

# Birationally rigid Fano threefold hypersurfaces

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**Abstract.** We prove that every quasi-smooth hypersurface in the 95 families of weighted Fano threefold hypersurfaces is birationally rigid.

**Keywords:** Fano hypersurface; weighted projective space; birationally rigid; birational involution.

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# 1 Introduction

## 1.1 Introduction

Let  $V$  be a smooth projective variety. If its canonical class  $K_V$  is pseudo-effective, then Minimal model program produces a birational model  $W$  of the variety  $V$ , so-called *minimal model*, which has mild singularities (terminal and  $\mathbb{Q}$ -factorial) and the canonical class  $K_W$  is nef. This has been verified in dimension 3 and in any dimension for varieties of general type (see [2, Theorem 1.1]). Meanwhile, if the canonical class  $K_V$  is not pseudoeffective, then Minimal model program yields a birational model  $U$  of  $V$ , so-called *Mori fibred space*. It also has terminal and  $\mathbb{Q}$ -factorial singularities and it admits a fiber structure  $\pi: U \rightarrow Z$  of relative Picard rank 1 such that the divisor  $-K_U$  is ample on fibers. This has been proved in all dimensions (see [2, Corollary 1.3.3]).

Mori fibred spaces, alongside the minimal models, represent the terminal objects in Minimal model program. If the canonical class is pseudo-effective and its minimal models exist, then they are unique up to flops. However, this is not the case when the canonical class is not pseudo-effective, since Mori fibred spaces are usually not unique terminal objects in Minimal model program. Nevertheless, some Mori fibred spaces behave very much the same as minimal models. To distinguish them, Corti introduced

**Definition 1.1.1** ([22, Definition 1.3]). Let  $\pi: U \rightarrow Z$  be a Mori fibred space. It is called *birationally rigid* if, for a birational map  $\xi: U \dashrightarrow U'$  to a Mori fibred space  $\pi': U' \rightarrow Z'$ , there exist a birational automorphism  $\tau: U \dashrightarrow U$  and a birational map  $\sigma: Z \dashrightarrow Z'$  such that the birational map  $\xi \circ \tau$  induces an isomorphism between the generic fibres of the Mori fibrations  $\pi: U \rightarrow Z$  and  $\pi': U' \rightarrow Z'$  and the diagram

$$\begin{array}{ccccc}
 U & \xrightarrow{\tau} & U & \xrightarrow{\xi} & U' \\
 \pi \downarrow & & & & \downarrow \pi' \\
 Z & \xrightarrow{\sigma} & & & Z'
 \end{array}$$

commutes.

Fano varieties of Picard rank one with at most terminal  $\mathbb{Q}$ -factorial singularities are the basic examples of Mori fibred spaces. For them, Definition 1.1.1 can be simplified as follows:

**Definition 1.1.2.** Let  $V$  be a Fano variety of Picard rank 1 with at most terminal  $\mathbb{Q}$ -factorial singularities. Then the Fano variety  $V$  is called *birationally rigid* if the following property holds.

- If there is a birational map  $\xi: V \dashrightarrow U$  to a Mori fibred space  $U \rightarrow Z$ , then the Fano variety  $V$  is biregular to  $U$  (and hence  $Z$  must be a point).

If, in addition, the birational automorphism group of  $V$  coincides with its biregular automorphism group, then  $V$  is called *birationally super-rigid*.

Birationally rigid Fano varieties behave very much like *canonical models*. Their birational geometry is very simple. In particular, they are non-rational. The first example of a birationally rigid Fano variety is due to Iskovskikh and Manin. In 1971, they proved

**Theorem 1.1.3** ([30]). *A smooth quartic hypersurface in  $\mathbb{P}^4$  is birationally super-rigid.*

In fact, Iskovskikh and Manin only proved that smooth quartic hypersurfaces in  $\mathbb{P}^4$  do not admit non-biregular birational automorphisms and, therefore, are non-rational. In late nineties, Corti observed in [21] that their proof implies Theorem 1.1.3. Inspired by this observation, Pukhlikov generalized Theorem 1.1.3 as

**Theorem 1.1.4** ([37]). *A general hypersurface of degree  $n \geq 4$  in  $\mathbb{P}^n$  is birationally super-rigid.*

Shortly after Theorem 1.1.4 was proved, Reid suggested to Corti and Pukhlikov to generalize Theorem 1.1.3 for singular threefolds. Together they proved

**Theorem 1.1.5** ([23]). *Let  $X$  be a quasi-smooth hypersurface of degrees  $d$  with only terminal singularities in weighted projective space  $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ , where  $d = \sum a_i$ . Suppose that  $X$  is a general threefold in this family. Then  $X$  is birationally rigid.*

The singular threefolds in Theorem 1.1.5 have a long history. In 1979 Reid discovered the 95 families of  $K3$  surfaces in three dimensional weighted projective spaces (see [38]). After this, Fletcher, who was a Ph.D. student of Reid, discovered the 95 families of weighted Fano threefold hypersurfaces in his Ph.D. dissertation in 1988. These are quasi-smooth hypersurfaces of degrees  $d$  with only terminal singularities in weighted projective spaces  $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ , where  $d = \sum a_i$ . The 95 families are determined by the quadruples of non-decreasing positive integers  $(a_1, a_2, a_3, a_4)$ . All Reid's 95 families of  $K3$  surfaces arises as anticanonical divisors in Fletcher's 95 families of Fano threefolds. Because of this, the latter 95 families are often called the 95 families of Fletcher and Reid.

Quite often, we need to know the non-rationality of an explicitly given Fano variety (which does not follow from the non-rationality of a general member in its family).

**Example 1.1.6.** Recently Prokhorov classified all finite simple subgroups in the birational automorphism group  $\text{Bir}(\mathbb{P}^3)$  of the three-dimensional projective space. Up to isomorphism,  $A_5$ ,  $\text{PSL}_2(\mathbb{F}_7)$ ,  $A_6$ ,  $A_7$ ,  $\text{PSL}_2(\mathbb{F}_8)$  and  $\text{PSU}_4(\mathbb{F}_2)$  are all non-abelian finite simple subgroups in  $\text{Bir}(\mathbb{P}^3)$  ([36, Theorem 1.3]). Prokhorov's proof implies more. Up to conjugation,  $\text{Bir}(\mathbb{P}^3)$  contains unique subgroup  $\text{PSL}_2(\mathbb{F}_8)$  and exactly two subgroups  $\text{PSU}_4(\mathbb{F}_2)$ . For the alternating group  $A_7$ , he proved that  $\text{Bir}(\mathbb{P}^3)$  contains exactly one such subgroup provided that the threefold

$$\sum_{i=0}^6 x_i = \sum_{i=0}^6 x_i^2 = \sum_{i=0}^6 x_i^3 = 0 \subset \text{Proj}(\mathbb{C}[x_0, \dots, x_6]) \cong \mathbb{P}^6 \quad (1.1.7)$$

is not rational. This threefold is the unique complete intersection of a quadric and a cubic hypersurfaces in  $\mathbb{P}^5$  that admits a faithful action of  $A_7$ . Back in nineties Iskovskikh and Pukhlikov proved that general threefold in this family is birationally rigid (see [31]). The threefold (1.1.7) is smooth. However, it does not satisfy the generality assumptions imposed in [31]. Only last year Beauville proved in [1] that the threefold (1.1.7) is not rational. It is still unknown whether it is birationally rigid or not.

It took more than ten years to prove Theorem 1.1.4 for *every* smooth hypersurface in  $\mathbb{P}^n$  of degree  $n \geq 4$ , which was conjectured in [37]. This was done by de Fernex who proved

**Theorem 1.1.8** ([25]). *Every smooth hypersurface of degree  $n \geq 4$  in  $\mathbb{P}^n$  is birationally super-rigid.*

The goal of this paper is to prove Theorem 1.1.5 for *all* quasi-smooth hypersurfaces in each of the 95 families of Fletcher and Reid, which was conjectured in [23]. To be precise, we prove

**Theorem 1.1.9.** *Let  $X$  be a quasi-smooth hypersurface of degrees  $d$  with only terminal singularities in weighted projective space  $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ , where  $d = \sum a_i$ . Then  $X$  is birationally rigid.*

**Corollary 1.1.10.** *Let  $X$  be a quasi-smooth hypersurface of degrees  $d$  with only terminal singularities in weighted projective space  $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ , where  $d = \sum a_i$ . Then  $X$  is not rational.*

As an intermediate step in the proof of Theorem 1.1.9, we prove

**Theorem 1.1.11.** *Every quasi-smooth hypersurface in the families of the 95 families of Fletcher and Reid whose general members are birationally super-rigid is birationally super-rigid.*

The families corresponding to Theorem 1.1.11 are those in the list of Fletcher and Reid with entry numbers No. 1, 3, 10, 11, 14, 19, 21, 22, 28, 29, 34, 35, 37, 39, 49, 50, 51, 52, 53, 55, 57, 59, 62, 63, 64, 66, 67, 70, 71, 72, 73, 75, 77, 78, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94 and 95.

The 95 families of Fletcher and Reid contain the family (No. 1) of quartic hypersurfaces in  $\mathbb{P}^4$  and the family (No. 3) of hypersurfaces of degree 6 in  $\mathbb{P}(1, 1, 1, 1, 3)$ , i.e., double covers of  $\mathbb{P}^3$  ramified along sextic surfaces. However, we do not consider these two families in the present paper since every smooth quartic threefold and every smooth double covers of  $\mathbb{P}^3$  ramified along sextic surfaces (see [28]) are already proved to be birationally super-rigid.

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## 1.2 Set-up

In this section we briefly explain how to prove the main theorem. Before we proceed, we introduce a terminology that is frequently used in birational geometry as well as in the present paper.

**Definition 1.2.1.** Let  $U$  be a normal  $\mathbb{Q}$ -factorial variety and  $\mathcal{M}_U$  a mobile linear system on  $U$ . Let  $a$  be a non-negative rational number. An irreducible subvariety  $Z$  of  $U$  is called a center of non-canonical singularities (or simply non-canonical center) of the pair  $(U, a\mathcal{M}_U)$  if there is a birational morphism  $\mu: W \rightarrow U$  and an  $\mu$ -exceptional divisor  $E_1 \subset W$  such that

$$K_W + a\mu_*^{-1}(\mathcal{M}_U) = \mu^*(K_U + a\mathcal{M}_U) + \sum_{i=1}^m c_i E_i,$$

where  $E_i$  is an  $\mu$ -exceptional divisor,  $c_1 < 0$  and  $\mu(E_1) = Z$ .

The following result is known as the classical Nöther–Fano inequality.

**Theorem 1.2.2** ([21, Theorem 4.2]). *Let  $X$  be a terminal  $\mathbb{Q}$ -factorial Fano variety with  $\text{Pic}(X) \cong \mathbb{Z}$ .*

- *If the log pair  $(X, \frac{1}{n}\mathcal{M})$  has canonical singularities for every positive integer  $n$  and every mobile linear subsystem  $\mathcal{M} \subset |-nK_X|$ , then  $X$  is birationally super-rigid.*
- *If for every positive integer  $n$  and every mobile linear system  $\mathcal{M} \subset |-nK_X|$  there exists  $\tau \in \text{Bir}(X)$  such that the log pair  $(X, \frac{1}{n\tau}\tau(\mathcal{M}))$  has canonical singularities, where  $n_\tau$  is the positive integer such that  $\tau(\mathcal{M}) \subset |-n_\tau K_X|$ , then  $X$  is birationally rigid.*

The Nöther–Fano inequality will be the main key to the proof of the main theorem.

To prove the main theorem, we suppose that a given hypersurface  $X$  in one of the families has a mobile linear system  $\mathcal{M} \subset |-nK_X|$  for some positive integer  $n$  such that the log pair  $(X, \frac{1}{n}\mathcal{M})$  is not canonical. Then we must have a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .

The center of a non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$  can be, a priori, a smooth point, an irreducible curve, or a singular point of the Fano variety  $X$ .

In Section 2.1 we show that a smooth point of  $X$  cannot be a center.

In Section 2.2 we show that a curve contained in the smooth locus of  $X$  cannot be a center. Then Theorem 2.2.1 implies that a singular point of  $X$  must be a center.

Now Theorem 1.1.11 can be proved by excluding all the singular points of  $X$  as a center. 50 families out of the 95 families are those that are considered in Theorem 1.1.11. In Section 4.2, we verify that a singular point of  $X$  cannot be a center if the hypersurface  $X$  belongs to one of the families considered in Theorem 1.1.11.

To prove Theorem 1.1.9, we have only to consider the 45 families that are not considered in Theorem 1.1.11 since birational super-rigidity implies birational rigidity. For a given singular point of a hypersurface  $X$  in one of the 45 families we show that either a singular point of

$X$  cannot be a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$  or there exists a birational automorphism  $\tau$  of  $X$  such that  $\tau(\mathcal{M}) \subset |-n_\tau K_X|$  for some positive integer  $n_\tau < n$ . It then follows from Theorem 1.2.2 that the given hypersurface  $X$  is birationally rigid.

### 1.3 Notations

Let us describe the notations we will use in the rest of the article. Unless otherwise mentioned, these notations are fixed from now until the end of the article.

- In the weighted projective space  $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ , we assume that  $a_1 \leq a_2 \leq a_3 \leq a_4$ . For weighted homogeneous coordinates, we always use  $x, y, z, t$  and  $w$  with  $\text{wt}(x) = 1$ ,  $\text{wt}(y) = a_1$ ,  $\text{wt}(z) = a_2$ ,  $\text{wt}(t) = a_3$  and  $\text{wt}(w) = a_4$ .
- In each family, we always let  $X$  be a quasi-smooth hypersurface of degree  $d$  in the weighted projective space  $\mathbb{P}(1, a_1, a_2, a_3, a_4)$  with only terminal singularities, where  $d = \sum_{i=1}^4 a_i$ .
- On the threefold  $X$ , a given mobile linear system is denoted by  $\mathcal{M}$ .
- For a given mobile linear system  $\mathcal{M}$ , we always assume that  $\mathcal{M} \sim_{\mathbb{Q}} -nK_X$ .
- $S_x$  is the surface on the hypersurface  $X$  cut by the equation  $x = 0$ .
- $S_y$  is the surface on the hypersurface  $X$  cut by the equation  $y = 0$ .
- $S_z$  is the surface on the hypersurface  $X$  cut by the equation  $z = 0$ .
- $S_t$  is the surface on the hypersurface  $X$  cut by the equation  $t = 0$ .
- $S_w$  is the surface on the hypersurface  $X$  cut by the equation  $w = 0$ .
- $L_{tw}$  is the one-dimensional stratum on  $\mathbb{P}(1, a_1, a_2, a_3, a_4)$  defined by  $x = y = z = 0$ , and the other one-dimensional strata are labelled similarly.
- $O_x := [1 : 0 : 0 : 0 : 0]$ .
- $O_y := [0 : 1 : 0 : 0 : 0]$ .
- $O_z := [0 : 0 : 1 : 0 : 0]$ .
- $O_t := [0 : 0 : 0 : 1 : 0]$ .
- $O_w := [0 : 0 : 0 : 0 : 1]$ .
- The entry number of each family of Fletcher and Reid is given in the Big Table of [23]. Indeed, it is given in the lexicographic order of  $(d, a_1, a_2, a_3, a_4)$ .

## 2 Excluding smooth points and curves

### 2.1 Smooth points

In this section we show that smooth points of  $X$  cannot be centers of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$ .

