternatively, this result also follows by combining [5, Proposition 7.1] with [20] and Example 2.8.

Proposition 3.1. *Suppose that $-K_S - C$ is ample. Then (S, C, β) does not admit a KEE metric for all sufficiently small β.*

Remark 3.2. By [5, Corollary 2.3] C is rational.

Proof. Pick any positive $\gamma < \epsilon(S, Z, -K_S - C)$. By definition,

$$-K_S - C \sim_{\mathbb{R}} \gamma Z + H$$

for some ample $\mathbb{R}$-divisor H. Letting $Z := C$ then

$$L_\beta \sim_{\mathbb{R}} -K_S - C + \beta C \sim_{\mathbb{R}} \Big(\gamma + \beta\Big) C + H,$$

which implies that $\epsilon(S, L_\beta, Z) \geq \gamma + \beta > \gamma$.

Pick some $c \in (0, \gamma]$. Let us use notation and assumptions of §2.3. Then $\mathcal{L}_{\beta,c}$ is p-ample by Lemma 2.2. By Remark 3.2, $L_\beta . C = -(K_S + (1-\beta)C).C = 2 + \beta C^2$. Therefore, using Proposition 2.7, with $c = \gamma$,

$$\begin{aligned}
\mathrm{F}(\mathcal{X}, \mathcal{L}_{\beta,c}, \mathcal{D}, \beta) &= -3\gamma^2 L_\beta.C + 2\gamma^3 C^2 + \beta(6\gamma L_\beta.C - 3\gamma^2 C^2) \\
&= -\gamma^2 L_\beta.C - 2\gamma^2(L_\beta - \gamma C).C + \beta(6\gamma L_\beta.C - 3\gamma^2 C^2) \\
&= -\gamma^2(2 + \beta C^2) - 2\gamma^2(L_\beta - \gamma C).C + \beta(6\gamma L_\beta.C - 3\gamma^2 C^2) \\
&< -2\gamma^2 + \beta(6\gamma L_\beta.C - 4\gamma^2 C^2).
\end{aligned}$$

so $\lim_{\beta\to 0^+} \mathrm{F}(\mathcal{X}, \mathcal{L}_{\beta,c}, \mathcal{D}, \beta) \leq -2\gamma^2 < 0$. Thus, Theorem 2.1 implies the desired result. □

Remark 3.3. One cannot drop the ampleness condition in Proposition 3.1. Indeed, if $-K_S - C$ is not ample, then it follows from the classification in [5] and Lemma 4.3 (i) below that $\epsilon(S, L_\beta, C) \leq \beta$ so the arguments used in the proof of Proposition 3.1 are no longer valid.

4. Flop-slope stability

We follow the notation and assumptions of §2.3. In addition, denote by $O_1, \ldots, O_r$, distinct points on the curve Z, and let

$$\pi_O \colon S' \to S$$

be the blow-up of the union of these points, whose exceptional curves are

$$C_1', \ldots, C_r' \subset S',$$

with $\pi_O(C_i') = O_i$. Denote by

$$C', Z' \subset S'$$

the π_O-proper transforms of the curves $C, Z \subset S$, respectively. Let

$$p_{S'} \colon S' \times \mathbb{P}^1 \to S', \quad p'_{\mathbb{P}^1} \colon S' \times \mathbb{P}^1 \to \mathbb{P}^1,$$

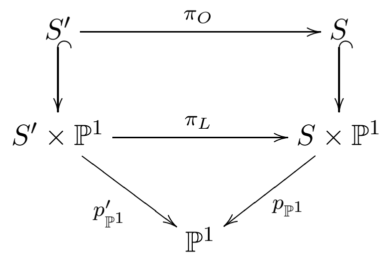
be the natural projections. Put

$$L_i := \{O_i\} \times \mathbb{P}^1 \subset S \times \mathbb{P}^1, \tag{4.1}$$

and let

$$\pi_L \colon S' \times \mathbb{P}^1 \to S \times \mathbb{P}^1$$

be the blow-up of the union of the smooth disjoint curves $L_1, \dots, L_r$. From now on, by abuse of notation, we identify S and S' with the fibers of $p_{\mathbb{P}^1}$ and $p'_{\mathbb{P}^1}$ over the point $[0:1] \in \mathbb{P}^1$, respectively. The blow-up $\pi_O \colon S' \to S$ is induced by the blow-up π_L. In sum, there exists a commutative diagram:



Let

$$\pi_{Z'} \colon \mathcal{X}' \to S' \times \mathbb{P}^1$$

be the blow-up of the curve $Z' \subset S' \subset S' \times \mathbb{P}^1$, and let

$$E_{Z'} \subset \mathcal{X}'$$

be the $\pi_{Z'}$-exceptional divisor. If Z is rational, then $E_{Z'} \cong \mathbb{F}_k$ where $k = |Z'^2|$ (to see this recall (2.4)). Denote by

$$S'_0 \subset \mathcal{X}'$$

the $\pi_{Z'}$-proper transform of the surface $S' \subset S' \times \mathbb{P}^1$. Put

$$p' := p'_{\mathbb{P}^1} \circ \pi_{Z'} : \mathcal{X}' \to \mathbb{P}^1.$$

Then,

$$S'_0 \cong S' \tag{4.2}$$

and

$$S'_0 \cup E_{Z'}$$

is the fiber of p' over the point $[0:1]$ (the "central fiber"). Denote by

$$C_1, \dots, C_r \subset \mathcal{X}'$$

the $\pi_{Z'}$-proper transform on of the curves $C'_1, \dots, C'_r \subset S' \subset S' \times \mathbb{P}^1$, respectively. Then $C_i \cong \mathbb{P}^1$.

Lemma 4.1. *The normal bundle of C_i in $\mathcal{X}'$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.*

Proof. Since C_i is rational, by Grothendieck's lemma [12, p. 516] $N_{C_i|\mathcal{X}'} = \mathcal{O}(a) \oplus \mathcal{O}(b)$. Thus,

$$0 \to T_{C_i} \to T_{\mathcal{X}'}|_{C_i} \to \mathcal{O}(a) \oplus \mathcal{O}(b) \to 0,$$

implies (considering the first Chern classes) that

$$a + b + 2 - 2g(C_i) = c_1(\mathcal{X}').C_i = -K_{\mathcal{X}'}.C_i. \tag{4.3}$$

Note that $C_i.E_{Z'} = 1$ since C'_i and Z' intersect transversally at one point downstairs (in $S' \subset S' \times \mathbb{P}^1$). In addition, $K_{\mathcal{X}'} = \pi^\star_{Z'} K_{S' \times \mathbb{P}^1} + E_{Z'}$. Thus,

$$K_{\mathcal{X}'}.C_i = \pi^\star_{Z'} K_{S' \times \mathbb{P}^1}.C_i + 1 = K_{S' \times \mathbb{P}^1}.C'_i + 1 = K_{S' \times}.C'_i + 1 = 2g(C'_i) - 2 - (C'_i)^2 + 1 = 0.$$