Let  $X \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$  be a quasi-smooth weighted hypersurface of degree  $d = \sum a_i$  with terminal singularities. Suppose that a smooth point  $p$  on  $X$  is a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ . Then we obtain

$$\text{mult}_p(\mathcal{M}^2) > 4n^2$$

by [22, Corollary 3.4].

Let  $s$  be an integer not greater than  $\frac{4}{-K_X^3}$ . Suppose that we find a divisor  $H$  in  $|-sK_X|$  such that

- it passes through the point  $p$ ,
- it contains no 1-dimensional component of the base locus of the linear system  $\mathcal{M}$  that passes through the point  $p$ .

Then we can obtain the following contradictory inequality:

$$-sn^2K_X^3 = H \cdot \mathcal{M}^2 \geq \text{mult}_p(H) \cdot \text{mult}_p(\mathcal{M}^2) > 4n^2.$$

In order to show that the pair  $(X, \frac{1}{n}\mathcal{M})$  is canonical at a smooth point, we mainly try to find such a divisor.

Before we proceed, set  $\widehat{a}_2 = \text{lcm}\{a_1, a_3, a_4\}$ ,  $\widehat{a}_3 = \text{lcm}\{a_1, a_2, a_4\}$  and  $\widehat{a}_4 = \text{lcm}\{a_1, a_2, a_3\}$ .

**Lemma 2.1.1.** *Suppose that the hypersurface  $X$  satisfies one of the following:*

- $X$  does not pass through the point  $O_w$  and  $d \cdot \widehat{a}_4 \leq 4a_1a_2a_3a_4$ ;
- $X$  does not pass through the point  $O_t$  and  $d \cdot \widehat{a}_3 \leq 4a_1a_2a_3a_4$ ;
- $X$  does not pass through the point  $O_z$  and  $d \cdot \widehat{a}_2 \leq 4a_1a_2a_3a_4$ .

*Then a smooth point of  $X$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .*

*Proof.* For simplicity we suppose that the hypersurface  $X$  satisfies the first condition. The proofs for the other cases are the same.

Let  $\pi_4 : X \rightarrow \mathbb{P}(1, a_1, a_2, a_3)$  be the regular projection centered at the point  $O_w$ . The linear system  $|\mathcal{O}_{\mathbb{P}(1, a_1, a_2, a_3)}(\widehat{a}_4)|$  is base point free. Choose a general member in the linear system  $|\mathcal{O}_{\mathbb{P}(1, a_1, a_2, a_3)}(\widehat{a}_4)|$  that passes through the point  $\pi_4(p)$ . Then its pull-back by the finite morphism  $\pi_4$  can play the role of the divisor  $H$  in the explanation at the beginning.  $\square$

The condition above is satisfied by all the families except the families

$$\text{No. } 2, 5, 12, 13, 20, 23, 25, 33, 40, 58, 61, 76.$$

**Lemma 2.1.2.** *Suppose that the threefold  $X$  satisfies the following conditions:*

- The points  $O_w$  and  $O_t$  are singular points of  $X$ ;
- $X$  passes through the point  $O_w$  and  $d \cdot \widehat{a}_4 \leq 4a_1a_2a_3a_4$ ;
- $X$  passes through the point  $O_t$  and  $d \cdot \widehat{a}_3 \leq 4a_1a_2a_3a_4$ .

In addition, we suppose that the curve  $L_{tw}$  is not contained in the hypersurface  $X$ . Then a smooth point of  $X$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .

*Proof.* Let  $\pi_4 : X \dashrightarrow \mathbb{P}(1, a_1, a_2, a_3)$  be the projection centered at the point  $O_w$  and  $\pi_3 : X \dashrightarrow \mathbb{P}(1, a_1, a_2, a_4)$  the projection centered at the point  $O_t$ . The linear systems  $|\mathcal{O}_{\mathbb{P}(1, a_1, a_2, a_3)}(\widehat{a}_4)|$  and  $|\mathcal{O}_{\mathbb{P}(1, a_1, a_2, a_4)}(\widehat{a}_3)|$  are base point free. Choose a general member in the linear  $|\mathcal{O}_{\mathbb{P}(1, a_1, a_2, a_3)}(\widehat{a}_4)|$  that passes through the point  $\pi_4(p)$  and a general member in the linear  $|\mathcal{O}_{\mathbb{P}(1, a_1, a_2, a_4)}(\widehat{a}_3)|$  that passes through the point  $\pi_3(p)$ . Then either the pull-back of the former divisor by the projection  $\pi_4$  or the pull-back of the latter divisor by the projection  $\pi_3$  can play the role of the divisor  $H$  in the explanation at the beginning.  $\square$

The conditions above are satisfied by the families

No. 23, 40, 61, 76.

The family No. 2 and special members of the families No. 5, 12, 13, 20, 25, 33, 58 remain.

We first consider the special members of the families  $N = 5, 12, 13, 20, 25$ . These members must contain the curve  $L_{tw}$ , *i.e.*, the defining polynomials of  $X$  do not contain the monomial  $t^2w$ .

**Lemma 2.1.3.** *Suppose that the hypersurface  $X$  satisfies the following conditions:*

- The points  $O_w$  and  $O_t$  are singular points of  $X$ ;
- $X$  passes through the point  $O_w$  and  $d \cdot \widehat{a}_4 \leq 4a_1a_2a_3a_4$ ;
- $X$  passes through the point  $O_t$  and  $d \cdot \widehat{a}_3 \leq 4a_1a_2a_3a_4$ .

*Suppose that the curve  $L_{tw}$  is contained in the hypersurface  $X$ . If a smooth point of  $X$  is a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ , then the point lies on the curve  $L_{tw}$ .*

*Proof.* The proof of Lemma 2.1.2 immediately shows the statement.  $\square$

**Lemma 2.1.4.** *Suppose that the curve  $L_{tw}$  is contained in the hypersurface  $X$ . In addition, we suppose that  $(a_3, a_4) = 1$ ,  $a_3a_4 > d$ , and there are non-negative integers  $m_1$  and  $m_2$  such that  $m_1a_1 + m_2a_2 = a_3a_4$ . Then any smooth point on  $L_{tw}$  of  $X$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$  if  $-a_3a_4K_X^3 \leq 4$ .*

*Proof.* Suppose that the hypersurface  $X$  is defined by  $F(x, y, z, t, w) = 0$ . Let  $p$  be a smooth point on the curve  $L_{tw}$ . Then there are non-zero constants  $\lambda$  and  $\mu$  such that the surface cut by  $\lambda t^{a_4} + \mu w^{a_3} = 0$  contains the point  $p$ . We then consider the linear system  $\mathcal{H}$  on  $X$  generated by  $x^{a_3a_4}$ ,  $y^{m_1}z^{m_2}$ , and  $\lambda t^{a_4} + \mu w^{a_3}$ . The base locus of this linear system consists of the locus cut by  $x = yz = \lambda t^{a_4} + \mu w^{a_3} = 0$ . Since the polynomial  $\lambda t^{a_4} + \mu w^{a_3}$  is irreducible and  $d < a_3a_4$ , neither  $F(0, 0, z, t, w)$  nor  $F(0, y, 0, t, w)$  can divide  $\lambda t^{a_4} + \mu w^{a_3}$ . Therefore, the base locus of the linear system  $\mathcal{H}$  is of dimension at most 0. Then a general member of this linear system is able to play the role of the divisor  $H$  in the explanation at the beginning.  $\square$



Combining the two lemmas, we can show that the pair  $(X, \frac{1}{n}\mathcal{M})$  is canonical at every smooth point for the families No. 33 and 58.

For the special cases of families No. 5, 12, 13, 20, 25, we may assume that the polynomial is given by

$$w^2z + w(tg_{a_3} + g_{2a_3}) + t^3y + t^2h_{a_1+a_3} + th_{a_1+2a_3} + h_{a_1+3a_3} = 0 \quad \text{for the families No. 12, 20,}$$

$$w^2y + w(tg_{a_3} + g_{2a_3}) + t^3z + t^2h_{a_2+a_3} + th_{a_2+2a_3} + h_{a_2+3a_3} = 0 \quad \text{for the families No. 5, 13, 25,}$$

where  $g_m$  and  $h_k$  are quasi-homogeneous polynomials of degrees  $m$  and  $k$  respectively in variables  $x$ ,  $y$ , and  $z$ .

**Lemma 2.1.5.** *For the families No. 13, 25, a smooth point of  $X$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .*

*Proof.* The following method works for both the families exactly in the same way. For this reason, we demonstrate the method only for the family No. 25.

Suppose that the pair  $(X, \frac{1}{n}\mathcal{M})$  is not canonical at some smooth point  $p$ . Then Lemma 2.1.3 shows that the point  $p$  must lie on the curve  $L_{tw}$ .

Consider the pencil  $|-K_X|$ . Its base locus consists of two reduced and irreducible curves. One is the curve  $L_{tw}$  and the other is the curve  $C$  defined by the equations  $x = y = t^3 - az^4 = 0$ , where  $a$  is a non-zero constant. Note that the curve  $L_{tw}$  is quasi-smooth everywhere and  $C$  is quasi-smooth outside the point  $O_w$ . They intersect only at the point  $O_w$ . Choose a general member  $S$  in the pencil  $|-K_X|$ . Note that  $S$  is a  $K3$  surface with  $A_3$  and  $A_6$  singularities at  $O_t$  and  $O_w$ , respectively. Then the pair  $(X, S + \frac{1}{n}\mathcal{M})$  is not log canonical at the point  $p$ . By Inversion of adjunction, we see that the pair  $(S, \frac{1}{n}\mathcal{M}|_S)$  is not log canonical at the point  $p$ . Let  $D_y$  be the divisor on  $S$  defined by the equation  $y = 0$ . Then  $D_y = L_{tw} + C$ .

We have the following intersection numbers on the surface  $S$ :

$$L_{tw}^2 = -\frac{11}{28}, \quad C^2 = -\frac{2}{7}, \quad C \cdot L_{tw} = \frac{3}{7}, \quad D_y \cdot L_{tw} = \frac{1}{28}, \quad D_y \cdot C = \frac{3}{21}.$$

We write

$$\mathcal{M}_S := \frac{1}{n}\mathcal{M}|_S = aL_{tw} + bC + \Delta,$$

where  $a$  and  $b$  are non-negative rational numbers and  $\Delta$  is an effective divisor whose support contains neither  $L_{tw}$  nor  $C$ . We then obtain

$$\frac{3}{21} = C \cdot \mathcal{M}_S = aL_{tw} \cdot C + bC^2 + \Delta \cdot C \geq \frac{3a}{7} - \frac{2b}{7}.$$

On the other hand, we obtain

$$\frac{5}{28} = \mathcal{M}_S^2 = aL_{tw} \cdot D_y + bC \cdot D_y + \Delta \cdot D_y \geq \frac{a}{28} + \frac{3b}{21}.$$

Combining these two inequalities we see that  $a \leq 1$ . Therefore, the pair  $(S, L_{tw} + bC + \Delta)$  is not log canonical at the point  $p$ , and hence the pair  $(L_{tw}, (bC + \Delta)|_{L_{tw}})$  is not log canonical at the point  $p$ . Consequently, we see that

$$\text{mult}_p((bC + \Delta)|_{L_{tw}}) > 1.$$

However,

$$(bC + \Delta) \cdot L_{tw} = (\mathcal{M}_S - aL_{tw}) \cdot L_{tw} = \frac{1}{28} + \frac{11a}{28} \leq \frac{12}{28}.$$

This completes the proof.  $\square$

**Lemma 2.1.6.** *For the families No. 12, 20, any smooth point of  $X$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .*

*Proof.* The following method works for both the families exactly in the same way. Even the method is the same as that of Lemma 2.1.5 with some slight difference. For this reason, we demonstrate the method only for the family No. 20.

Suppose that the pair  $(X, \frac{1}{n}\mathcal{M})$  is not canonical at some smooth point  $p$ . Then Lemma 2.1.3 shows that the point  $p$  must lie on the curve  $L_{tw}$ .

Consider the pencil  $|-K_X|$ . Its base locus consists of two reduced and irreducible curves. One is the curve  $L_{tw}$  and the other is the curve  $C$  defined by the equations  $x = y = w^2 - azt = 0$ , where  $a$  is a non-zero constant. Note that the curves  $L_{tw}$  and  $C$  are quasi-smooth everywhere. They intersect only at the point  $O_t$ .

Choose a general member  $S$  in the pencil  $|-K_X|$ . Note that  $S$  is a  $K3$  surface with  $A_2$ ,  $A_3$  and  $A_4$  singularities at  $O_z$ ,  $O_t$  and  $O_w$ , respectively. It also contains the curve  $L_{tw}$ . Then the pair  $(X, S + \frac{1}{n}\mathcal{M})$  is not log canonical at the point  $p$ . By Inversion of adjunction, we see that the pair  $(S, \frac{1}{n}\mathcal{M}|_S)$  is not log canonical at the point  $p$ . Let  $D_y$  be the divisor on  $S$  defined by the equation  $y = 0$ . Then  $D_y = L_{tw} + C$ .

We have the following intersection numbers on the surface  $S$ :

$$L_{tw}^2 = -\frac{9}{20}, \quad C^2 = -\frac{1}{3}, \quad C \cdot L_{tw} = \frac{1}{2}, \quad D_y \cdot L_{tw} = \frac{1}{20}, \quad D_y \cdot C = \frac{1}{6}.$$

We write

$$\mathcal{M}_S := \frac{1}{n}\mathcal{M}|_S = aL_{tw} + bC + \Delta,$$

where  $a$  and  $b$  are non-negative rational numbers and  $\Delta$  is an effective divisor whose support contains neither  $L_{tw}$  nor  $C$ . We then obtain

$$\frac{1}{6} = C \cdot \mathcal{M}_S = aL_{tw} \cdot C + bC^2 + \Delta \cdot C \geq \frac{a}{2} - \frac{b}{3}.$$

On the other hand, we obtain

$$\frac{13}{60} = \mathcal{M}_S^2 = aL_{tw} \cdot D_y + bC \cdot D_y + \Delta \cdot D_y \geq \frac{a}{20} + \frac{b}{6}.$$

Combining these two inequalities we see that  $a \leq 1$ . Therefore, the pair  $(S, L_{tw} + bC + \Delta)$  is not log canonical at the point  $p$ , and hence the pair  $(L_{tw}, (bC + \Delta)|_{L_{tw}})$  is not log canonical at the point  $p$ . Consequently, we see that

$$\text{mult}_p((bC + \Delta)|_{L_{tw}}) > 1.$$

However,

$$(bC + \Delta) \cdot L_{tw} = (\mathcal{M}_S - aL_{tw}) \cdot L_{tw} = \frac{1}{20} + \frac{9a}{20} \leq \frac{1}{2}.$$

This completes the proof.  $\square$

**Lemma 2.1.7.** *For the family No. 5, a smooth point of  $X$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .*

*Proof.* By suitable coordinate change we may assume that the hypersurface  $X$  in  $\mathbb{P}(1, 1, 1, 2, 3)$  is given by

$$w^2y + w(tg_2 + g_4) + t^3z + t^2h_3 + th_5 + h_7 = 0,$$

where  $g_m$  and  $h_k$  are quasi-homogeneous polynomials of degrees  $m$  and  $k$  respectively in variables  $x, y$ , and  $z$ .

Suppose that the pair  $(X, \frac{1}{n}\mathcal{M})$  is not canonical at some smooth point  $p$ . Then Lemma 2.1.3 shows that the point  $p$  must lie on the curve  $L_{tw}$ .