Thus, from (4.3) we conclude that $a + b = -2$. Next,

$$0 \to N_{C_i|S'_0} \to N_{C_i|\mathcal{X}'} \to N_{S'_0|\mathcal{X}'}|_{C_i} \to 0. \tag{4.4}$$

Observe that $N_{C_i|S_0'} = \mathcal{O}_{\mathbb{P}^1}(-1)$ since C_i is a -1-curve in S_0'. Thus, taking first Chern classes and using the previous paragraph, we must have $N_{S_0'|\mathcal{X}'}|_{C_i} = \mathcal{O}_{\mathbb{P}^1}(-1)$. The long exact sequence associated to (4.4) gives

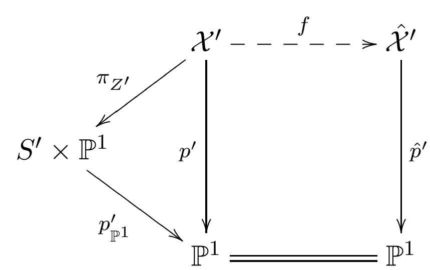
$$0 = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0,$$

implying that $a, b < 0$; thus, $a = b = -1$. □

Thus, as described in Appendix A, we can simultaneously flop the curves $C_1, \ldots, C_r \subset \mathcal{X}'$. Denote this composition of simple flops by $f\colon \mathcal{X}' \dashrightarrow \hat{\mathcal{X}}'$. Moreover, there exists a surjective morphism

$$\hat{p}'\colon \hat{\mathcal{X}}' \to \mathbb{P}^1$$

that makes the diagram



commute. Note that $\hat{p}'$ is flat [14, Proposition 9.7]. Let us show how to obtain $\hat{\mathcal{X}}'$ even more explicitly by blowing up the threefold $\mathcal{X}$. This will also show that $\hat{\mathcal{X}}'$ is projective.

Remark 4.2. Recall from §2.3 that we have a blow up $\pi_Z\colon \mathcal{X} \to S \times \mathbb{P}^1$ of the curve $Z \subset S \subset S \times \mathbb{P}^1$, and we denoted the π_Z-exceptional divisor by E_Z. If Z is rational, then $E_Z \cong \mathbb{F}_{|Z^2|}$.

Denote by

$$\tilde{L}_1, \ldots, \tilde{L}_r \subset \mathcal{X}$$

the π_Z-proper transforms of the curves $L_1, \ldots, L_r$ (defined in (4.1)). Then, each $\tilde{L}_i$ intersects E_Z in a unique point, because each curve L_i intersects the curve Z transversally by the point O_i. Then there exists a birational morphism

$$\pi_{\tilde{L}}\colon \hat{\mathcal{X}}' \to \mathcal{X}$$

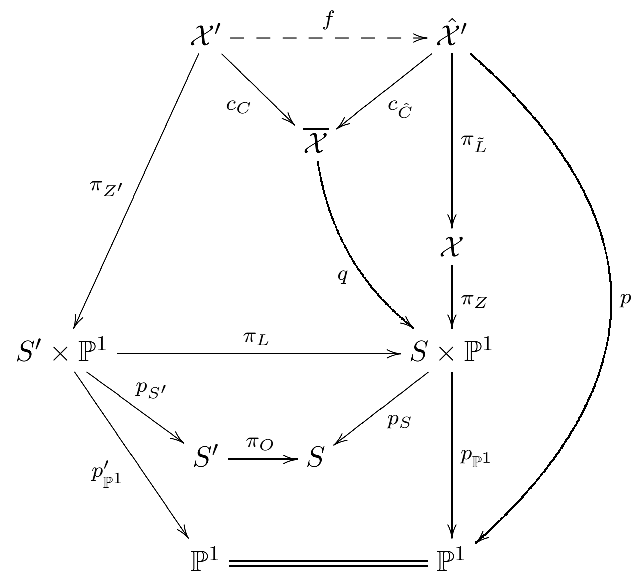
that is in fact the blow-up of the union of disjoint smooth curves $\tilde{L}_1 \cup \ldots \cup \tilde{L}_r$. In particular, the threefold $\hat{\mathcal{X}}'$ is projective.

Denote by

$$\hat{C}_1, \ldots, \hat{C}_r \subset \hat{\mathcal{X}}'$$

the $\pi_{\tilde{L}}$-proper transform of the fibers of the morphism $\pi_Z|_{E_Z}\colon E_Z \to Z$ over the points $O_1, \ldots, O_r$ in Z, respectively. Then there exists a commutative diagram,

(4.5)



such that q is the blow-up of the (singular curve) $Z + L_1 + \cdots + L_r$, c_C is the contraction of the curves $C_1, \ldots, C_r$ to the r singular points (ordinary double points) of the threefold $\bar{\mathcal{X}}$, $c_{\hat{C}}$ contracts the curves $\hat{C}_1, \ldots, \hat{C}_r$ on the threefold $\hat{\mathcal{X}}'$ to the same points.

Recall from §2.3 that S is equipped with an ample divisor L_β. Let L'_β be an ample $\mathbb{R}$-divisor on the surface S' such that

$$L'_\beta \sim_{\mathbb{R}} \pi_O^*(L_\beta) - \sum_{i=1}^{r} \delta_i C'_i \tag{4.6}$$

for some real numbers $\delta_1, \ldots, \delta_r$. Then all numbers $\delta_1, \ldots, \delta_r$ must be positive. Denote by $\epsilon(S', Z', L'_\beta)$ the Seshadri constant of (S', Z') with respect to L'_β. Denote by $\tau(S', Z', L'_\beta)$ the pseudoeffective threshold of (S', Z') with respect to L'_β, i.e. the number

$$\sup\{c > 0 \,:\, L'_\beta - cZ' \text{ is big}\}.$$

Let c be a positive real number.

Lemma 4.3. *(i) One has $\epsilon(S, Z, L_\beta) \geq \epsilon(S', Z', L'_\beta)$ and $\epsilon(S', Z', L'_\beta) \leq \delta_i$ for every i.*
(ii) If $c < \epsilon(S, Z, L_\beta)$ and $c \geq \delta_i$ for every i, then the divisor $L'_\beta - cZ'$ is big and, in particular, $\tau(S', Z', L'_\beta) > \epsilon(S', Z', L'_\beta)$.