Consider the 2-dimensional linear system  $|-K_X|$ . Its base locus consists of the reduced and irreducible curve  $L_{tw}$ . The curve  $L_{tw}$  is a quasi-smooth curve passing through the singular points  $O_t$  and  $O_w$ .

Choose a general member  $S$  in the pencil  $|-K_X|$ . It is a  $K3$  surface with  $A_1$  and  $A_2$  singularities at  $O_t$  and  $O_w$ , respectively. It also contains the rational curve  $L_{tw}$ . Therefore, the self-intersection number of  $L_{tw}$  on  $S$  is  $-5/6$ . Let  $D_y$  be the divisor on  $S$  defined by the equation  $y = 0$ . Then we can easily see that  $D_y = L_{tw} + R$ , where  $R$  is the curve defined by the equations  $y = z = f_6(x, w, t) = 0$ , where  $f_6$  is a quasi-homogeneous polynomial of degree 6. The two curves  $L_{tw}$  and  $R$  meet only at the point  $O_w$ .

We write

$$\mathcal{M}_S := \frac{1}{n}\mathcal{M}|_S = aL_{tw} + \Delta,$$

where  $a$  is a non-negative rational number and  $\Delta$  is an effective divisor whose support does not contain the curve  $L_{tw}$ . The pair  $(X, S + \frac{1}{n}\mathcal{M})$  is not log canonical at the point  $p$ . By Inversion of adjunction, we see that the pair  $(S, \frac{1}{n}\mathcal{M}|_S)$  is not log canonical at the point  $p$ . We then obtain

$$1 = R \cdot \mathcal{M}_S = aL_{tw} \cdot R + \Delta \cdot R \geq a.$$

Therefore, the pair  $(S, L_{tw} + \Delta)$  is not log canonical at the point  $p$ , and hence the pair  $(L_{tw}, \Delta|_{L_{tw}})$  is not log canonical at the point  $p$ . Consequently, we see that

$$\text{mult}_p(\Delta|_{L_{tw}}) > 1.$$

However,

$$\Delta \cdot L_{tw} = (\mathcal{M}_S - aL_{tw}) \cdot L_{tw} = \frac{1}{6} + \frac{5a}{6} \leq 1.$$

This completes the proof.  $\square$

**Lemma 2.1.8.** *For the family No. 2, a smooth point of  $X$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .*

*Proof.* For authors' convenience we reproduce the proof from p.211 in [23].

By suitable coordinate change we may assume that the hypersurface  $X$  in  $\mathbb{P}(1, 1, 1, 1, 2)$  is given by

$$w^2x + wf_3 + f_5 = 0,$$

where  $f_m$  is a quasi-homogeneous polynomial of degree  $m$  in variables  $x, y, z$  and  $t$ .

Suppose that the pair  $(X, \frac{1}{n}\mathcal{M})$  is not canonical at some smooth point  $p$ . Then the point  $p$  must lie on the curve  $L$  contracted by the projection  $\pi_4 : X \dashrightarrow \mathbb{P}^3$ . By additional coordinate

change, we may assume that the line  $L$  is defined by the equations  $x = y = z = 0$ , i.e.,  $L = L_{tw}$ .

Let  $S$  be a general element in  $|-K_X|$  containing the curve  $L_{tw}$ . Then the surface  $S$  is a  $K3$  surface with an  $A_1$  singularity at the point  $O_w$ . The self-intersection number of  $L_{tw}$  on  $S$  is  $-3/2$ .

We write

$$\mathcal{M}_S := \frac{1}{n}\mathcal{M}|_S = aL_{tw} + \mathcal{L},$$

where  $a$  is a non-negative rational number and  $\mathcal{L}$  is a mobile linear system on  $S$ .

Choose another line  $R$  that is contacted by the projection  $\pi_4$ . Note that such a line is given by a point on the zero set in  $\mathbb{P}^3$  defined by  $x = f_3 = f_6 = 0$ . Then we see that the intersection number  $L_{tw}$  and  $R$  is  $1/2$ . We then obtain

$$\frac{1}{2} = R \cdot \mathcal{M}_S = aL_{tw} \cdot R + \mathcal{L} \cdot R \geq \frac{a}{2},$$

and hence  $a \leq 1$ .

The pair  $(X, S + \frac{1}{n}\mathcal{M})$  is not log canonical at the point  $p$ . By Inversion of adjunction, we see that the pair  $(S, \mathcal{M}_S)$  is not log canonical at the point  $p$ . We then obtain from [22, Theorem 3.1]

$$4(1-a) < \mathcal{L}^2 = (\mathcal{M}_S - aL_{tw})^2 = \mathcal{M}_S^2 - 2a\mathcal{M}_S \cdot L_{tw} + a^2L_{tw}^2 = \frac{5}{2} - a - \frac{3a^2}{2}.$$

However, this inequality cannot be satisfied with any value of  $a$ . This completes the proof.  $\square$

## 2.2 Curves

In this section we show that an irreducible curve on  $X$  can not be a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$  provided that no point on this curve is a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$ . Indeed, the proof comes from [23, pp. 206-207] and it is based on the following *local* result of Kawamata:

**Theorem 2.2.1** ([32, Lemma 7]). *Let  $(U, p)$  be a germ of a threefold terminal quotient singularity of type  $\frac{1}{r}(1, a, r-a)$  where  $r \geq 2$  and  $a$  is coprime to  $r$  and let  $\mathcal{M}_U$  be a mobile linear system on  $U$ . Suppose that  $(U, \lambda\mathcal{M}_U)$  is not canonical at  $p$  for a positive rational number  $\lambda$ . Let  $f: W \rightarrow U$  be the weighted blowup at the point  $p$  with weights  $(1, a, r-a)$ . Then*

$$\mathcal{M}_W = f^*(\mathcal{M}_U) - mE$$

for some positive rational number  $m > \frac{1}{r\lambda}$ , where  $E$  is the exceptional divisor of  $f$  and  $\mathcal{M}_W$  is the proper transform of  $\mathcal{M}_U$ . In particular,

$$K_W + \lambda\mathcal{M}_W = f^*(K_U + \lambda\mathcal{M}_U) + \left(\frac{1}{r} - \lambda m\right)E,$$

where  $\frac{1}{r} - \lambda m < 0$ , and hence the point  $p$  is a center of non-canonical singularities of the log pair  $(U, \lambda\mathcal{M}_U)$ .

Theorem 2.2.1 and the mobility of the linear system  $\mathcal{M}$  imply the following *global* properties:

**Corollary 2.2.2** ([23, Lemma 5.2.1]). *Let  $\Lambda$  be a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$ . In case when  $\Lambda$  is a singular point of type  $\frac{1}{r}(1, a, r - a)$ , let  $f: Y \rightarrow X$  be the weighted blow up at  $\Lambda$  with weights  $(1, a, r - a)$ . In case when  $\Lambda$  is a smooth curve contained in the smooth locus of  $X$ , let  $f: Y \rightarrow X$  be the blow up along  $\Lambda$ . Then the 1-cycle  $(-K_Y)^2 \in N_1(Y)$  lies in the interior of the Mori cone of  $Y$ :*

$$(-K_Y)^2 \in \text{Int}(\overline{\text{NE}(Y)}).$$

**Corollary 2.2.3** ([23, Corollary 5.2.3]). *Under the same notations as in Corollary 2.2.2, we have  $H \cdot (-K_Y)^2 > 0$  for a non-zero nef divisor  $H$  on  $Y$ .*

Let  $L$  be an irreducible curve on  $X$ . Suppose that  $L$  is a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ . Then it follows from Theorem 2.2.1 that every singular point of  $X$  contained in  $L$  (if any) must be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ . Later we will show that every singular point of  $X$  is not a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ . Moreover, it will be done regardless of the fact that the log pair  $(X, \frac{1}{n}\mathcal{M})$  is canonical outside of this singular point. Therefore it is enough to exclude only irreducible curves contained in the smooth locus of  $X$ .

Suppose that  $L$  is contained in the smooth locus of  $X$ . Pick two general members  $H_1$  and  $H_2$  in the mobile linear system  $\mathcal{M}$ . Then we see

$$-n^2 K_X^3 = -K_X \cdot H_1 \cdot H_2 \geq (\text{mult}_L(\mathcal{M}))^2 (-K_X \cdot L) > -n^2 K_X \cdot L.$$

On the other hand, we have  $\text{mult}_L(\mathcal{M}) > n$ . Therefore,  $-K_X \cdot L < -K_X^3$ .

Since the curve  $L$  is contained in the smooth locus of  $X$ , we have  $-K_X \cdot L \geq 1$ . Therefore the curve  $L$  can exist only on the hypersurface  $X$  with  $-K_X^3 > 1$  as a curve of degree less than  $-K_X^3$ . Such conditions can be satisfied only in the following cases:

- quasi-smooth hypersurface of degree 5 in  $\mathbb{P}(1, 1, 1, 1, 2)$  with a curve  $L$  of degree 1 or 2;
- quasi-smooth hypersurface of degree 6 in  $\mathbb{P}(1, 1, 1, 2, 2)$  with a curve  $L$  of degree 1;
- quasi-smooth hypersurface of degree 7 in  $\mathbb{P}(1, 1, 1, 2, 3)$  with a curve  $L$  of degree 1.

Let  $f: Y \rightarrow X$  be the blow up of the ideal sheaf of the curve  $L$ . Then  $Y$  is smooth whenever the curve  $L$  is smooth. As explained in [23, page 207] (it is independent of generality), in each of the three cases listed above, there exists a non-zero nef divisor  $M$  on  $Y$  such that  $M \cdot K_Y^2 \leq 0$ . Corollary 2.2.3 therefore shows that the curve  $L$  must be singular. Consequently, the curve  $L$  must be an irreducible curve of degree 2 in a quasi-smooth hypersurface of degree 5 in  $\mathbb{P}(1, 1, 1, 1, 2)$ . More precisely, the curve  $L$  has either an ordinary double point (which implies that  $Y$  has an ordinary double point on  $E$ ) or  $L$  has a cusp (which implies that  $Y$  has an isolated double point that is locally given by  $x^2 + y^2 + z^2 + t^3 = 0$  in  $\mathbb{C}^4$ ). In both the cases, we can proceed exactly as explained in [23, page 207] (the very end of the proof of [23, Theorem 5.1.1]) to obtain a contradiction.

We are therefore able to draw a conclusion that the hypersurface  $X$  has a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$  at a singular point.

### 3 Excluding singular points

#### 3.1 Method

Let  $p$  be a singular point of type  $\frac{1}{r}(1, a, r-a)$  on  $X$ , where  $r$  and  $a$  are relatively prime positive integers. From now on, the weighted blow up of  $X$  at the point  $p$  with weights  $(1, a, r-a)$  will be denoted by  $f: Y \rightarrow X$ . Its exceptional divisor and its anticanonical divisor will be denoted by  $E$  and  $B$  respectively. We denote by  $\mathcal{M}_Y$  the proper transform of the linear system  $\mathcal{M}$  via  $f$ . The pull-back of  $-K_X$  will be denoted by  $A$ . The surface  $S$  is the proper transform of the surface on  $X$  defined by the equation  $x = 0$ . Since the Picard group of  $X$  is generated by  $-K_X$ , the surface  $S$  is always irreducible. The surface  $S$  may be assumed to be  $\mathbb{Q}$ -linearly equivalent to  $B$  if one of the following conditions holds:

- $a_1 = 1$ ;
- $d - 1$  is not divisible by  $r$ .

If  $a_1 > 1$  and  $d - 1$  is divisible by  $r$ , then one can check that it is always  $\mathbb{Q}$ -linearly equivalent to either  $B$  or  $B - E$ .

Before we explain how to show that  $p$  is not a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ , let us prove

**Lemma 3.1.1.** *Suppose that  $p$  is a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ . Then the one-cycle  $B \cdot S \in N_1(Y)$  lies in the interior of the Mori cone of  $Y$ :*

$$B \cdot S \in \text{Int}(\overline{\text{NE}}(Y)).$$

*Proof.* It follows from Theorem 2.2.1 that

$$\mathcal{M}_Y \sim_{\mathbb{Q}} nB - \epsilon E$$

for some positive rational number  $\epsilon$ . Since

$$S \cdot \mathcal{M}_Y = S \cdot (nB - \epsilon E)$$

is an effective one-cycle, the one-cycle  $B \cdot S$  must lie in the interior of the Mori cone of  $Y$ , because the one-cycles  $S \cdot E$  and  $S \cdot B$  are not proportional in  $N_1(Y)$ , and the one-cycle  $S \cdot E$  generates the extremal ray contracted by  $f$ .  $\square$

We have started with the assumption that the pair  $(X, \frac{1}{n}\mathcal{M})$  is not canonical. What we have proved so far shows that one of the singular points of  $X$  must be a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$ . To exclude singular points with  $B^3 \leq 0$ , we mainly apply the following lemma. It is a slightly modified version of [23, Lemma 5.4.3].

**Lemma 3.1.2.** *Suppose that  $B^3 \leq 0$  and there is an index  $i$  such that*

1. *there is a surface  $T$  on  $Y$  such that  $T \sim_{\mathbb{Q}} a_i A - \frac{m}{r} E$  such that  $a_i \geq m > 0$ ;*
2. *the intersection  $\Gamma = S \cap T$  consists of irreducible curves that are numerically proportional each other;*

3.  $T \cdot \Gamma \leq 0$ .

Then the point  $p$  is not a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$ .

*Proof.* Let  $S \cap T = \sum e_i C_i$ , where  $e_i > 0$  and  $C_i$ 's are distinct irreducible and reduced curves. Let  $R$  be the extremal ray of  $\overline{\text{NE}}(Y)$  contracted by  $f: Y \rightarrow X$ .

Since the curves  $C_i$  are numerically proportional each other, each irreducible curve  $C_i$  defines the same ray in the Mori cone of  $Y$ . We first claim that the ray  $Q$  defined by  $\Gamma$  is the other ray of  $\overline{\text{NE}}(Y)$ , so that  $\overline{\text{NE}}(Y) = R + Q$ . Since  $C_i \not\subset E$  for each  $i$ , we have  $E \cdot C_i \geq 0$ . Therefore, we have the inequalities

$$a_i B \cdot \Gamma \leq T \cdot \Gamma \leq 0,$$

If the surface  $T$  is nef, then  $T \cdot \Gamma = 0$  and hence  $\Gamma$  is in the boundary of  $\overline{\text{NE}}(Y)$ . If the surface  $T$  is not nef, then there is a curve  $C$  with  $T \cdot C < 0$ . The curve  $C$  is not contained in the exceptional divisor  $E$  since

$$T \cdot R = \left( a_i B + \frac{a_i - m}{r} E \right) \cdot R = (mB + (a_i - m)A) \cdot R = mB \cdot R \geq 0.$$

Then

$$S \cdot C \leq B \cdot C = \frac{1}{a_i} \left( T - \frac{a_i - m}{r} E \right) \cdot C < 0,$$

and hence  $C \subset S \cap T$ . Therefore, the curve  $C$  must be one of the component of  $\Gamma$  and hence  $\Gamma$  defines the other boundary ray of  $\overline{\text{NE}}(Y)$ .

If  $B \cdot S \in \text{Int}(\overline{\text{NE}}(Y))$ , then the ray

$$Q = \mathbb{R}_+ \left[ S \cdot \left( a_i B + \frac{a_i - m}{r} E \right) \right]$$

cannot be a boundary of  $\overline{\text{NE}}(Y)$ . Therefore, Lemma 3.1.1 implies that the point  $p$  cannot be a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$ .  $\square$

*Remark 3.1.3.* The condition  $T \cdot \Gamma \leq 0$  is equivalent to the inequality  $ra(r-a)a_i^2 A^3 \leq km^2$ , where  $k = 1$  if  $S \sim_{\mathbb{Q}} B$ ;  $k = r + 1$  otherwise.

We have singular points with  $B^3 \leq 0$  to which we cannot apply Lemma 3.1.2 in a simple way. The paper [23] has dealt with such singular points in a special way. However, we are dealing with every quasi-smooth hypersurface, not only a general one and the method of [23] is too complicated for us to analyze the irreducible components of the intersections  $\Gamma$ , which is inevitable for our purpose. We here present another method that enables us to avoid such difficulty. This method is based on

**Lemma 3.1.4.** *Suppose that there is a nef divisor  $T$  on  $Y$  with  $T \cdot S \cdot B \leq 0$ . Then the point  $p$  cannot be a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$ .*

*Proof.* Suppose that  $p$  is a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$ . Then it follows from Theorem 2.2.1 that

$$\mathcal{M}_Y = f^*(\mathcal{M}) - n\mu E$$

with some rational number  $\mu > \frac{1}{r}$ , where  $\mathcal{M}_Y$  is the proper transform of the mobile linear system  $\mathcal{M}$  by the morphism  $f$ . The intersection of the surface  $S$  and a general surface  $M_Y$  in the mobile linear system  $\mathcal{M}_Y$  gives us an effective 1-cycle. However,

$$T \cdot S \cdot M_Y = nT \cdot S \cdot (A - \mu E) < nT \cdot S \cdot (A - \frac{1}{r}E) = nT \cdot S \cdot B \leq 0.$$

This contradicts the condition that  $T$  is nef.  $\square$

*Remark 3.1.5.* For the divisor  $T$  equivalent to  $cA - \frac{m}{r}E = cB + \frac{c-m}{r}E$  for some positive integers  $c$  and  $m$ , the condition  $T \cdot S \cdot B \leq 0$  is equivalent to the inequality  $ra(r-a)cA^3 \leq km$ , where  $k = 1$  if  $S \sim_{\mathbb{Q}} B$ ;  $k = r + 1$  otherwise.

To apply Lemma 3.1.4, we construct a nef divisor  $T$  in  $|cB + bE|$  for some integers  $c \geq 0$  and  $b \leq \frac{c}{r}$ . To construct a nef divisor  $T$  the following will be useful.

**Lemma 3.1.6.** *Let  $\mathcal{L}_X$  be a mobile linear subsystem in  $|-cK_X|$  for some positive integer  $c$ . Denote the proper transforms of the base curves of the linear system  $\mathcal{L}_X$  on  $Y$  by  $\tilde{C}_1, \dots, \tilde{C}_s$  (if any). Let  $T$  be the proper transform of a general surface in  $\mathcal{L}_X$ .*

- *The divisor  $T$  belongs to  $|cB + bE|$  for some integer  $b$  not greater than  $\frac{c}{r}$ .*
- *The divisor  $T$  is nef if  $T \cdot \tilde{C}_i \geq 0$  for every  $i$ . In particular, it is nef if the base locus of  $\mathcal{L}_X$  contains no curves.*

*Proof.* Let  $H$  be a general surface in  $\mathcal{L}_X$ . Since  $f^*(H) \sim_{\mathbb{Q}} cA - \frac{m}{r}E$  for some non-negative integer  $m$  and  $B \sim_{\mathbb{Q}} A - \frac{1}{r}E$ , we obtain  $T \sim_{\mathbb{Q}} cB + \frac{c-m}{r}E$ . The number  $b := \frac{c-m}{r}$  must be an integer because the divisor class group of  $Y$  is generated by  $B$  and  $E$ .

Suppose that  $T$  is not nef. Then there exists a curve  $\tilde{C} \subset Y$  such that  $T \cdot \tilde{C} < 0$ , which implies that the curve  $\tilde{C}$  is contained in the base locus of the proper transform of the linear system  $\mathcal{L}_X$ . Since  $b \leq \frac{c}{r}$ , the divisor  $T|_E$  is nef, and hence  $\tilde{C} \not\subset E$ . We then draw an absurd conclusion that  $\tilde{C}$  is one of the curves  $\tilde{C}_1, \dots, \tilde{C}_s$ .  $\square$

The singular points with the special care by [23] are exactly the points for which our method works. However, in spite of our new methods, we encounter special cases that cannot not be excluded by the methods proposed so far. To deal with these special cases, we apply the following two lemmas.

**Lemma 3.1.7.** *Suppose that the surface  $S$  is  $\mathbb{Q}$ -linearly equivalent to  $B$  and there is a normal surface  $T$  on  $Y$  such that the support of the one-cycle  $S|_T$  consists of curves on  $T$  whose intersection form is negative-definite. Then the singular point  $p$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .*

*Proof.* Put  $S|_T = \sum c_i C_i$  where  $c_i$ 's are positive numbers and  $C_i$ 's are irreducible and reduced curves on the normal surface  $T$ . Suppose that the point  $p$  is a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ . Then we have

$$K_Y + \frac{1}{n}\mathcal{M}_Y + cE = f^*(K_X + \frac{1}{n}\mathcal{M}) \sim_{\mathbb{Q}} 0,$$



where  $c$  is a positive constant and  $\mathcal{M}_Y$  is the proper transform of the mobile linear system  $\mathcal{M}$ . Therefore, we obtain  $\mathcal{M}_Y + ncE \sim_{\mathbb{Q}} nS$ , and hence

$$(\mathcal{M}_Y + ncE)|_T \sim_{\mathbb{Q}} n \sum c_i C_i.$$

However, this cannot be true since the divisor  $\sum c_i C_i$  is negative-definite and  $nc > 0$ .  $\square$

**Lemma 3.1.8.** *Suppose that there is a one-dimensional family of irreducible curves  $C_\lambda$  on  $Y$  with  $E \cdot C_\lambda > 0$  and  $-K_Y \cdot C_\lambda \leq 0$ . Then the singular point  $p$  cannot be a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$ .*

*Proof.* We have

$$K_Y + \frac{1}{n}\mathcal{M}_Y = f^*(K_X + \frac{1}{n}\mathcal{M}) + cE$$

with a negative number  $c$ , where  $\mathcal{M}_Y$  is the proper transform of the mobile linear system  $\mathcal{M}$ . Suppose that there is a one-dimensional family of curves  $C_\lambda$  on  $Y$  with  $E \cdot C_\lambda > 0$  and  $-K_Y \cdot C_\lambda \leq 0$ . Then for each member  $C_\lambda$ , we have

$$\mathcal{M}_Y \cdot C_\lambda = -nK_Y \cdot C_\lambda + cnE \cdot C_\lambda \leq cnE \cdot C_\lambda < 0,$$

and hence the curve  $C_\lambda$  is contained in the base locus of the linear system  $\mathcal{M}_Y$ . This is a contradiction since the linear system  $\mathcal{M}_Y$  is mobile.  $\square$

Notice that Lemmas 3.1.4, 3.1.7 and 3.1.8 do not require  $B^3$  to be non-positive. Therefore, these lemmas can be applied to exclude the singular points with  $B^3 > 0$ .

## 4 Proof I

### 4.1 How to read the tables I

In this section we exclude singular points of the hypersurfaces in the families corresponding to Theorem 1.1.11. Before we proceed, we introduce a lemma that follows from Lemma 3.1.8 .

**Lemma 4.1.1.** *Suppose that the hypersurface  $X$  is given by a quasi-homogeneous equation*

$$w^2 + x_i t^k + w f_{d-a_4}(x, x_i, x_j, t) + f_d(x, x_i, x_j, t) = 0$$

*of degree  $d$ , where one of the variables  $x_i$  and  $x_j$  is  $y$  and the other is  $z$ . Let  $a_i$  and  $a_j$  be the weights of the variables  $x_i$  and  $x_j$ , respectively. If  $2a_4 = 3a_3 + a_i$ , then the singular point  $O_t$  cannot be a center of non-canonical singularities of  $(X, \frac{1}{n}\mathcal{M})$ .*

*Proof.* The singular point  $O_t$  is of type  $\frac{1}{a_3}(1, a_j, a_4 - a_3)$ . Let  $T$  be the proper transform of the surface  $S_{x_i}$  cut by  $x_i = 0$ . Due to the monomial  $w^2$ , we see that the surface  $S_{x_i}$  has multiplicity  $\frac{2(a_4 - a_3)}{a_3}$  at the point  $O_t$ . Therefore, the surface  $T$  belongs to  $|a_i B - E|$  since  $2a_4 = 3a_3 + a_i$ .

Let  $C_\lambda$  be a curve on the surface  $S_{x_i}$  defined by

$$\begin{cases} x_i = 0, \\ x_j = \lambda x^{a_j} \end{cases}$$

for a sufficiently general complex number  $\lambda$ . Then the curve  $C_\lambda$  is a curve of degree  $d$  in  $\mathbb{P}(1, a_3, a_4)$  defined by the equation

$$w^2 + wf_{d-a_4}(x, 0, \lambda x^{a_j}, t) + f_d(x, 0, \lambda x^{a_j}, t) = 0.$$

Denote by  $\tilde{C}_\lambda$  the proper transform of the curve  $C_\lambda$  on  $Y$ . Then

$$-K_Y \cdot \tilde{C}_\lambda = a_j B^2 \cdot (a_i B - E) = a_1 a_2 A^3 - \frac{2a_j(a_4 - a_3)}{a_3^3} E^3 = \frac{2}{a_3} - \frac{2}{a_3} = 0.$$

If the curve  $\tilde{C}_\lambda$  is reducible, it consists of two irreducible components that are numerically equivalent since the two components of the curve  $C_\lambda$  are symmetric with respect to the biregular quadratic involution of  $X$ . Then each component of  $\tilde{C}_\lambda$  intersects  $-K_Y$  trivially. Consequently, Lemma 3.1.8 implies the statement.  $\square$

In fact, this lemma excludes all the singular points with  $B^3 > 0$  that appear in Theorem 1.1.11.

In what follows, we exclude all the singular points on quasi-smooth hyper surfaces in the families that appear in Theorem 1.1.11.

For each family we present a table that has information on a defining equation of the hypersurface  $X$ , its singularities, the sign of  $B^3$ , the linear system containing a surface  $T$ , the defining equation for the surface  $f(T)$ , and a term that determines the vanishing order of the surface  $f(T)$  at the singular point  $p$ .

Also, each table shows which method is applied to each of the singular points by the symbols  $\textcircled{b}$ ,  $\textcircled{n}$ ,  $\textcircled{s}$ ,  $\textcircled{t}$  and  $\textcircled{p}$ .

The following explain the method corresponding to each of the symbols.

$\textcircled{b}$  : Apply Lemma 3.1.2.

The condition  $T \cdot \Gamma \leq 0$  can be easily checked by the items on the table (see Remark 3.1.3). The condition on the 1-cycle  $\Gamma$  can be immediately checked. This can be done on the hypersurface  $X$  even though the cycle lies on the threefold  $Y$ . Indeed, in the cases where this method is applied, the surface  $T$  is given in such a way that the cycle  $\Gamma$  has no component on the exceptional divisor  $E$ .

$\textcircled{n}$  : Apply Lemma 3.1.4.

Using Lemma 3.1.6 we check that the given divisor  $T$  is nef. The non-positivity of  $T \cdot S \cdot B$  can be immediately verified from the items on the table (see Remark 3.1.5).

$\textcircled{s}$  : Apply Lemma 3.1.7.

We take a general member  $H$  in the linear system generated by the polynomials given in the slot for the item  $T$  in the table. We can easily show that the surface  $H$  is normal by checking it has only isolated singularities. The surface  $S$  cuts out the proper transform  $T$  of the surface  $H$  by the morphism  $f$  into a reducible curve. We check that this reducible curve forms a negative-definite divisor on the normal surface  $T$ .

$\textcircled{t}$  : Apply Lemma 3.1.8.

We find a 1-dimensional family of irreducible curves  $C_\lambda$  on the given surface  $T$  such that  $-K_Y \cdot C_\lambda \leq 0$ . The surface  $T$  can be given as the proper transform of a general member of the linear system generated by the polynomials given in the slot for the item  $T$  in the table.

Ⓟ : Apply Lemma 4.1.1.

In this section, this method excludes all the singular points with  $B^3 > 0$ . Lemma 4.1.1 shows that for such singular points, we can always find a 1-dimensional family of irreducible curves  $C_\lambda$  on the given surface  $T$  such that  $-K_Y \cdot C_\lambda \leq 0$ .

In each table, we present a defining equation of the hypersurface in the family. For this we use the following notations and conventions.

- The Roman alphabets  $a, b, c, d, e$  with numeric subscripts or without subscripts are constants.
- The Greek alphabets  $\alpha, \beta$  with numeric subscripts or without subscripts are constants.
- The same Roman alphabets with distinct numeric subscripts, e.g.,  $a_1, a_2, a_3$ , in an equation are constants one of which is not zero.
- The same Greek alphabets with distinct numeric subscripts, e.g.,  $\alpha_1, \alpha_2, \alpha_3$ , in an equation are distinct constants.
- $f_m(x_{i_1}, \dots, x_{i_k}), g_m(x_{i_1}, \dots, x_{i_k})$  and  $h_m(x_{i_1}, \dots, x_{i_k})$  are quasi-homogeneous polynomials of degree  $m$  in variables  $x_{i_1}, \dots, x_{i_k}$  in a given weighted projective space  $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ .
- If a monomial appears individually in the equation, then the monomial is assumed not to be contained in any other terms. For example, in the equation  $w^2 + t^3 + wf_6(x, y, z, t) + f_{12}(x, y, z, t)$ , the polynomial  $f_{12}$  does not contain the monomial  $t^3$ .
- The singularity types are often given as a form  $\frac{1}{r}(w_{x_{k_1}}^1, w_{x_{k_2}}^2, w_{x_{k_3}}^3)$ , where the subscript  $x_{k_i}$  is the homogeneous coordinate function which induces a local parameter corresponding to the weight  $w^i$ .

## 4.2 Super-rigid families

<b>No. 10:</b> $X_{10} \subset \mathbb{P}(1, 1, 1, 3, 5)$					$A^3 = 2/3$
$w^2 + wf_5(x, y, z, t) + t^3z + t^2f_4(x, y, z) + tf_7(x, y, z) + f_{10}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{3}(1_x, 1_y, 2_w)$ Ⓟ	+	$B - E$	$z$	$w^2$	

<b>No. 11:</b> $X_{10} \subset \mathbb{P}(1, 1, 2, 2, 5)$					$A^3 = 1/2$
$w^2 + \prod_{i=1}^5 (t - \alpha_i z) + wf_5(x, y, z, t) + f_{10}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_z O_t = 5 \times \frac{1}{2}(1_x, 1_y, 1_w)$ Ⓟ	0	$B$	$y$	$y$	

The curve defined by  $x = y = 0$  is irreducible since the defining polynomial of  $X_{10}$  contains

the monomial  $w^2$  and a reduced polynomial  $\prod_{i=1}^5(t - \alpha_i z)$  of degree 10. Therefore, the 1-cycle  $\Gamma$  is irreducible.