Proof. The inequality $\epsilon(S, Z, L_\beta) \geq \epsilon(S', Z', L'_\beta)$ is obvious. The inequality $\epsilon(S', Z', L'_\beta) \leq \delta_i$ follows from $L'_\beta . C'_i = \delta_i$ and $Z'.C'_i = 1$. Suppose that $c < \epsilon(S, Z, L_\beta)$. Then $L_\beta - cZ$ is ample. Since

$$L'_\beta - cZ' \sim_{\mathbb{R}} \pi_O^*\big(L_\beta - cZ\big) + \sum_{i=1}^{r} (c - \delta_i) C'_i,$$

we see that the divisor $L'_\beta - cZ'$ is big provided that $c \geq \delta_i$ for every i. □

Let $\mathcal{D}'$ be the proper transform of the divisor $\mathcal{D}$ on $\mathcal{X}'$. Put

$$\mathcal{L}'_\beta := (p_{S'} \circ \pi_{Z'})^\star (L'_\beta) - cE_{Z'}. \tag{4.7}$$

If $c < \epsilon(S', Z', L'_\beta)$, then $\mathcal{L}'_\beta$ is p'-ample by Lemma 2.2.

Remark 4.4. If $\mathcal{L}'_\beta$ is p'-ample, then the triple $(\mathcal{X}', \mathcal{L}'_\beta, \mathcal{D}')$ is the test configuration obtained via deformation to the normal cone of Z' in S'.

Definition 4.5. *Denote by $R' \subset S' \times \mathbb{P}^1$, $R_{\mathcal{X}'} \subset \mathcal{X}'$, and $R_{\hat{\mathcal{X}}'} \subset \hat{\mathcal{X}}'$ the proper transforms of the surface $Z \times \mathbb{P}^1 \subset S \times \mathbb{P}^1$ with respect to the maps $\pi_L, \pi_L \circ \pi_{Z'}$, and $\pi_{\tilde{L}} \circ \pi_Z$, respectively.*

Lemma 4.6. *Suppose that $\epsilon(S', Z', L'_\beta) < c < \epsilon(S, Z, L_\beta)$ and $c \geq \delta_i$ for every i. Then $\mathcal{L}'_\beta$ is p'-big. Moreover, the curves $C_1, \ldots, C_r$ are the only curves in $\mathcal{X}'$ that are mapped by p' to points and have negative intersections with $\mathcal{L}'_\beta$.*

Proof. One has

$$\begin{aligned}
\mathcal{L}'_\beta &\sim_{\mathbb{R}} (p_{S'} \circ \pi_{Z'})^\star(L'_\beta) - cE_{Z'} \\
&\sim_{\mathbb{R}} (p_{S'} \circ \pi_{Z'})^\star(L'_\beta - cZ') + c(p_{S'} \circ \pi_{Z'})^\star(Z') - cE_{Z'} \\
&\sim_{\mathbb{R}} (p_{S'} \circ \pi_{Z'})^\star(L'_\beta - cZ) + c\pi_{Z'}^\star(R') - cE_{Z'} \\
&\sim_{\mathbb{R}} (p_{S'} \circ \pi_{Z'})^\star(L'_\beta - cZ) + cR_{\mathcal{X}'}.
\end{aligned}$$

Since $L'_\beta - cZ'$ is big by Lemma 4.3, we see that $(p_{S'} \circ \pi_{Z'})^\star L'_\beta - cE_{Z'}$ is $p_{\mathbb{P}^1} \circ \pi$-big.

Let Γ be an irreducible curve in $\mathcal{X}'$ such that $p(\Gamma)$ is the point $[0 : 1]$ and $\mathcal{L}'_\beta.\Gamma < 0$. Let us show that Γ is one of the curves $C_1, \ldots, C_r$. If $\pi_{Z'}(\Gamma)$ is a point, then

$$\mathcal{L}'_\beta.\Gamma = \Big((p_{S'} \circ \pi_{Z'})^\star(L'_\beta) - cE_{Z'}\Big).\Gamma = -cE_{Z'}.\Gamma > 0.$$

So, $\pi_{Z'}(\Gamma)$ is not a point. Thus, if $\Gamma \subset E_{Z'}$, then

$$0 > \mathcal{L}'_\beta.\Gamma = \Big((p_{S'} \circ \pi_{Z'})^\star(L'_\beta) - cE_{Z'}\Big).\Gamma \geq L'_\beta.Z' - cE_{Z'}.\Gamma = L'_\beta.Z' + cS'_0.\Gamma > cS'_0.\Gamma,$$

which implies that $S'_0.\Gamma < 0$. Thus, $\Gamma \subset S'_0$. Then,

$$\mathcal{L}'_\beta.\Gamma = \Big((p_{S'} \circ \pi_{Z'})^\star(L'_\beta) - cE_{Z'}\Big).\Gamma = (L'_\beta - cZ').\Gamma,$$

where we used that $S'_0 \cong S'$. On the other hand, (4.6) gives

$$L'_\beta - cZ' \sim_{\mathbb{R}} \pi_O^*\Big(L_\beta - cZ\Big) + \sum_{i=1}^r (c - \delta_i)C'_i,$$

where $L_\beta - cZ$ is ample on S. Since $c \geq \delta_i$ for every i by assumption, we see that the curve Γ must be one of the curves $C'_1, \ldots, C'_r$. ☐

A sufficient condition for the $\hat{p}'$-ampleness of the divisor $\hat{\mathcal{L}}'_\beta$ is given by

Lemma 4.7. *Suppose that $\epsilon(S', Z', L'_\beta) < c < \epsilon(S, Z, L_\beta)$ and $c \geq \delta_i$ for every i. Then $\hat{\mathcal{L}}'_\beta$ is $\hat{p}'$-ample.*

Proof. Since L'_β is ample, there is a constant $\gamma_0 > 0$ (that depend only on L'_β) such that

$$L'_\beta.\Omega' \geq \gamma_0$$

for every curve Ω' in S'. Similarly, there is a constant $\gamma_1 > 0$ (that depend on L'_β and c alone) such that

$$\Big(L_\beta - cZ\Big).\Omega \geq \gamma_1$$

for every curve $\Omega \subset S$, because $c < \epsilon(X, Z, L_\beta)$. Put

$$\gamma = \min\big\{c, \gamma_0, \gamma_1, \delta_1, \ldots, \delta_r, c - \delta_1, \ldots, c - \delta_r\big\}.$$

Let Γ be an irreducible curve in $\hat{\mathcal{X}}'$ that is contracted by $\hat{p}'$ to a point. To show that $\hat{\mathcal{L}}'_\beta$ is $\hat{p}'$-ample, it is enough to prove that $\hat{\mathcal{L}}_\beta.\Gamma \geq \gamma$.

Denote by $\hat{S}_0$ and $\hat{E}_Z$ the proper transforms of the surfaces S_0 and E_Z on the threefold $\hat{\mathcal{X}}'$, respectively. If $\Gamma \not\subset \hat{E}_Z \cup \hat{S}_0$, then

$$\hat{\mathcal{L}}'_\beta.\Gamma \geq \gamma_0 \geq \gamma.$$

Thus, we may assume that $\Gamma \subset \hat{E}_Z \cup \hat{S}_0$. One the other hand, it follows from (4.5) that

$$\hat{S}_0 \cong S$$

and $\hat{\mathcal{L}}'_\beta|_{\hat{S}_0} \sim_{\mathbb{R}} L_\beta - cZ$. Thus, if $\Gamma \subset \hat{S}_0$, then

$$\hat{\mathcal{L}}'_\beta.\Gamma \geq \gamma_1 \geq \gamma.$$

Hence, we may assume that $\Gamma \subset \hat{E}_Z$.

Denote by $\hat{F}_1, \ldots, \hat{F}_r$ the exceptional divisors of $\pi_{\tilde{L}}$. We may assume that $\pi_{\tilde{L}}(\hat{F}_i) = \tilde{L}_i$ for every i. Using (4.7) and (4.6) gives

$$\hat{\mathcal{L}}'_\beta \sim_{\mathbb{R}} (p_S \circ \pi_Z \circ \pi_{\tilde{L}})^\star(L_\beta) - \sum_{i=1}^r \delta_i \hat{F}_i - c\hat{E}_Z. \tag{4.8}$$

If $\pi_{\tilde{L}}(\Gamma)$ is a point $\tilde{L}_i \cap E_Z$, then

$$\hat{\mathcal{L}}'_\beta.\Gamma = \delta_i \geq \gamma.$$

If $\Gamma = \hat{C}_i$, then

$$\hat{\mathcal{L}}'_\beta.\Gamma = c - \delta_i \geq \gamma.$$

If Γ is contracted by $\pi_Z \circ \pi_{\tilde{L}}$ to a point in Z that is different from $O_1, \ldots, O_r$, then

$$\hat{\mathcal{L}}'_\beta.\Gamma = c \geq \gamma.$$

Thus, we may assume that $\pi_{\tilde{L}} \circ \pi_Z(\Gamma) = Z$. In particular, we see that

$$\Gamma \text{ is not contained in any divisor } \hat{F}_i. \tag{4.9}$$

Rewriting (4.8) and using the fact that $\hat{E}_Z \cup_i \hat{F}_i$ is the exceptional divisor of $\pi_Z \circ \pi_{\tilde{L}}$ gives (recall Definition 4.5),

$$\begin{aligned}
\hat{\mathcal{L}}'_\beta &\sim_{\mathbb{R}} (p_S \circ \pi_Z \circ \pi_{\tilde{L}})^\star(L_\beta - cZ) + c(p_S \circ \pi_Z \circ \pi_{\tilde{L}})^\star Z - \sum_{i=1}^r \delta_i \hat{F}_i - c\hat{E}_Z \\
&\sim_{\mathbb{R}} (p_S \circ \pi_Z \circ \pi_{\tilde{L}})^\star(L_\beta - cZ) + c(\pi_Z \circ \pi_{\tilde{L}})^\star(Z \times \mathbb{P}^1) - \sum_{i=1}^r \delta_i \hat{F}_i - c\hat{E}_Z \\
&\sim_{\mathbb{R}} (p_S \circ \pi_Z \circ \pi_{\tilde{L}})^\star(L_\beta - cZ) + cR_{\hat{\mathcal{X}}'} + \sum_{i=1}^r (c - \delta_i)\hat{F}_i.
\end{aligned}$$

Thus, if $\Gamma \not\subset R_{\hat{\mathcal{X}}'}$, then since Γ is a finite cover of Z, degree consideration give

$$\begin{aligned}
\hat{\mathcal{L}}'_\beta.\Gamma &= (p_S \circ \pi_Z \circ \pi_{\tilde{L}})^\star(L_\beta - cZ).\Gamma + cR_{\hat{\mathcal{X}}'}.\Gamma + \sum_{i=1}^r (c - \delta_i)\hat{F}_i.\Gamma \\
&\geq (L_\beta - cZ).Z + cR_{\hat{\mathcal{X}}'}.\Gamma + \sum_{i=1}^r (c - \delta_i)\hat{F}_i.\Gamma \\
&\geq (L_\beta - cZ).Z \geq \gamma_1 \geq \gamma,
\end{aligned}$$

where we also used (4.9). Thus, we may assume that $\Gamma \subset R_{\hat{\mathcal{X}}'}$. Then Γ is the proper transform of the curve $E_Z \cap R_{\mathcal{X}}$. Since the surfaces S_0 and $R_{\mathcal{X}}$ are disjoint, we have $S_0.\pi_{\tilde{L}}(\Gamma) = 0$. Then

$$\begin{aligned}
\hat{\mathcal{L}}'_\beta.\Gamma &= (p_S \circ \pi_{\tilde{L}} \circ \pi_Z)^\star(L_\beta).\Gamma - \sum_{i=1}^r \delta_i \hat{F}_i.\Gamma - c\hat{E}_Z.\Gamma \\
&= L_\beta.Z - \sum_{i=1}^r \delta_i \hat{F}_i.\Gamma - c\hat{E}_Z.\Gamma \\
&= L_\beta.Z - \sum_{i=1}^r \delta_i - c\hat{E}_Z.\Gamma \\
&= L_\beta.Z - \sum_{i=1}^r \delta_i - cE_Z.\pi_{\tilde{L}}(\Gamma) \\
&= L_\beta.Z - \sum_{i=1}^r \delta_i + cS_0.\pi_{\tilde{L}}(\Gamma) \\
&= L_\beta.Z - \sum_{i=1}^r \delta_i = L'_\beta.Z' \geq \gamma_0 \geq \gamma.
\end{aligned}$$

This completes the proof of the lemma. □

Let $\hat{\mathcal{D}}'$ be the proper transform of the divisor $\mathcal{D}$ on the threefold $\hat{\mathcal{X}}'$, and let $\hat{\mathcal{L}}'_\beta$ be the proper transform of (the class in $\mathrm{Pic}(\hat{\mathcal{X}}') \otimes \mathbb{R}$ of) the divisor $\mathcal{L}'_\beta$ on the threefold $\hat{\mathcal{X}}'$ (note that $\hat{\mathcal{L}}'_\beta$ is well-defined, since f is an isomorphism in codimension one).

Corollary 4.8. *Suppose that $\epsilon(S', Z', L'_\beta) < c < \epsilon(S, Z, L_\beta)$ and $c \geq \delta_i$ for every i. Then the quadruple $(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_\beta, \hat{\mathcal{D}}', \beta)$ is a test configuration.*

Next, we compute the generalized Futaki invariant of the flopped test configuration.