<b>No. 14:</b> $X_{12} \subset \mathbb{P}(1, 1, 1, 4, 6)$					$A^3 = 1/2$
$w^2 + t^3 + w f_6(x, y, z, t) + f_{12}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t O_w = 1 \times \frac{1}{2}(1_x, 1_y, 1_z) \textcircled{B}$	0	$B$	$y$	$y$	

The curve defined by  $x = y = 0$  is irreducible because we have the monomials  $w^2$  and  $t^3$  in the homogenous polynomial defining  $X_{12}$ . Therefore, the 1-cycle  $\Gamma$  is irreducible.

<b>No. 19:</b> $X_{12} \subset \mathbb{P}(1, 2, 3, 3, 4)$					$A^3 = 1/6$
$(w - \alpha_1 y^2)(w - \alpha_2 y^2)(w - \alpha_3 y^2) + z(z - \beta_1 t)(z - \beta_2 t) + x f_{11}(x, y, z, t, w) + y f_{10}(y, z, t, w)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_y O_w = 3 \times \frac{1}{2}(1_x, 1_z, 1_t) \textcircled{B}$	–	$3B + E$	$t$	$t$	
$O_z O_t = 4 \times \frac{1}{3}(1_x, 2_y, 1_w) \textcircled{B}$	0	$2B$	$y$	$y$	

The divisor  $T$  for each singular point of type  $\frac{1}{2}(1, 1, 1)$  is nef since the linear system generated by  $xy, z, t$  has no base curve.

The 1-cycle  $\Gamma$  for each singular point of type  $\frac{1}{3}(1, 2, 1)$  is irreducible since the curve cut by  $x = y = 0$  are irreducible.

<b>No. 21:</b> $X_{14} \subset \mathbb{P}(1, 1, 2, 4, 7)$					$A^3 = 1/4$
$w^2 + z(t - \alpha_1 z^2)(t - \alpha_2 z^2)(t - \alpha_3 z^2) + w f_7(x, y, z, t) + t^2 f_6(x, y, z) + t f_{10}(x, y, z) + f_{14}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{4}(1_x, 1_y, 3_w) \textcircled{B}$	+	$2B - E$	$z$	$w^2$	
$O_z O_t = 3 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{B}$	–	$B$	$y$	$y$	

The curve defined by  $x = y = 0$  is irreducible, and hence the 1-cycle  $\Gamma$  for the singularities of type  $\frac{1}{2}(1, 1, 1)$  is irreducible.

<b>No. 22:</b> $X_{14} \subset \mathbb{P}(1, 2, 2, 3, 7)$					$A^3 = 1/6$
$w^2 + t^4 z + h_{14}(y, z) + w f_7(x, y, z, t) + t^3 g_5(x, y, z) + t^2 g_8(x, y, z) + t g_{11}(x, y, z) + x g_{13}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{3}(1_x, 2_y, 1_w) \textcircled{B}$	0	$2B$	$y$	$y$	
$O_y O_z = 7 \times \frac{1}{2}(1_x, 1_t, 1_w) \textcircled{B}$	–	$2B$	$y - \alpha_i z$	$w^2$	

Note that the homogenous polynomial  $h_{14}$  cannot be divisible by  $z$  since the hypersurface  $X_{14}$  is quasi-smooth. Therefore, we may write

$$h_{14}(y, z) = \prod_{i=1}^7 (y - \alpha_i z).$$

The curve defined by  $x = y = 0$  is irreducible because we have the monomials  $w^2$  and  $t^4 z$  in the quasi-homogeneous polynomial defining  $X_{14}$ . The curves defined by  $x = y - \alpha_i z = 0$  are also irreducible for the same reason. Therefore, the 1-cycle  $\Gamma$  for each singular point is irreducible.

<b>No. 28:</b> $X_{15} \subset \mathbb{P}(1, 3, 3, 4, 5)$					$A^3 = 1/12$
$w^3 + t^3 z + h_{15}(y, z) + w^2 f_5(x, y, z, t) + w f_{10}(x, y, z, t) + t^2 g_7(x, y, z) + t g_{11}(x, y, z) + x g_{14}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = 1 \times \frac{1}{4}(1_x, 3_y, 1_w) \textcircled{D}$	0	$3B$	$y$	$y$	
$O_y O_z = 5 \times \frac{1}{3}(1_x, 1_t, 2_w) \textcircled{D}$	-	$3B$	$y - \alpha_i z$	$t^3 z$	

Note that the homogenous polynomial  $h_{15}$  cannot be divisible by  $z$  since the hypersurface  $X_{15}$  is quasi-smooth. Therefore, we may write

$$h_{15}(y, z) = \prod_{i=1}^5 (y - \alpha_i z).$$

The curve defined by  $x = y = 0$  is irreducible because we have the monomials  $w^2$  and  $t^4 z$  in the quasi-homogeneous polynomial defining  $X_{15}$ . The curves defined by  $x = y - \alpha_i z = 0$  are also irreducible for the same reason. Therefore, the 1-cycle  $\Gamma$  for each singular point is irreducible.

<b>No. 29:</b> $X_{16} \subset \mathbb{P}(1, 1, 2, 5, 8)$					$A^3 = 1/5$
$(w - \alpha_1 z^4)(w - \alpha_2 z^4) + t^3 y + a t^2 z^3 + w f_8(x, y, z, t) + t^2 f_6(x, y, z) + t f_{11}(x, y, z) + f_{16}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{5}(1_x, 2_z, 3_w) \textcircled{D}$	+	$B - E$	$y$	$w^2$	
$O_z O_w = 2 \times \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{D}$	-	$B$	$y$	$y$	$a \neq 0$
$O_z O_w = 2 \times \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{S}$	-	$B$	$x, y$	$x, y$	$a = 0$

If the constant  $a$  is nonzero, then the 1-cycle  $\Gamma$  for each singular point of type  $\frac{1}{2}(1, 1, 1)$  is irreducible.

Suppose that  $a = 0$ . We have only to consider one of the singular points of type  $\frac{1}{2}(1, 1, 1)$ . The other singular point can be excluded in the same way. Moreover, we may assume that the singular point is located at the point  $O_z$  by a suitable coordinate change.

We take a general surface  $H$  from the pencil  $|-K_{X_{16}}|$  and then let  $T$  be the proper transform

of the surface. It is a K3 surface only with du Val singularities. The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves on the normal surface  $T$ . One is the proper transform  $\tilde{L}_{zt}$  on  $T$  of the curve  $L_{zt}$  on  $H$ . The other is the proper transform  $\tilde{C}$  on  $T$  of the curve  $C$  defined by  $x = y = w - (\alpha_2 - \alpha_1)z^4 = 0$ .

From the intersection numbers

$$(\tilde{L}_{zt} + \tilde{C}) \cdot \tilde{L}_{zt} = -K_Y \cdot \tilde{L}_{zt} = -\frac{2}{5}, \quad (\tilde{L}_{zt} + \tilde{C})^2 = B^3 = -\frac{3}{10}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{zt}^2 = -\frac{2}{5} - \tilde{L}_{zt} \cdot \tilde{C}, \quad \tilde{C}^2 = \frac{1}{10} - \tilde{L}_{zt} \cdot \tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zt}^2 & \tilde{L}_{zt} \cdot \tilde{C} \\ \tilde{L}_{zt} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} - \tilde{L}_{zt} \cdot \tilde{C} & \tilde{L}_{zt} \cdot \tilde{C} \\ \tilde{L}_{zt} \cdot \tilde{C} & \frac{1}{10} - \tilde{L}_{zt} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{yw} \cdot \tilde{C} = \frac{4}{5}$ .

<b>No. 34:</b> $X_{18} \subset \mathbb{P}(1, 1, 2, 6, 9)$					$A^3 = 1/6$
$w^2 + t^3 + z^9 + wf_9(x, y, z, t) + f_{18}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t O_w = 1 \times \frac{1}{3}(1_x, 1_y, 2_z) \textcircled{D}$	0	$B$	$y$	$y$	
$O_z O_t = 3 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{D}$	-	$B$	$y$	$y$	

The curve defined by  $x = y = 0$  is irreducible always since we have the monomials  $w^2$  and  $t^3$ . Therefore, the 1-cycle  $\Gamma$  for each singular point is irreducible.

<b>No. 35:</b> $X_{18} \subset \mathbb{P}(1, 1, 3, 5, 9)$					$A^3 = 2/15$
$w^2 + t^3 z + wf_9(x, y, z, t) + t^2 f_8(x, y, z) + t f_{13}(x, y, z) + f_{18}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{5}(1_x, 1_y, 4_w) \textcircled{D}$	+	$3B - E$	$z$	$w^2$	
$O_z O_w = 2 \times \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{D}$	-	$B$	$y$	$y$	

The 1-cycle  $\Gamma$  for the singular points of type  $\frac{1}{2}(1, 1, 1)$  is irreducible since we have  $w^2$  and  $t^3 z$ .

<b>No. 37:</b> $X_{18} \subset \mathbb{P}(1, 2, 3, 4, 9)$					$A^3 = 1/12$
$(w - \beta_1 z^3)(w - \beta_2 z^3) + y \prod_{i=1}^4 (t - \alpha_i y^2) + at^3 z^2 + wf_9(x, y, z, t) + t^3 f_6(x, y, z) + t^2 f_{10}(x, y, z) + t f_{14}(x, y, z) + f_{18}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{4}(1_x, 3_z, 1_w) \textcircled{D}$	0	$2B$	$y$	$w^2$	

$O_z O_w = 2 \times \frac{1}{3}(1_x, 2_y, 1_t)$ ⑥	–	$2B$	$y$	$y$	$a \neq 0$
$O_z O_w = 2 \times \frac{1}{3}(1_x, 2_y, 1_t)$ ⑤	–	$2B$	$x^2, y$	$x^2, y$	$a = 0$
$O_y O_t = 4 \times \frac{1}{2}(1_x, 1_z, 1_w)$ ⑥	–	$4B + E$	$t - \alpha_i y^2$	$w^2$	

For the singular point  $O_t$ , the 1-cycle  $\Gamma$  can be reducible. In case, we see that  $\Gamma$  consists of the proper transforms of the curves defined by  $x = y = w - \beta_1 z^3 = 0$  and  $x = y = w - \beta_2 z^3 = 0$ . These two irreducible components are symmetric with respect to the biregular involution of  $X_{18}$ . In addition, the point  $O_t$  is the intersection point of these two curves. Consequently, the components of  $\Gamma$  are numerically equivalent to each other.

For each singular point of type  $\frac{1}{3}(1, 2, 1)$ , the 1-cycle  $\Gamma$  is irreducible if the constant  $a$  is not zero. Suppose that the constant  $a$  is zero. We have only to consider one of the singular points of type  $\frac{1}{3}(1, 2, 1)$ . The other singular point can be excluded in the same way. To this purpose, we may put  $\beta_1 = 0$  and consider the point singular point  $O_z$ . We may also assume that the defining equation of  $X_{18}$  contains neither  $xz^3t^2$  nor  $x^2z^4t$  by changing the coordinate  $w$ .

We take a general surface  $H$  from the pencil  $|-2K_{X_{18}}|$  and then let  $T$  be the proper transform of the surface. Note that the surface  $H$  is normal. However, it is not quasi-smooth at the point  $O_t$ . The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves on the normal surface  $T$ . They are the proper transforms  $\tilde{L}_{zt}$  and  $\tilde{C}$  on  $T$  of the curve  $L_{zt}$  and the curve  $C$  defined by  $x = y = w - \beta_2 z^3 = 0$ , respectively. From the intersection numbers

$$(\tilde{L}_{zt} + \tilde{C}) \cdot \tilde{L}_{zt} = -K_Y \cdot \tilde{L}_{zt} = -\frac{1}{4}, \quad (\tilde{L}_{zt} + \tilde{C})^2 = 2B^3 = -\frac{1}{6}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{zt}^2 = -\frac{1}{4} - \tilde{L}_{zt} \cdot \tilde{C}, \quad \tilde{C}^2 = \frac{1}{12} - \tilde{L}_{zt} \cdot \tilde{C}.$$

To compute the intersection number  $\tilde{L}_{zt} \cdot \tilde{C}$ , we consider the divisor  $D_w$  on  $H$  cut by the equation  $w = 0$ . We easily see that  $D_w = 2L_{zt} + R$ , where  $R$  is a curve whose support does not contain  $L_{zt}$ . The curve  $R$  and  $L_{zt}$  intersects at the point  $O_z$ . Let  $\tilde{R}$  be the proper transform of  $R$ . Then we have  $\tilde{L}_{zt} \cdot \tilde{R} = 0$  since they are disconnected on  $T$ . From the intersection

$$(2\tilde{L}_{zt} + \tilde{R}) \cdot \tilde{L}_{zt} = (9B + E) \cdot \tilde{L}_{zt} = -\frac{5}{4}$$

we obtain  $\tilde{L}_{zt}^2 = -\frac{5}{8}$ . With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zt}^2 & \tilde{L}_{zt} \cdot \tilde{C} \\ \tilde{L}_{zt} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} - \tilde{L}_{zt} \cdot \tilde{C} & \tilde{L}_{zt} \cdot \tilde{C} \\ \tilde{L}_{zt} \cdot \tilde{C} & \frac{1}{12} - \tilde{L}_{zt} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{zt} \cdot \tilde{C} = \frac{3}{8}$ .

For each singular point of type  $\frac{1}{2}(1, 1, 1)$ , the 1-cycle  $\Gamma$  may be reducible. In case, the curve  $\Gamma$  consists of the proper transforms of the curves defined by  $x = t - \alpha_i y^2 = w + by^3z + cz^3 = 0$  and  $x = t - \alpha_i y^2 = w + dy^3z + ez^3 = 0$ , where  $b, c, d, e$  are constants. These two irreducible components are also symmetric with respect to the biregular involution of  $X_{18}$ . In addition, the singular point is the intersection point of these two curves. Therefore, the components of  $\Gamma$  are numerically equivalent.

<b>No. 39:</b> $X_{18} \subset \mathbb{P}(1, 3, 4, 5, 6)$				$A^3 = 1/20$	
$(w - \alpha_1 y^2)(w - \alpha_2 y^2)(w - \alpha_3 y^2) + yt^3 + z^3 w + at^2 z^2 + by^2 z^3 + w^2 f_6(x, y, z, t) +$ $wf_{12}(x, y, z, t) + t^2 f_8(x, y, z) + t f_{13}(x, y, z) + f_{18}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{5}(1_x, 4_z, 1_w) \textcircled{B}$	0	$3B$	$y$	$w^3$	
$O_z = \frac{1}{4}(1_x, 3_y, 1_t) \textcircled{B}$	–	$3B$	$y$	$y$	$a \neq 0$
$O_z = \frac{1}{4}(1_x, 3_y, 1_t) \textcircled{S}$	–	$3B$	$x^3, y$	$x^3, y$	$a = 0$
$O_z O_w = 1 \times \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{U}$	–	$5B + 2E$	$t$	$t$	
$O_y O_w = 3 \times \frac{1}{3}(1_x, 1_z, 2_t) \textcircled{U}$	–	$5B + E$	$t$	$t$	$b \neq 0$
$O_y O_w = 3 \times \frac{1}{3}(1_x, 1_z, 2_t) \textcircled{S}$	–	$5B + E$	$x^5, xz, t$	$t$	$b = 0$

For the singular point  $O_t$ , the 1-cycle  $\Gamma$  may be reducible. However, in case, it consists of two irreducible components. One is the proper transform  $\tilde{L}_{zt}$  of the curve  $L_{zt}$ , and the other is the proper transform  $\tilde{C}$  of the curve defined by  $x = y = w^2 + z^3 = 0$ . We can easily check that

$$E \cdot \tilde{C} = 2E \cdot \tilde{L}_{zt} = \frac{1}{2}, \quad B \cdot \tilde{C} = 2B \cdot \tilde{L}_{zt} = 0.$$

Therefore, the irreducible curves  $\tilde{L}_{zt}$  and  $\tilde{C}$  are numerically proportional on  $Y$ .