Proposition 4.9. *Suppose that*

$$L'_\beta \sim_{\mathbb{R}} -K_{S'} - (1-\beta)C'.$$

Then (recall (2.3)),

$$\begin{aligned}
\mathrm{F}(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_\beta, \hat{\mathcal{D}}', \beta) &= \mathrm{F}(\mathcal{X}', \mathcal{L}'_\beta, \mathcal{D}', \beta) - 2\sum_{i=1}^r (\mathcal{L}'_\beta.C_i)^3 - 3(1-\beta)\sum_{i=1}^r (\mathcal{L}'_\beta.C_i)^2(\mathcal{D}'.C_i) \\
&= 2(\mathcal{L}'_\beta)^3 + 3\Big(K_{\mathcal{X}'} - (p')^\star\big(K_{\mathbb{P}^1}\big) + (1-\beta)\mathcal{D}'\Big).(\mathcal{L}'_\beta)^2 \\
&\quad - 2\sum_{i=1}^r (\mathcal{L}'_\beta.C_i)^3 - 3(1-\beta)\sum_{i=1}^r (\mathcal{L}'_\beta.C_i)^2(\mathcal{D}'.C_i).
\end{aligned}$$

Proof. Recall from §2.1 that

$$\mathrm{F}(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_\beta, \hat{\mathcal{D}}', \beta) = 2(\hat{\mathcal{L}}'_\beta)^3 + 3\Big(K_{\hat{\mathcal{X}}'} - (\hat{p}')^\star\big(K_{\mathbb{P}^1}\big) + (1-\beta)\hat{\mathcal{D}}'\Big).(\hat{\mathcal{L}}'_\beta)^2.$$

The assertion now follows from (2.3) and Lemma A.3, together with the fact that, as in (4.3), $K_{\mathcal{X}'}.C'_i = 0$, while $p^\star K_{\mathbb{P}^1}.C'_i = 0$ since the C_i are contained in the central fiber of p. □

5. Proof of Theorem 1.6

According to [5, Theorem 1.4], all ALF surfaces (S, C) such that $-K_S - C$ is big satify either (i) $-K_S - C$ is ample, or (ii) S is obtained from an ALF surface (s, c) such that $-K_s - c$ is ample by blowing-up s at $r > 0$ distinct points on c and letting C denote the proper transform of c. Proposition 3.1 already established Theorem 1.6 in the case (i) holds. To complete the proof of Theorem 1.6 it remains to handle case (ii).

To that end, we switch back to the notation and assumptions of §4. We suppose that (S, C) is such that $-K_S - C$ is ample (hence ALF), and that (S', C') is still ALF, i.e.,

$$L'_\beta := -K_{S'} - (1-\beta)C'$$

is ample for all sufficiently small β. Note that $-K_{S'} - C' = \pi_O^\star(-K_S - C)$ is big being the pull-back under a birational map of an ample class. Thus, (S', C') satisfies the assumptions of Theorem 1.6. However, it is not possible to slope destabilize this latter pair in the same way as was done for (S, C) in §3. Indeed,

$$L'_\beta \sim_{\mathbb{R}} \pi_O^\star(-K_S - (1-\beta)C) - \beta \sum_{i=1}^{r} C'_i, \tag{5.1}$$

so that by Lemma 4.3 (i) (putting $\delta_i = \beta$ and $Z = C$), $\epsilon(S', Z', L'_\beta) \le \beta$, and in particular using Proposition 2.7 one checks that the generalized Futaki invariant $F(\mathcal{X}', \mathcal{L}'_{\beta,c}, \mathcal{D}, \beta)$ of the degeneration to the normal cone is positive for $c \in (0, \beta)$, and so (S', C') is not slope destabilized in this way. In what follows, we apply the results of §4 to destabilize our pair nevertheless.

Before proving Theorem 1.6, let us consider a model example.

Example 5.1. Suppose that $S = \mathbb{P}^2$ and C is a smooth conic. Then $\epsilon(S', L'_\beta, Z') = \beta$. Thus, if $c < \beta$, then $\mathcal{L}'_{\beta,c}$ is p'-ample. By Proposition 2.7, we have

$$\mathrm{F}(\mathcal{X}', \mathcal{L}'_{\beta,c}, \mathcal{D}', \beta) = \big(6\beta c - 3c^2\big)\big(2 + \beta(4-r)\big) + \big(2c^3 - 3c^2\beta\big)\big(4 - r\big).$$

In particular, this invariant is always positive for β sufficiently small (depending on r). On the other hand,

$$\tau(S', L'_\beta, Z') = \epsilon(S, L_\beta, Z) = \frac{1}{2} + \beta.$$

Thus, if $\beta < c < \frac{1}{2} + \beta$, then $\hat{\mathcal{L}}'_\beta$ is $\hat{p}'$-ample by Lemma 4.7. By Proposition 4.9, one has

$$\begin{aligned}\mathrm{F}(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_{\beta,c}, \hat{\mathcal{D}}', \beta) &= \mathrm{F}(\mathcal{X}', \mathcal{L}'_{\beta,c}, \mathcal{D}', \beta) + 2r(c-\beta)^3 = \\ &= \big(6\beta c - 3c^2\big)\big(2 + \beta(4-r)\big) + \big(2c^3 - 3c^2\beta\big)\big(4-r\big) + 2r(c-\beta)^3\end{aligned}$$

(see Appendix A). If we put $c = \frac{1}{2} + \beta$, then

$$\lim_{\beta \to 0^+} \mathrm{F}(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_\beta, \hat{\mathcal{D}}', \beta) = -\frac{1}{2}.$$

Recalling the discussion at the beginning of this section, Theorem 1.6 follows from the following result.

Proposition 5.2. *The triple (S', C', β) is flop-slope unstable for all sufficiently small β.*

Proof. Let $\epsilon(S, Z, -K_S - C)$ be the Seshadri constant of $Z \subset S$ with respect to $-K_S - C$. Pick any positive $\gamma < \epsilon(S, Z, -K_S - C)$. Then

$$-K_S - C \sim_{\mathbb{R}} \gamma Z + H$$

for some ample $\mathbb{R}$-divisor H. Then

$$L_\beta \sim_{\mathbb{R}} -K_S - C + \beta C \sim_{\mathbb{R}} (\gamma + \beta)C + H,$$

hence $\epsilon(S, L_\beta, Z) \geq \gamma + \beta$, so $\epsilon(S, L_\beta, Z) > \beta \geq \epsilon(S', L'_\beta, Z')$ by Lemma 4.3 (i). By taking β small, we may suppose that $\gamma > \beta$. Letting c be a real number such that

$$\epsilon(S', Z', L'_\beta) \leq \beta < c \leq \gamma < \epsilon(S, L_\beta, Z), \tag{5.2}$$

Lemma 4.7 implies that $\hat{\mathcal{L}}'_{\beta,c}$ is $\hat{p}'$-ample. By Proposition 4.9, we have

$$\mathrm{F}(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_{\beta,c}, \hat{\mathcal{D}}', \beta) = \mathrm{F}(\mathcal{X}', \mathcal{L}'_{\beta,c}, \mathcal{D}', \beta) - 2\sum_{i=1}^{r}(\mathcal{L}'_{\beta,c}.C_i)^3 - 3\sum_{i=1}^{r}(\mathcal{L}'_{\beta,c}.C_i)^2(\mathcal{D}'.C_i)$$