The 1-cycle  $\Gamma$  for the singular point  $O_z$  with  $a \neq 0$  is irreducible due to  $w^3$  and  $t^2 z^2$ .

For the singular point  $O_z$  with  $a = 0$  we may assume that the defining equation of  $X_{18}$  contains neither  $xz^3 t$  nor  $x^2 z^4$  by changing the coordinate  $w$ .

We take a general surface  $H$  from the pencil  $| -3K_{X_{18}} |$  and then let  $T$  be the proper transform of the surface. Note that the surface  $H$  is normal. However, it is not quasi-smooth at the point  $O_t$ .

The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves  $\tilde{L}_{zt}$  and  $\tilde{C}$  on the normal surface  $T$ . The curve  $\tilde{L}_{zt}$  is the proper transform of the curve  $L_{zt}$  and the curve  $\tilde{C}$  is the proper transform of the curve  $C$  defined by  $x = y = w^2 + z^3 = 0$ . From the intersection numbers

$$(\tilde{L}_{zt} + \tilde{C}) \cdot \tilde{L}_{zt} = -K_Y \cdot \tilde{L}_{zt} = -\frac{1}{5}, \quad (\tilde{L}_{zt} + \tilde{C})^2 = 3B^3 = -\frac{1}{10}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{zt}^2 = -\frac{1}{5} - \tilde{L}_{zt} \cdot \tilde{C}, \quad \tilde{C}^2 = \frac{1}{10} - \tilde{L}_{zt} \cdot \tilde{C}.$$

To compute the intersection number  $\tilde{L}_{zt} \cdot \tilde{C}$ , we consider the divisor  $D_w$  on  $H$  cut by the equation  $w = 0$ . We easily see that  $D_w = 3L_{zt} + R$ , where  $R$  is a curve whose support does not contain  $L_{zt}$ . The curve  $R$  and  $L_{zt}$  intersects at the point  $O_z$ . Let  $\tilde{R}$  be the proper transform of  $R$ . Then we have  $\tilde{L}_{zt} \cdot \tilde{R} = 0$  since they are disconnected on  $T$ . From the intersection

$$(3\tilde{L}_{zt} + \tilde{R}) \cdot \tilde{L}_{zt} = -6K_Y \cdot \tilde{L}_{zt} = -\frac{6}{5}$$

we obtain  $\tilde{L}_{zt}^2 = -\frac{2}{5}$ . With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zt}^2 & \tilde{L}_{zt} \cdot \tilde{C} \\ \tilde{L}_{zt} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} - \tilde{L}_{zt} \cdot \tilde{C} & \tilde{L}_{zt} \cdot \tilde{C} \\ \tilde{L}_{zt} \cdot \tilde{C} & \frac{1}{10} - \tilde{L}_{zt} \cdot \tilde{C} \end{pmatrix}$$



is negative-definite since  $\tilde{L}_{zt} \cdot \tilde{C} = \frac{1}{5}$ .

For the singular point of type  $\frac{1}{2}(1, 1, 1)$  we consider the linear system generated by  $x^{15}$ ,  $y^5$  and  $t^3$  on the hypersurface  $X_{18}$ . Its base locus is cut out by  $x = y = t = 0$ . Since we have the monomial  $z^3w$ , the base locus does not contain curves. Therefore the proper transform of a general member in the linear system is nef by Lemma 3.1.6 and it belongs to  $|15B + 6E|$ . Consequently, the surface  $T$  is nef since  $3T \sim_{\mathbb{Q}} 15B + 6E$ .

For the singular points of type  $\frac{1}{3}(1, 1, 2)$  we may assume that  $\alpha_1 = 0$  and consider the singular point  $O_y$ . The other points can be dealt with in the same way.

Since  $\alpha_1 = 0$ , the defining equation of  $X_{18}$  does not contain the monomial  $y^6$ . We may also assume that it does not contain the monomials  $x^6y^4$ ,  $x^3y^5$ ,  $x^2y^4z$  and  $xy^4t$ .

For the singular point  $O_y$  with  $b \neq 0$  we consider the linear system generated by  $x^{30}$ ,  $t^5$  and  $w^6$  on the hypersurface  $X_{18}$ . Its base locus is cut out by  $x = t = w = 0$ . Since  $b \neq 0$ , its base locus does not contain curves. Therefore the proper transform of a general member in the linear system is nef by Lemma 3.1.6 and it belongs to  $|30B + 6E|$ . Consequently, the surface  $T$  is nef since  $6T \sim_{\mathbb{Q}} 30B + 6E$ .

For the singular point  $O_y$  with  $b = 0$  we take a general surface  $H$  from the linear system generated by  $x^5$ ,  $xz$  and  $t$ . Then  $H$  is normal. Moreover, the surface  $H$  is smooth at the point  $x = t = w = z^3 + \alpha_1\alpha_2y^4 = 0$ . Indeed, the defining equation of  $X_{18}$  must contain at least one of the monomials  $xz^2y^3$ ,  $ty^3z$ ; otherwise  $X_{18}$  would be singular at the point  $x = t = w = z^3 + \alpha_1\alpha_2y^4 = 0$ . Plugging in  $\lambda xz + \mu x^5$  with general complex numbers  $\lambda$  and  $\mu$  instead of  $t$  into the defining equation of  $X_{18}$ , we obtain the defining equation of  $H$  in  $\mathbb{P}(1, 3, 4, 6)$ . It must contain the monomial  $xz^2y^3$ . Therefore, the surface  $H$  is smooth at the point  $x = t = w = z^3 + \alpha_1\alpha_2y^4 = 0$ .

Let  $T$  be the proper transform of the surface  $H$ . The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves  $\tilde{L}_{yz}$  and  $\tilde{C}$  on the normal surface  $T$ . The curve  $\tilde{L}_{yz}$  is the proper transform of the curve  $L_{yz}$  and the curve  $\tilde{C}$  is the proper transform of the curve  $C$  defined by

$$x = t = z^3 + (w - \alpha_2y^2)(w - \alpha_3y^2) = 0.$$

The curves  $L_{yz}$  and  $C$  intersect at the point defined by  $x = t = w = z^3 + \alpha_1\alpha_2y^4 = 0$ . From the intersection numbers

$$(\tilde{L}_{yz} + \tilde{C}) \cdot \tilde{L}_{yz} = -K_Y \cdot \tilde{L}_{yz} = -\frac{1}{4}, \quad (\tilde{L}_{yz} + \tilde{C})^2 = B^2 \cdot (5B + E) = -\frac{1}{12}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{yz}^2 = -\frac{1}{4} - \tilde{L}_{yz} \cdot \tilde{C}, \quad \tilde{C}^2 = \frac{1}{6} - \tilde{L}_{yz} \cdot \tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yz}^2 & \tilde{L}_{yz} \cdot \tilde{C} \\ \tilde{L}_{yz} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} - \tilde{L}_{yz} \cdot \tilde{C} & \tilde{L}_{yz} \cdot \tilde{C} \\ \tilde{L}_{yz} \cdot \tilde{C} & \frac{1}{6} - \tilde{L}_{yz} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{yz} \cdot \tilde{C} \geq 1$ .

<b>No. 49:</b> $X_{21} \subset \mathbb{P}(1, 3, 5, 6, 7)$	$A^3 = 1/30$
$w^3 + t^3y + z^3(a_1t + a_2xz) + w^2f_7(x, y, z, t) + wf_{14}(x, y, z, t) + t^2f_9(x, y, z) +$ $tf_{15}(x, y, z) + f_{21}(x, y, z)$	

Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{6}(1_x, 5_z, 1_w) \textcircled{B}$	0	$3B$	$y$	$w^3$	
$O_z = \frac{1}{5}(1_x, 3_y, 2_w) \textcircled{B}$	0	$3B$	$y$	$y$	$a_1 \neq 0$
$O_z = \frac{1}{5}(3_y, 1_t, 2_w) \textcircled{B}$	0	$3B$	$y$	$y$	$a_1 = 0$
$O_y O_t = 3 \times \frac{1}{3}(1_x, 2_z, 1_w) \textcircled{D}$	–	$5B + E$	$z$	$z$	

For the singular points of types  $\frac{1}{6}(1, 5, 1)$  and  $\frac{1}{5}(1, 3, 2)$ , the 1-cycle  $\Gamma$  is the proper transform of the curve defined by  $x = y = w^3 + a_1 z^3 t = 0$ . It is irreducible even though it can be non-reduced.

For the singular points of type  $\frac{1}{3}(1, 2, 1)$ , consider the linear system on  $X_{21}$  generated by  $x^2 y$  and  $z$ . Its base curves are defined by  $x = z = 0$  and  $y = z = 0$ . The curve defined by  $x = z = 0$  is irreducible because of the monomials  $w^3$  and  $yt^3$ . For its proper transform  $\tilde{C}$ , we have  $T \cdot \tilde{C} = \frac{1}{6}$ . Therefore, the divisor  $T$  is nef since the curve defined by  $y = z = 0$  does not pass through any singular point of type  $\frac{1}{3}(1, 2, 1)$ .

<b>No. 50:</b> $X_{22} \subset \mathbb{P}(1, 1, 3, 7, 11)$					$A^3 = 2/21$
$w^2 + t^3 y + z^5(a_1 t + a_2 x z^2 + a_3 y z^2) + w f_{11}(x, y, z, t) + t^2 f_8(x, y, z) + t f_{15}(x, y, z) + f_{22}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{7}(1_x, 3_z, 4_w) \textcircled{D}$	+	$B - E$	$y$	$w^2$	
$O_z = \frac{1}{3}(1_x, 1_y, 2_w) \textcircled{B}$	–	$B$	$y$	$y$	$a_1 \neq 0$
$O_z = \frac{1}{3}(1_x, 1_t, 2_w) \textcircled{B}$	–	$B - E$	$a_2 x + a_3 y$	$w^2$	$a_1 = 0$

If  $a_1 = 0$ , then  $a_2 \neq 0$ : otherwise the hypersurface  $X_{22}$  would be singular at the point defined by  $x = y = w = 0$  and  $t^3 + a_3 z^7 = 0$ .

If  $a_1 \neq 0$ , the 1-cycle  $\Gamma$  for the singular point  $O_z$  is irreducible because of the monomials  $w^2$  and  $z^5 t$ . If  $a_1 = 0$ , the 1-cycle  $\Gamma$  for the singular point  $O_z$  is still irreducible even though it is not reduced.

<b>No. 51:</b> $X_{22} \subset \mathbb{P}(1, 1, 4, 6, 11)$					$A^3 = 1/12$
$w^2 + t^3 z + z^4 t + w f_{11}(x, y, z, t) + t^2 f_{10}(x, y, z) + t f_{16}(x, y, z) + f_{22}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{6}(1_x, 1_y, 5_w) \textcircled{D}$	+	$4B - E$	$z$	$w^2$	
$O_z = \frac{1}{4}(1_x, 1_y, 3_w) \textcircled{D}$	0	$B$	$y$	$y$	
$O_z O_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{D}$	–	$B$	$y$	$y$	

For the singular point  $O_z$ , we can easily see that the surface  $T$  is nef since the base locus of the linear system  $| -K_{X_{22}} |$  is the irreducible curve cut by  $x = y = 0$  and  $B^3 = 0$ .

For the singular point of type  $\frac{1}{2}(1, 1, 1)$ , the intersection  $\Gamma$  is irreducible since we have the monomials  $w^2$  and  $t^3 z$ .

<b>No. 52:</b> $X_{22} \subset \mathbb{P}(1, 2, 4, 5, 11)$					$A^3 = 1/20$
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$w^2 + yt^4 + y \prod_{i=1}^5 (z - \alpha_i y^2) + wf_{11}(x, y, z, t) + f_{22}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{5}(1_x, 4_z, 1_w) \textcircled{B}$	0	$4B$	$z$	$z$	
$O_z = \frac{1}{4}(1_x, 1_t, 3_w) \textcircled{B}$	–	$5B + E$	$t$	$t$	
$O_y O_z = 5 \times \frac{1}{2}(1_x, 1_t, 1_w) \textcircled{B}$	–	$4B + E$	$z - \alpha_i y^2$	$w^2$	

For the singular points of types  $\frac{1}{5}(1, 4, 1)$  and  $\frac{1}{2}(1, 1, 1)$ , the 1-cycle  $\Gamma$  is always irreducible because the defining polynomial of  $X_{22}$  contains the monomials  $w^2$ ,  $yt^4$  and  $yz^5$ .

For the singularity  $O_z$ , consider the linear system on  $X_{22}$  generated by  $xz$  and  $t$ . Its base curves are defined by  $x = t = 0$  and  $z = t = 0$ . The curve defined by  $x = t = 0$  is irreducible because of the monomials  $w^2$  and  $yz^5$ . Its proper transform intersects the divisor  $T$  positively. Therefore, the divisor  $T$  is nef since the curve defined by  $z = t = 0$  does not pass through  $O_z$ .

<b>No. 53:</b> $X_{24} \subset \mathbb{P}(1, 1, 3, 8, 12)$ <span style="float: right;"><math>A^3 = 1/12</math></span>					
$w^2 + t^3 + z^8 + wf_{12}(x, y, z, t) + f_{24}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t O_w = 1 \times \frac{1}{4}(1_x, 1_y, 3_z) \textcircled{B}$	0	$B$	$y$	$y$	
$O_z O_w = 2 \times \frac{1}{3}(1_x, 1_y, 2_t) \textcircled{B}$	–	$B$	$y$	$y$	

The 1-cycle  $\Gamma$  for each singular point is irreducible because of the monomials  $w^2$ ,  $z^8$  and  $t^3$ .

<b>No. 55:</b> $X_{24} \subset \mathbb{P}(1, 2, 3, 7, 12)$ <span style="float: right;"><math>A^3 = 1/21</math></span>					
$(w - \alpha_1 y^6)(w - \alpha_2 y^6) + zt^3 + wf_{12}(x, y, z, t) + t^2 f_{10}(x, y, z) + t f_{17}(x, y, z) + f_{24}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{7}(1_x, 2_y, 5_w) \textcircled{D}$	+	$3B - E$	$z$	$w^2$	
$O_z O_w = 2 \times \frac{1}{3}(1_x, 2_y, 1_t) \textcircled{B}$	–	$2B$	$y$	$y$	
$O_y O_w = 2 \times \frac{1}{2}(1_x, 1_z, 1_t) \textcircled{B}$	–	$7B + 3E$	$t$	$t$	

The intersection  $\Gamma$  for each singular point of type  $\frac{1}{3}(1, 2, 1)$  is irreducible because of the monomials  $w^2$  and  $zt^3$ .

For each singular point of type  $\frac{1}{2}(1, 1, 1)$ , the divisor  $T$  is nef. Indeed, the base curve of the linear system on  $X_{24}$  generated by  $xy^3$ ,  $y^2z$  and  $t$  is cut out by  $y = t = 0$ . It does not pass through any singular point of type  $\frac{1}{2}(1, 1, 1)$ . Therefore, the surface  $T$  must be nef.