Moreover, by Proposition 2.7

$$\mathrm{F}(\mathcal{X}', \mathcal{L}'_{\beta,c}, \mathcal{D}', \beta) = (6\beta c - 3c^2)L'_\beta.C' + (2c^3 - 3c^2\beta)(C')^2.$$

Note that using (4.7) and (5.1),

$$\mathcal{L}'_{\beta,c}.C_i = \big((p_{S'} \circ \pi_{Z'})^\star L'_\beta - cE_{Z'}\big).C_i = L'_\beta.C'_i - cZ'.C'_i = \beta - c.$$

In addition, before the blow-up $\pi_{Z'}$, the intersection of $\mathcal{D} = C' \times \mathbb{P}^1$ and $S' \subset S' \times \mathbb{P}^1$ is precisely $Z' \subset S' \subset S' \times \mathbb{P}^1$ (this is precisely where we use that $Z' = C'$). Thus, after blowing-up Z', the surfaces $\mathcal{D}'$ and $S'_0 \cong S'$ (recall (4.2)) no longer intersect. Since C_i is contained in S'_0,

$$\mathcal{D}'.C_i = 0.$$

Combining these facts,

$$\mathrm{F}(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_{\beta,c}, \hat{\mathcal{D}}', \beta) = (6\beta c - 3c^2)L'_\beta.C' + (2c^3 - 3c^2\beta)(C')^2 + 2r(c-\beta)^3.$$

By Remark 3.2, C and hence also C' are rational, so $L'_\beta.C' = -(K_{S'} - (1-\beta)C').C' = 2 + \beta C'^2$. Thus, putting $c = \gamma$ and grouping most terms of order β together yields,

$$\begin{aligned}
\mathrm{F}(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_{\beta,\gamma}, \hat{\mathcal{D}}', \beta) &= -\gamma^2 L'_\beta.C' - 2\gamma^2\big(L'_\beta - \gamma C'\big).C' + 2r\gamma^3 \\
&\quad + \beta\big(6\gamma L'_\beta.C' - 3\gamma^2 C'^2 - 6r\gamma^2 + 6r\beta\gamma - 2r\beta^2\big) \\
&= -\gamma^2(2 + \beta C'^2) - 2\gamma^2\Big(\pi_O^\star(L_\beta - \gamma C) + (\gamma - \beta)\sum_{i=1}^{r} C'_i\Big).C' + 2r\gamma^3 \\
&\quad + \beta\big(6\gamma L'_\beta.C' - 3\gamma^2 C'^2 - 6r\gamma^2 + 6r\beta\gamma - 2r\beta^2\big) \\
&= -\gamma^2(2 + \beta C'^2) - 2\gamma^2\big(L_\beta - \gamma C\big).C - 2\gamma^2(\gamma - \beta)r + 2r\gamma^3 \\
&\quad + \beta\big(6\gamma L'_\beta.C' - 3\gamma^2 C'^2 - 6r\gamma^2 + 6r\beta\gamma - 2r\beta^2\big) \\
&= -2\gamma^2 - 2\gamma^2\big(L_\beta - \gamma C\big).C + \beta\big(6\gamma L'_\beta.C' - 4\gamma^2 C'^2 - 4r\gamma^2 + 6r\beta\gamma - 2r\beta^2\big),
\end{aligned} \tag{5.3}$$

so by (5.2),

$$\mathrm{F}(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_{\beta,c}, \hat{\mathcal{D}}', \beta) < -2\gamma^2 + \beta\big(6\gamma L'_\beta.C' - 4\gamma^2 C'^2 - 4r\gamma^2 + 6r\beta\gamma - 2r\beta^2\big). \tag{5.4}$$

implying that $\lim_{\beta \to 0^+} \mathrm{F}(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_{\beta,c}, \hat{\mathcal{D}}', \beta) \leq -2\gamma^2 < 0$. □

6. Further examples

We close by illustrating the advantage of using flop-slope stability over slope stability with two simple examples.

6.1. $\mathbb{F}_1$. According to Ross–Thomas [26, Examples 5.27,5.35] (cf. Panov–Ross [24, Example 3.8]), $\mathbb{F}_1$ is (for $\beta = 1$) slope destabilized by the -1-curve. More generally, by Li–Sun [20], the Futaki invariant of the slope test configuration of the triple $(\mathbb{F}_1, C, \beta)$ with C smooth in $|-K_{\mathbb{F}_1}|$ and with respect to the -1-curve equals $-3c^2\beta - 2c^3 + 3c^2 + 6c\beta$, which for $c = 2\beta$ (the Seshadri constant in this case), gives $4\beta^2(6-7\beta)$, showing that there exists no KEE metric when $\beta \in (6/7, 1]$. However, $\mathbb{F}_1$ is *not* destabilized by any fiber of its natural projection to $\mathbb{P}^1$ [24, Theorem 1.3]. We now show that $\mathbb{F}_1$ *is* destabilized by a fiber after one flop, and this even holds for $\beta \in (12/13, 1]$.

To show this, it is most convenient to carry over the notation and assumptions of §4. Thus, we let S be $\mathbb{P}^2$, C be a smooth cubic, and Z be a line. Then $S' = \mathbb{F}_1$ is the blow-up of S at a point $O_1 \in Z \cap C$, C' is an elliptic (anticanonical) curve, and Z' is a fiber of the natural projection $\mathbb{F}_1 \to \mathbb{P}^1$. In addition $\mathcal{D}$ is $C' \times \mathbb{P}^1$ and $\mathcal{D}'$ is its proper transform with respect to the blow-up of $Z' \subset S' \times \mathbb{P}^1$. Let $L'_\beta := -K_{S'} - (1-\beta)C' = \beta C'$, so $\epsilon(S', L'_\beta, Z') = \beta$. As $L'_\beta . Z' = 2\beta$ and $Z'^2 = 0$, Proposition 2.7 gives

$$\mathrm{F}(\mathcal{X}', \mathcal{L}'_{\beta,c}, \mathcal{D}', \beta) = 6c\beta(2-c). \tag{6.1}$$

Thus, if $c < \beta$, then $\mathrm{F}(\mathcal{X}', \mathcal{L}'_{\beta,c}, \mathcal{D}') > 0$. On the other hand, we have

$$\tau(S', L'_\beta, Z') = \epsilon(S, L_\beta, Z) = 3\beta.$$

Thus, it follows from Lemma 4.7 that $\hat{\mathcal{L}}_\beta$ is ample for every $c \in (\beta, 3\beta)$. By Proposition 4.9 and (6.1),

$$\begin{aligned} \mathrm{F}(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_{\beta,c}, \hat{\mathcal{D}}', \beta) &= \mathrm{F}(\mathcal{X}', \mathcal{L}'_{\beta,c}, \mathcal{D}', \beta) - 2(\beta-c)^3 - 3(1-\beta)(\beta-c)^2 \\ &= 6c\beta(2-c) - 2(\beta-c)^3 - 3(1-\beta)(\beta-c)^2. \end{aligned} \tag{6.2}$$

If $c = 3\beta$, then $\mathrm{F}(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_{\beta,c}, \hat{\mathcal{D}}') = 24\beta^2 - 26\beta^3$, which implies that $\mathrm{F}(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_{\beta,c}, \hat{\mathcal{D}}', \beta) < 0$ (for some $c \in (\beta, 3\beta)$) provided that $\beta > \frac{12}{13}$.