<b>No. 57:</b> $X_{24} \subset \mathbb{P}(1, 3, 4, 5, 12)$ <span style="float: right;"><math>A^3 = 1/30</math></span>					
$(w - \alpha_1 y^4)(w - \alpha_2 y^4) + zt^4 + z^6 + wf_{12}(x, y, z, t) + f_{24}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{5}(1_x, 3_y, 2_w) \textcircled{B}$	0	$3B$	$y$	$y$	
$O_z O_w = 2 \times \frac{1}{4}(1_x, 3_y, 1_t) \textcircled{B}$	–	$3B$	$y$	$y$	

$O_y O_w = 2 \times \frac{1}{3}(1_x, 1_z, 2_t) \textcircled{\text{D}}$	—	$4B + E$	$z$	$z$	
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The intersections  $\Gamma$  for the singular points of types  $\frac{1}{5}(1, 3, 2)$  and  $\frac{1}{4}(1, 3, 1)$  are irreducible because of the monomials  $w^2$  and  $zt^4$ .

For each singular point of type  $\frac{1}{3}(1, 1, 2)$  we consider the linear system generated by  $x^{20}$ ,  $z^5$  and  $t^4$  on the hypersurface  $X_{24}$ . Its base locus is cut out by  $x = z = t = 0$ . Since the defining equation of  $X_{24}$  contains the monomial  $wy^4$ , its base locus does not contain curves. Therefore the proper transform of a general member in the linear system is nef by Lemma 3.1.6 and it belongs to  $|20B + 5E|$ . Consequently, the surface  $T$  is nef since  $5T \sim_{\mathbb{Q}} 20B + 5E$ .

<b>No. 59:</b> $X_{24} \subset \mathbb{P}(1, 3, 6, 7, 8)$					$A^3 = 1/42$
$w^3 + yt^3 + \prod_{i=1}^4 (z - \alpha_i y^2) + w^2 f_8(x, y, z, t) + w f_{16}(x, y, z, t) + f_{24}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{7}(1_x, 6_z, 1_w) \textcircled{\text{D}}$	0	$3B$	$y$	$w^3$	
$O_z O_w = 1 \times \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{\text{D}}$	—	$3B + E$	$y$	$y$	
$O_y O_z = 4 \times \frac{1}{3}(1_x, 1_t, 2_w) \textcircled{\text{D}}$	—	$6B + E$	$z - \alpha_i y^2$	$yt^3$	

The 1-cycle  $\Gamma$  for each singular point is irreducible due to the monomials  $w^3$ ,  $z^4$  and  $yt^3$ .

<b>No. 62:</b> $X_{26} \subset \mathbb{P}(1, 1, 5, 7, 13)$					$A^3 = 2/35$
$w^2 + zt^3 + yz^5 + w f_{13}(x, y, z, t) + f_{26}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{7}(1_x, 1_y, 6_w) \textcircled{\text{D}}$	+	$5B - E$	$z$	$w^2$	
$O_z = \frac{1}{5}(1_x, 2_t, 3_w) \textcircled{\text{E}}$	+	$B - E$	$y$	$w^2$	

For the singular point  $O_z$ , let  $C_\lambda$  be a curve on the surface  $S_y$  defined by

$$\begin{cases} y = 0, \\ t = \lambda x^7 \end{cases}$$

for a sufficiently general complex number  $\lambda$ . Denote by  $\tilde{C}_\lambda$  the proper transform of the curve  $C_\lambda$  on  $Y$ . Then

$$-K_Y \cdot \tilde{C}_\lambda = (B - E)(7B + E)B = 0.$$

If the curve  $\tilde{C}_\lambda$  is reducible, it consists of two irreducible components that are numerically equivalent since the two components of the curve  $C_\lambda$  are symmetric with respect to the biregular quadratic involution of  $X_{26}$ . Then each component of  $\tilde{C}_\lambda$  intersects  $-K_Y$  trivially.

<b>No. 63:</b> $X_{26} \subset \mathbb{P}(1, 2, 3, 8, 13)$					$A^3 = 1/24$
$w^2 + y(t - \alpha_1 y^4)(t - \alpha_2 y^4)(t - \alpha_3 y^4) + z^6(a_1 t + a_2 y z^2) + w f_{13}(x, y, z, t) + t^2 f_{10}(x, y, z) + t f_{18}(x, y, z) + f_{26}(x, y, z)$					

Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{8}(1_x, 3_z, 5_w) \textcircled{D}$	+	$2B - E$	$y$	$w^2$	
$O_z = \frac{1}{3}(1_x, 2_y, 1_w) \textcircled{B}$	-	$2B$	$y$	$y$	$a_1 \neq 0$
$O_z = \frac{1}{3}(1_x, 2_t, 1_w) \textcircled{B}$	-	$2B$	$y$	$w^2$	$a_1 = 0$
$O_y O_t = 3 \times \frac{1}{2}(1_x, 1_z, 1_w) \textcircled{D}$	-	$3B + E$	$z$	$z$	

For each of the singular points of types  $\frac{1}{3}(1, 2, 1)$  and  $\frac{1}{2}(1, 1, 1)$ , the 1-cycle  $\Gamma$  is always irreducible, because of the monomials  $w^2$ ,  $yt^3$  and  $z^6t$ , even though it is possibly non-reduced.

No. 64: $X_{26} \subset \mathbb{P}(1, 2, 5, 6, 13)$					$A^3 = 1/30$
$w^2 + y \prod_{i=1}^4 (t - \alpha_i y^3) + z^4(a_1 t + a_2 x) + y f_{24}(y, z, t, w) + x g_{25}(x, y, z, t, w)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{6}(1_x, 5_z, 1_w) \textcircled{B}$	0	$2B$	$y$	$w^2$	
$O_z = \frac{1}{5}(1_x, 2_y, 3_w) \textcircled{B}$	0	$2B$	$y$	$y$	$a_1 \neq 0$
$O_z = \frac{1}{5}(2_y, 1_t, 3_w) \textcircled{B}$	0	$2B$	$y$	$y$	$a_1 = 0$
$O_y O_t = 4 \times \frac{1}{2}(1_x, 1_z, 1_w) \textcircled{B}$	-	$6B + 2E$	$t - \alpha_i y^3$	$w^2$	

For each of the singular points the 1-cycle  $\Gamma$  is irreducible due to the monomials  $w^2$  and  $z^4t$ .

No. 66: $X_{27} \subset \mathbb{P}(1, 5, 6, 7, 9)$					$A^3 = 1/70$
$w^3 + wz^3 + zt^3 + y^4t + ay^3z^2 + w^2f_9(x, y, z, t) + wf_{18}(x, y, z, t) + f_{27}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{7}(1_x, 5_y, 2_w) \textcircled{B}$	0	$5B$	$y$	$y$	
$O_z = \frac{1}{6}(1_x, 5_y, 1_t) \textcircled{B}$	-	$5B$	$y$	$y$	
$O_y = \frac{1}{5}(1_x, 1_z, 4_w) \textcircled{B}$	-	$7B + E$	$t$	$y^3z^2$	$a \neq 0$
$O_y = \frac{1}{5}(1_x, 1_z, 4_w) \textcircled{S}$	-	$7B$	$x^7, t$	$x^7, wz^3$	$a = 0$
$O_z O_w = 1 \times \frac{1}{3}(1_x, 2_y, 1_t) \textcircled{B}$	-	$5B + E$	$y$	$y$	

We may assume that the polynomial  $f_{27}$  contains neither  $xy^4z$  nor  $x^2y^5$  by changing the coordinate  $t$  in an appropriate way.

For the singular points except the point  $O_y$ , the 1-cycles  $\Gamma$  are always irreducible because of the monomials  $w^3$ ,  $zt^3$  and  $z^3w$ .

For the singularity  $O_y$  with  $a \neq 0$ , we consider the linear system generated by  $x^2y$ ,  $xz$  and  $t$ . Its base curve  $C$  is cut out by  $x = t = 0$ . It is irreducible because of the monomials  $w^3$  and  $y^3z^2$ . Since we have  $T \cdot \tilde{C} = (7B + E)^2 \cdot B = \frac{1}{2}$ , where  $\tilde{C}$  is the proper transform of  $C$ , the divisor  $T$  is nef.

For the singularity  $O_y$  with  $a = 0$ , we may assume that the defining equation of  $X_{27}$  contains neither  $xy^4z$  nor  $x^2y^5$  by changing the coordinate  $t$ .

Take a general member  $H$  in the linear system generated by  $x^7$  and  $t$ . Then it is a normal surface of degree 27 in  $\mathbb{P}(1, 5, 6, 9)$ . Let  $T$  be the proper transform of the surface  $H$ . The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves  $\tilde{L}_{yz}$

and  $\tilde{C}$  on the normal surface  $T$ . The curve  $\tilde{L}_{yz}$  is the proper transform of the curve  $L_{yz}$  and the curve  $\tilde{C}$  is the proper transform of the curve  $C$  defined by  $x = t = w^2 + z^3 = 0$ . From the intersection numbers

$$(\tilde{L}_{yz} + \tilde{C}) \cdot \tilde{L}_{yz} = -K_Y \cdot \tilde{L}_{yz} = -\frac{1}{6}, \quad (\tilde{L}_{yz} + \tilde{C})^2 = 7B^3 = -\frac{1}{4}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{yz}^2 = -\frac{1}{6} - \tilde{L}_{yz} \cdot \tilde{C}, \quad \tilde{C}^2 = -\frac{1}{12} - \tilde{L}_{yz} \cdot \tilde{C}.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yz}^2 & \tilde{L}_{yz} \cdot \tilde{C} \\ \tilde{L}_{yz} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} - \tilde{L}_{yz} \cdot \tilde{C} & \tilde{L}_{yz} \cdot \tilde{C} \\ \tilde{L}_{yz} \cdot \tilde{C} & -\frac{1}{12} - \tilde{L}_{yz} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{yz} \cdot \tilde{C}$  is non-negative.

<b>No. 67:</b> $X_{28} \subset \mathbb{P}(1, 1, 4, 9, 14)$					$A^3 = 1/18$
$w^2 + yt^3 + z^7 + wf_{14}(x, y, z, t) + f_{28}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{9}(1_x, 4_z, 5_w) \textcircled{\text{D}}$	+	$B - E$	$y$	$w^2$	
$O_z O_w = 1 \times \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{\text{D}}$	-	$B$	$y$	$y$	

The 1-cycle  $\Gamma$  for the singular point of type  $\frac{1}{2}(1, 1, 1)$  is irreducible since we have the monomials  $w^2$  and  $z^7$ .

<b>No. 70:</b> $X_{30} \subset \mathbb{P}(1, 1, 4, 10, 15)$					$A^3 = 1/20$
$w^2 + t^3 + tz^5 + wf_{15}(x, y, z, t) + f_{30}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_z = \frac{1}{4}(1_x, 1_y, 3_w) \textcircled{\text{D}}$	-	$B$	$y$	$y$	
$O_t O_w = 1 \times \frac{1}{5}(1_x, 1_y, 4_z) \textcircled{\text{D}}$	0	$B$	$y$	$y$	
$O_z O_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{\text{D}}$	-	$B$	$y$	$y$	

For each singular point the 1-cycle  $\Gamma$  is irreducible due to the monomial  $w^2$  and  $t^3$ .

<b>No. 71:</b> $X_{30} \subset \mathbb{P}(1, 1, 6, 8, 15)$					$A^3 = 1/24$
$w^2 + t^3 z + wf_{15}(x, y, z, t) + t^2 f_{14}(x, y, z) + t f_{22}(x, y, z) + f_{30}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{8}(1_x, 1_y, 7_w) \textcircled{\text{D}}$	+	$6B - E$	$z$	$w^2$	
$O_z O_w = 1 \times \frac{1}{3}(1_x, 1_y, 2_t) \textcircled{\text{D}}$	-	$B$	$y$	$y$	
$O_z O_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{\text{D}}$	-	$B$	$y$	$y$	

For the singular points of types  $\frac{1}{2}(1, 1, 1)$  and  $\frac{1}{3}(1, 1, 2)$ , the 1-cycles  $\Gamma$  are irreducible because of  $w^2$  and  $t^3z$ .

<b>No. 72:</b> $X_{30} \subset \mathbb{P}(1, 2, 3, 10, 15)$					$A^3 = 1/30$
$w^2 + t^3 + z^{10} + y^{15} + wf_{15}(x, y, z, t) + f_{30}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t O_w = 1 \times \frac{1}{5}(1_x, 2_y, 3_z) \textcircled{B}$	0	$2B$	$y$	$y$	
$O_z O_w = 2 \times \frac{1}{3}(1_x, 2_y, 1_t) \textcircled{B}$	–	$2B$	$y$	$y$	
$O_y O_t = 3 \times \frac{1}{2}(1_x, 1_z, 1_w) \textcircled{B}$	–	$3B + E$	$z$	$z$	

For each singular point the 1-cycle  $\Gamma$  is irreducible due to the monomials  $w^2$  and  $t^3$ .

<b>No. 73:</b> $X_{30} \subset \mathbb{P}(1, 2, 6, 7, 15)$					$A^3 = 1/42$
$w^2 + \prod_{i=1}^5(z - \alpha_i y^3) + yt^4 + wf_{15}(x, y, z, t) + f_{30}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{7}(1_x, 6_z, 1_w) \textcircled{B}$	0	$2B$	$y$	$w^2$	
$O_z O_w = 1 \times \frac{1}{3}(1_x, 2_y, 1_t) \textcircled{B}$	–	$2B$	$y$	$y$	
$O_y O_z = 5 \times \frac{1}{2}(1_x, 1_t, 1_w) \textcircled{B}$	–	$6B + 2E$	$z - \alpha_i y^3$	$w^2$	

For each singular point the 1-cycle  $\Gamma$  is irreducible due to the monomials  $w^2$ ,  $z^5$  and  $yt^4$ .

<b>No. 75:</b> $X_{30} \subset \mathbb{P}(1, 4, 5, 6, 15)$					$A^3 = 1/60$
$w^2 + y^6 t + z^6 + t^5 + wf_{15}(x, y, z, t) + f_{30}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_y = \frac{1}{4}(1_x, 1_z, 3_w) \textcircled{B}$	–	$5B + E$	$z$	$z$	
$O_t O_w = 1 \times \frac{1}{3}(1_x, 1_y, 2_z) \textcircled{B}$	–	$5B + E$	$z$	$z$	
$O_z O_w = 2 \times \frac{1}{5}(1_x, 4_y, 1_t) \textcircled{B}$	–	$4B$	$y$	$y$	
$O_y O_t = 2 \times \frac{1}{2}(1_x, 1_z, 1_w) \textcircled{B}$	–	$5B + 2E$	$z$	$z$	

For the singular point  $O_y$ , consider the linear system generated by  $xy$  and  $z$ . Its base curves are defined by  $x = z = 0$  and  $y = z = 0$ . The curve defined by  $y = z = 0$  does not pass through the point  $O_y$ . The curve defined by  $x = z = 0$  is irreducible. Moreover, its proper transform is the 1-cycle defined by  $(5B + E) \cdot B$ . Consequently, the divisor  $T$  is nef since  $(5B + E)^2 \cdot B > 0$ .

For the other singular points, we immediately see that the 1-cycles  $\Gamma$  are irreducible due to the monomials  $w^2$  and  $t^5$ .