In fact, one can show that $(\mathbb{F}_1, C', \beta)$ does not admit a KEE metric for $\beta \in (\frac{4}{5}, 1]$ [29]. On the other hand, $(\mathbb{F}_1, C', \beta)$ admits a KEE metric for $\beta \in (0, \frac{3}{10})$, and, moreover, if C' is a *general* curve in $|-K_{\mathbb{F}_1}|$, then $(\mathbb{F}_1, C', \beta)$ admits a KEE metric for $\beta \in (0, \frac{3}{7})$ [4, Corollary 1.16].

6.2. $\mathrm{Bl}_{O_1,O_2}\mathbb{P}^2$. We take, as in the previous subsection, $S = \mathbb{P}^2, C$ a smooth cubic, and Z a line, but now blow-up two points $O_1, O_2 \in C \cap Z$ to obtain S', and let C', Z' be the proper transforms of C, Z, respectively. According to Panov–Ross [24, Example 7.6], the surface S' (with $\beta = 1$) is slope stable. We will show that it is *not* flop-slope stable, and moreover this holds also for (S', C', β) with $\beta \in (21/25, 1]$. By comparison, Székelyhidi [29] constructed a destabilizing toric degeneration for $\beta \in (\frac{7}{9}, 1]$ in the case when C' does not contain either of the points $Z' \cap C'_1$ or $Z' \cap C'_2$, where C'_i are the exceptional curves of the blow-down map to $\mathbb{P}^2$. It is interesting to note that the value 21/25 also arises in the related smooth continuity method [28, Proposition 10],[19, Example 2].

By Proposition 2.7, we have

$$\mathrm{F}(\mathcal{X}', \mathcal{L}'_{\beta,c}, \mathcal{D}', \beta) = 3\beta c(2-c) - c^2(2c-3).$$

Here $c < \epsilon(S', L'_\beta, Z') = \beta$. Thus, $\mathrm{F}(\mathcal{X}', \mathcal{L}'_\beta, \mathcal{D}', \beta) > 0$ for every $c \in (0, \beta)$ (i.e., slope stable). On the other hand, we have

$$\tau(S', L'_\beta, Z') = \epsilon(S, L_\beta, Z) = 3\beta.$$

By Lemma 4.7, the divisor $\hat{\mathcal{L}}'_\beta$ is ample for $c \in (\beta, 3\beta)$. Note that $C'_i . Z' = 1$ and as in (4.3) (see also (A.1)) $K_{\mathcal{X}'} . C'_i = 0$. Therefore,

$$\mathcal{L}'_\beta . C_i = -\beta K_{S'} . C'_i - cZ' . C'_i = \beta - cZ' . C'_i = \beta - c,$$

and by Proposition 4.9, one has

$$\mathrm{F}(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_\beta, \hat{\mathcal{D}}', \beta) = \mathrm{F}(\mathcal{X}', \mathcal{L}'_\beta, \mathcal{D}', \beta) - 4(\beta - c)^3 - 6(\beta - c)^2(1 - \beta).$$

Plugging-in $c = 3\beta$ yields

$$\mathrm{F}(\hat{\mathcal{X}}', \hat{\mathcal{L}}'_\beta, \hat{\mathcal{D}}', \beta) = 9\beta^2(2 - 3\beta) - 9\beta^2(6\beta - 3) + 32\beta^3 - 24\beta^2(1 - \beta) = \beta^2(21 - 25\beta) < 0,$$

when $\beta > \frac{21}{25}$.

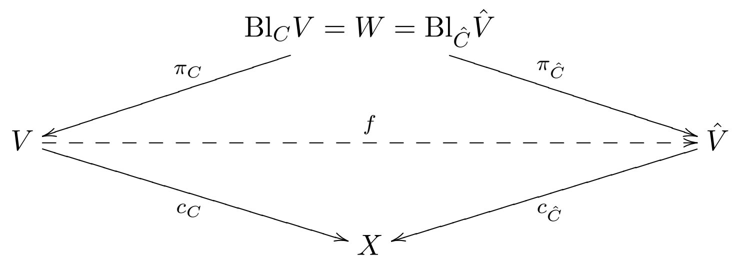
Note that (S', C', β) admits a KEE metric for $\beta \in (0, \frac{3}{7})$, and, moreover, if C' does not contain neither of the points $Z \cap C'_1$ and $Z \cap C'_2$, then a KEE metric exists for $\beta \in (0, \frac{1}{2})$ [4, Corollary 1.16].

Appendix A. Simple flops

Let V be a smooth projective variety, and let

$$C \subset V$$

be a smooth rational curve such that its normal bundle in V is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Then there exists a commutative diagram



such that the threefold $\hat{V}$ is smooth, the threefold X has an isolated ordinary double point, π_C is a blow-up of the curve C, $\pi_{\hat{C}}$ is the contraction of the π_C-exceptional surface, let us call it $E \cong \mathbb{P}^1 \times \mathbb{P}^1$, to a smooth rational curve, let us call it $\hat{C}$. We define the map f by declaring the diagram to be commutative. This defines f as a birational map away from C. It is important in this construction that $\pi_C \neq \pi_{\hat{C}}$, so that the map f is not an isomorphism. Finally, c_C and $c_{\hat{C}}$ are (small) contractions of the curves C and $\hat{C}$, respectively, to the isolated ordinary double point of X.

Remark A.1. The birational map $f \colon V \dashrightarrow \hat{V}$ is called the simple flop of the curve C. Sometimes it is called an Atiyah flop [1]. Later it was explicitly introduced by Kulikov in [18, §4.2] as *perestroika I.*

Note that the normal bundle of $\hat{C}$ in U is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. As in (4.3),

$$K_V.C = 0. \tag{A.1}$$

In fact, another way to see this equality is by noting that since c_C is an isomorphism away from codimension 2, then $K_V \sim_{\mathbb{Q}} c_C^\star K_V$, and of course $c_C^\star(K_X).C = 0$ since c_C contracts C. Similarly, $K_{\hat{V}}.\hat{C} = 0$ by construction. Moreover, we have $E|_E$ is a divisor on $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(-1, -1)$. Furthermore, the morphism $c_C \circ \pi_C = c_{\hat{C}} \circ \pi_{\hat{C}}$ is just the contraction of the surface E to the the isolated ordinary double point of X, i.e. its inverse map is the blow-up of this point.

Remark A.2. Note that in general $\hat{V}$ is not necessarily projective. However, it is not hard to see that $\hat{V}$ is projective in many cases, either by explicit construction or by using log MMP. In all our applications, $\hat{V}$ is projective by construction, see §4.

Given an irreducible reduced Weyl divisor D on V, we denote by $\hat{D}$ the unique divisor on $\hat{V}$ such that

$$\hat{D} := \overline{f(D \setminus C)}.$$

By linearity, we extend the same notation to all $\mathbb{R}$-divisors on V. The following formula may be known, but we provide a proof since we were not able to find a reference for it.

Lemma A.3. *Let $H_i, i = 1, 2, 3,$ be $\mathbb{R}$-divisors on V. Then,*

$$\hat{H}_1.\hat{H}_2.\hat{H}_3 = H_1.H_2.H_3 - (H_1.C)(H_2.C)(H_3.C).$$

Proof. Let $\tilde{H}_1$, $\tilde{H}_2$ and $\tilde{H}_3$ be the proper transforms of the divisors H_1, H_2 and H_3 on W, respectively. Recall that $E = \mathbb{P}^1 \times \mathbb{P}^1$ denotes the exceptional divisor of π_C (and of $\pi_{\hat{C}}$). Then,

$$\begin{cases} \tilde{H}_1 \sim_{\mathbb{R}} c_C^\star H_1 - m_1 E \sim_{\mathbb{R}} c_{\hat{C}}^\star \hat{H}_1 - \hat{m}_1 E, \\ \tilde{H}_2 \sim_{\mathbb{R}} c_C^\star H_2 - m_2 E \sim_{\mathbb{R}} c_{\hat{C}}^\star \hat{H}_2 - \hat{m}_2 E, \\ \tilde{H}_3 \sim_{\mathbb{R}} c_C^\star H_3 - m_3 E \sim_{\mathbb{R}} c_{\hat{C}}^\star \hat{H}_3 - \hat{m}_3 E, \end{cases}$$

for some real numbers $m_i, \hat{m}_i$. Put

$$r_i := H_i.C, \quad \hat{r}_i := \hat{H}_i.\hat{C}.$$

Then each $\tilde{H}_i|_E$ is a divisor (in $\mathbb{P}^1 \times \mathbb{P}^1$) of bi-degree

$$(r_i + m_i, m_i) = (\hat{m}_i, \hat{r}_i + \hat{m}_i);$$

this is because $E|_E = N_{E|V}$ is a line bundle of bi-degree $(-1, -1)$, while since $c_C(E) = C$ and $c_{\hat{C}}(E) = \hat{C}$,

(A.2) $$c_C^\star H_i|_E = H_i.c_C(E) \times (\text{fiber of projection of } \pi_C) = r_i \times (\text{bi-degree (1,0) curve}),$$

and

$$c_{\hat{C}}^\star \hat{H}_i|_E = \hat{H}_i.c_{\hat{C}}(E) \times (\text{fiber of projection of } \pi_{\hat{C}}) = \hat{r}_i \times (\text{bi-degree (0,1) curve}).$$

Thus,

(A.3) $$\hat{m}_i = r_i + m_i, \quad \text{and } \hat{r}_i = -r_i.$$

Now, $E^3 = E|_E.E|_E = c_1(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))^2 = 2$, and by (A.2),

$$c_C^\star H_i.E^2 = c_C^\star H_i|_E.E = -H_i|_E.C = -r_i.$$

On the other hand,

$$c_C^\star H_i.c_C^\star H_j.E = c_C^\star H_i|_E.c_C^\star H_j|_E = 0$$

since by (A.2), $c_C^\star H_i|_E$ and $c_C^\star H_j|_E$ are fibers of the same projection in $\mathbb{P}^1 \times \mathbb{P}^1 = E$. Altogether,

$$\begin{aligned} \tilde{H}_1.\tilde{H}_2.\tilde{H}_3 &= \big(c_C^\star H_1 - m_1 E\big).\big(c_C^\star H_2 - m_2 E\big).\big(c_C^\star H_3 - m_3 E\big) \\ &= H_1.H_2.H_3 + \big(m_1 m_2 c_C^\star H_3 + m_1 m_3 c_C^\star H_2 + m_2 m_3 c_C^\star H_1\big).E^2 - m_1 m_2 m_3 E^3 \\ &= H_1.H_2.H_3 - \big(m_1 m_2 r_3 + m_1 m_3 r_2 + m_2 m_3 r_1\big) - 2 m_1 m_2 m_3. \end{aligned}$$

Similarly,

$$\begin{aligned} \tilde{H}_1.\tilde{H}_2.\tilde{H}_3 &= \big(c_{\hat{C}}^\star \hat{H}_1 - \hat{m}_1 E\big).\big(c_{\hat{C}}^\star \hat{H}_2 - \hat{m}_2 E\big).\big(c_{\hat{C}}^\star \hat{H}_3 - \hat{m}_3 E\big) \\ &= \hat{H}_1.\hat{H}_2.\hat{H}_3 + \big(\hat{m}_1 \hat{m}_2 c_{\hat{C}}^\star \hat{H}_3 + \hat{m}_1 \hat{m}_3 c_{\hat{C}}^\star \hat{H}_2 + \hat{m}_2 \hat{m}_3 c_{\hat{C}}^\star \hat{H}_1\big).E^2 - \hat{m}_1 \hat{m}_2 \hat{m}_3 E^3 \\ &= \hat{H}_1.\hat{H}_2.\hat{H}_3 - \big(\hat{m}_1 \hat{m}_2 \hat{r}_3 + \hat{m}_1 \hat{m}_3 \hat{r}_2 + \hat{m}_2 \hat{m}_3 \hat{r}_1\big) - 2 \hat{m}_1 \hat{m}_2 \hat{m}_3. \end{aligned}$$

Thus,

$$\begin{aligned} &H_1.H_2.H_3 - m_1 m_2 r_1 - m_1 m_3 r_2 - m_2 m_3 r_1 - 2 m_1 m_2 m_3 \\ &\quad = \hat{H}_1.\hat{H}_2.\hat{H}_3 - \hat{m}_1 \hat{m}_2 \hat{r}_3 - \hat{m}_1 \hat{m}_3 \hat{r}_2 - \hat{m}_2 \hat{m}_3 \hat{r}_1 - 2 \hat{m}_1 \hat{m}_2 \hat{m}_3. \end{aligned}$$

By (A.3), this yields $\hat{H}_1.\hat{H}_2.\hat{H}_3 = H_1.H_2.H_3 - r_1r_2r_3$. □

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