<b>No. 77:</b> $X_{32} \subset \mathbb{P}(1, 2, 5, 9, 16)$					$A^3 = 1/45$
$w^2 + t^3 z + yz^6 + wf_{16}(x, y, z, t) + t^2 f_{14}(x, y, z) + t f_{23}(x, y, z) + f_{32}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{9}(1_x, 2_y, 7_w) \textcircled{B}$	+	$5B - E$	$z$	$w^2$	

$O_z = \frac{1}{5}(1_x, 4_t, 1_w) \textcircled{B}$	–	$2B$	$y$	$w^2$	
$O_y O_w = 2 \times \frac{1}{2}(1_x, 1_z, 1_t) \textcircled{U}$	–	$9B + 4E$	$t$	$t$	

For the singular point  $O_z$ , the 1-cycle  $\Gamma$  is irreducible due to the monomials  $w^2$  and  $t^3z$ .

For the singular points of type  $\frac{1}{2}(1, 1, 1)$ , we consider the linear system generated by  $xy^4$ ,  $y^2z$  and  $t$  on  $X_{32}$ . Its base curve is defined by  $y = t = 0$ . The curve defined by  $y = t = 0$  passes through no singular point of type  $\frac{1}{2}(1, 1, 1)$ . Consequently, the divisor  $T$  is nef.

<b>No. 78:</b> $X_{32} \subset \mathbb{P}(1, 4, 5, 7, 16)$					$A^3 = 1/70$
$w^2 + z^5t + yt^4 + wf_{16}(x, y, z, t) + t^3f_{11}(x, y, z) + t^2f_{18}(x, y, z) + tf_{25}(x, y, z) + f_{32}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{7}(1_x, 5_z, 2_w) \textcircled{B}$	0	$4B$	$y$	$w^2$	
$O_z = \frac{1}{5}(1_x, 4_y, 1_w) \textcircled{B}$	–	$4B$	$y$	$y$	
$O_y O_w = 2 \times \frac{1}{4}(1_x, 1_z, 3_t) \textcircled{U}$	–	$5B + E$	$z$	$z$	

For the singular points other than those of type  $\frac{1}{4}(1, 1, 3)$ , the 1-cycles  $\Gamma$  are always irreducible due to the monomials  $w^2$  and  $z^5t$ .

For the singular points of type  $\frac{1}{4}(1, 1, 3)$ , we consider the linear system generated by  $xy$  and  $z$  on  $X_{32}$ . Its base curves are defined by  $x = z = 0$  and  $y = z = 0$ . The curve defined by  $y = z = 0$  passes through no singular point of type  $\frac{1}{4}(1, 1, 3)$ . The curve defined by  $x = z = 0$  is irreducible because of the monomials  $w^2$  and  $yt^4$ . Moreover, its proper transform is the 1-cycle defined by  $(5B + E) \cdot B$ . Consequently, the divisor  $T$  is nef since  $(5B + E)^2 \cdot B > 0$ .

<b>No. 80:</b> $X_{34} \subset \mathbb{P}(1, 3, 4, 10, 17)$					$A^3 = 1/60$
$w^2 + zt^3 + z^6t + y^8(a_1t + a_2y^2z + a_3xy^4) + wf_{17}(x, y, z, t) + f_{34}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{10}(1_x, 3_y, 7_w) \textcircled{D}$	+	$4B - E$	$z$	$w^2$	
$O_z = \frac{1}{4}(1_x, 3_y, 1_w) \textcircled{B}$	–	$3B$	$y$	$y$	
$O_y = \frac{1}{3}(1_x, 1_z, 2_w) \textcircled{U}$	–	$4B + E$	$z$	$z$	$a_1 \neq 0$
$O_y = \frac{1}{3}(1_x, 1_t, 2_w) \textcircled{B}$	–	$4B$	$z$	$w^2$	$a_1 = 0, a_2 \neq 0$
$O_y = \frac{1}{3}(1_z, 1_t, 2_w) \textcircled{B}$	–	$4B + E$	$z$	$z$	$a_1 = a_2 = 0$
$O_z O_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{B}$	–	$3B + E$	$y$	$y$	

We may assume that the monomial  $xyt^3$  is not contained in the polynomial  $f_{34}$  by changing the coordinate  $z$  in a suitable way.

For each of the singular points, the 1-cycle  $\Gamma$  is always irreducible even though it is possibly non-reduced.

For the singular point  $O_y$  with  $a_1 \neq 0$ , we consider the linear system generated by  $xy$  and  $z$  on  $X_{34}$ . Its base curves are defined by  $x = z = 0$  and  $y = z = 0$ . The curve defined by  $y = z = 0$  does not pass through the point  $O_y$ . The curve defined by  $x = z = 0$  is irreducible because of the monomials  $w^2$  and  $y^8t$ . Moreover, its proper transform is the 1-cycle defined



by  $(4B + E) \cdot B$ . Consequently, the divisor  $T$  is nef since  $(4B + E)^2 \cdot B > 0$ .

<b>No. 81:</b> $X_{34} \subset \mathbb{P}(1, 4, 6, 7, 17)$					$A^3 = 1/84$
$w^2 + yz^5 + y^7z + t^4z + wf_{17}(x, y, z, t) + t^3f_{13}(x, y, z) + t^2f_{20}(x, y, z) + tf_{27}(x, y, z) + f_{34}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{7}(1_x, 4_y, 3_w) \text{ ⑥}$	0	$6B$	$z$	$w^2$	
$O_z = \frac{1}{6}(1_x, 1_t, 5_w) \text{ ⑥}$	–	$4B$	$y$	$zt^4$	
$O_y = \frac{1}{4}(1_x, 3_t, 1_w) \text{ ⑥}$	–	$7B + E$	$t$	$t$	
$O_yO_z = 2 \times \frac{1}{2}(1_x, 1_t, 1_w) \text{ ⑩}$	–	$7B + 3E$	$t$	$t$	

The 1-cycle  $\Gamma$  for the singular point  $O_t$  is irreducible due to the monomials  $w^2$  and  $y^5t^2$  even though it can be non-reduced.

For the singular point  $O_z$ , the 1-cycle  $\Gamma$  is irreducible due to the monomials  $w^2$  and  $zt^4$ .

For the singular point  $O_y$ , the 1-cycle  $\Gamma$  is irreducible due to the monomials  $w^2$ ,  $yz^5$  and  $y^7z$ .

For the singular points of type  $\frac{1}{2}(1, 1, 1)$ , we consider the linear system generated by  $xz$  and  $t$  on  $X_{34}$ . Its base curves are defined by  $x = t = 0$  and  $z = t = 0$ . The curve defined by  $z = t = 0$  passes through no singular point of type  $\frac{1}{2}(1, 1, 1)$ . The curve defined by  $x = t = 0$  is irreducible due to the monomials  $w^2$ ,  $yz^5$  and  $y^7z$ . Moreover, its proper transform is equivalent to the 1-cycle defined by  $(7B + 3E) \cdot B$ . Consequently, the divisor  $T$  is nef since  $(7B + 3E)^2 \cdot B > 0$ .

<b>No. 82:</b> $X_{36} \subset \mathbb{P}(1, 1, 5, 12, 18)$					$A^3 = 1/30$
$w^2 + t^3 + yz^7 + wf_{18}(x, y, z, t) + f_{36}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_z = \frac{1}{5}(1_x, 2_t, 3_w) \text{ ①}$	0	$B - E$	$y$	$w^2$	
$O_tO_w = 1 \times \frac{1}{6}(1_x, 1_y, 5_z) \text{ ⑥}$	0	$B$	$y$	$y$	

For the singular point  $O_z$ , let  $C_\lambda$  be a curve on the surface  $S_y$  defined by

$$\begin{cases} y = 0, \\ t = \lambda x^{12} \end{cases}$$

for a sufficiently general complex number  $\lambda$ . Denote by  $\tilde{C}_\lambda$  the proper transform of the curve  $C_\lambda$  on  $Y$ . Then

$$-K_Y \cdot \tilde{C}_\lambda = (B - E)(12B + 2E)B = 0.$$

If the curve  $\tilde{C}_\lambda$  is reducible, it consists of two irreducible components that are numerically equivalent since the two components of the curve  $C_\lambda$  are symmetric with respect to the biregular quadratic involution of  $X_{36}$ . Then each component of  $\tilde{C}_\lambda$  intersects  $-K_Y$  trivially.

For the singular point of type  $\frac{1}{6}(1, 1, 5)$ , the 1-cycle  $\Gamma$  is irreducible due to  $w^2$  and  $t^3$ .

<b>No. 83:</b> $X_{36} \subset \mathbb{P}(1, 3, 4, 11, 18)$					$A^3 = 1/66$
$(w - \alpha_1 y^6)(w - \alpha_2 y^6) + yt^3 + z^9 + wf_{18}(x, y, z, t) + t^2f_{14}(x, y, z) + tf_{25}(x, y, z) + f_{36}(x, y, z)$					

Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{11}(1_x, 4_z, 7_w) \textcircled{D}$	+	$3B - E$	$y$	$w^2$	
$O_z O_w = 1 \times \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{D}$	-	$3B + E$	$y$	$y$	
$O_y O_w = 2 \times \frac{1}{3}(1_x, 1_z, 2_t) \textcircled{D}$	-	$18B + 4E$	$w - \alpha_i y^6$	$yt^3$	

For each of the singular points, the 1-cycle  $\Gamma$  is irreducible since we have the monomials  $w^2$ ,  $yt^3$ , and  $z^9$ .

<b>No. 84:</b> $X_{36} \subset \mathbb{P}(1, 7, 8, 9, 12)$					$A^3 = 1/168$
$w^3 + wz^3 + t^4 + y^4(a_1z + a_2xy) + w^2f_{12}(x, y, z, t) + wf_{24}(x, y, z, t) + f_{36}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_z = \frac{1}{8}(1_x, 7_y, 1_t) \textcircled{D}$	-	$7B$	$y$	$y$	
$O_y = \frac{1}{7}(1_x, 2_t, 5_w) \textcircled{D}$	-	$8B$	$z$	$t^4$	$a_1 \neq 0$
$O_y = \frac{1}{7}(1_z, 2_t, 5_w) \textcircled{D}$	-	$12B + E$	$w$	$w$	$a_1 = 0$
$O_t O_w = 1 \times \frac{1}{3}(1_x, 1_y, 2_z) \textcircled{D}$	-	$8B + 2E$	$z$	$z$	
$O_z O_w = 1 \times \frac{1}{4}(1_x, 3_y, 1_t) \textcircled{D}$	-	$7B + E$	$y$	$y$	

For each singular point the 1-cycle  $\Gamma$  is always irreducible because of the monomials  $w^3$  and  $t^4$ . In particular, the intersection  $\Gamma$  for the singular point  $O_y$  with  $a_1 = 0$  is irreducible even though it is non-reduced.

<b>No. 85:</b> $X_{38} \subset \mathbb{P}(1, 3, 5, 11, 19)$					$A^3 = 2/165$
$w^2 + t^3z + yz^7 + y^9(a_1t + a_2y^2z) + wf_{19}(x, y, z, t) + t^2f_{16}(x, y, z) + tf_{27}(x, y, z) + f_{38}(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{11}(1_x, 3_y, 8_w) \textcircled{D}$	+	$5B - E$	$z$	$w^2$	
$O_z = \frac{1}{5}(1_x, 1_t, 4_w) \textcircled{D}$	-	$3B$	$y$	$zt^3$	
$O_y = \frac{1}{3}(1_x, 2_z, 1_w) \textcircled{D}$	-	$5B + E$	$z$	$z$	$a_1 \neq 0$
$O_y = \frac{1}{3}(1_x, 2_t, 1_w) \textcircled{D}$	-	$5B + E$	$z$	$w^2$	$a_1 = 0$

For the singular point  $O_z$ , the 1-cycle  $\Gamma$  is irreducible because of the monomials  $w^2$  and  $zt^3$ . For the singular point  $O_y$ , the 1-cycle  $\Gamma$  is irreducible because of the monomials  $w^2$  and  $y^9t$ . Note that in case when  $a_1 = 0$  the 1-cycle  $\Gamma$  is still irreducible but non-reduced.

<b>No. 86:</b> $X_{38} \subset \mathbb{P}(1, 5, 6, 8, 19)$					$A^3 = 1/120$
$w^2 + zt^4 + z^5t + y^6t + wf_{19}(x, y, z, t) + f_{38}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{8}(1_x, 5_y, 3_w) \textcircled{D}$	0	$5B$	$y$	$y$	
$O_z = \frac{1}{6}(1_x, 5_y, 1_w) \textcircled{D}$	-	$5B$	$y$	$y$	
$O_y = \frac{1}{5}(1_x, 1_z, 4_w) \textcircled{D}$	-	$6B + E$	$z$	$z$	

$O_z O_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{B}$	—	$5B + 2E$	$y$	$y$	
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For the singular points except the point  $O_y$ , the 1-cycles  $\Gamma$  are always irreducible because of the monomials  $w^2$ ,  $zt^4$  and  $z^5t$ .

For the singularity  $O_y$ , we consider the linear system generated by  $xy$  and  $z$  on  $X_{38}$ . Its base curves are defined by  $x = z = 0$  and  $y = z = 0$ . The curve defined by  $y = z = 0$  does not pass through the singular point  $O_y$ . The curve defined by  $x = z = 0$  is irreducible because of the monomials  $w^2$  and  $y^6t$ . Moreover, its proper transform is equivalent to the 1-cycle defined by  $(6B + E) \cdot B$ . Consequently, the divisor  $T$  is nef since  $(6B + E)^2 \cdot B > 0$ .

<b>No. 87:</b> $X_{40} \subset \mathbb{P}(1, 5, 7, 8, 20)$					$A^3 = 1/140$
$(w - \alpha_1 y^4)(w - \alpha_2 y^4) + t^5 + yz^5 + wf_{20}(x, y, z, t) + f_{40}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_z = \frac{1}{7}(1_x, 1_t, 6_w) \textcircled{B}$	—	$5B$	$y$	$t^5$	
$O_t O_w = 1 \times \frac{1}{4}(1_x, 1_y, 3_z) \textcircled{B}$	—	$7B + E$	$z$	$z$	
$O_y O_w = 2 \times \frac{1}{5}(1_x, 2_z, 3_t) \textcircled{B}$	—	$7B + E$	$z$	$z$	

The irreducibility of the 1-cycle  $\Gamma$  can be immediately checked for each singular points since we have the terms  $w^2$  and  $t^5$ .

For the singular points of type  $\frac{1}{5}(1, 2, 3)$ , we consider the linear system generated by  $x^2y$  and  $z$  on  $X_{40}$ . Its base curves are defined by  $x = z = 0$  and  $y = z = 0$ . The curve defined by  $y = z = 0$  passes through no singular points of type  $\frac{1}{5}(1, 2, 3)$ . The curve defined by  $x = z = 0$  is irreducible because of the monomials  $w^2$  and  $t^5$ . Moreover, its proper transform is equivalent to the 1-cycle defined by  $(7B + E) \cdot B$ . Consequently, the divisor  $T$  is nef since  $(7B + E)^2 \cdot B > 0$ .

<b>No. 88:</b> $X_{42} \subset \mathbb{P}(1, 1, 6, 14, 21)$					$A^3 = 1/42$
$w^2 + t^3 + z^7 + wf_{21}(x, y, z, t) + f_{42}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t O_w = 1 \times \frac{1}{7}(1_x, 1_y, 6_z) \textcircled{B}$	0	$B$	$y$	$y$	
$O_z O_w = 1 \times \frac{1}{3}(1_x, 1_y, 2_t) \textcircled{B}$	—	$B$	$y$	$y$	
$O_z O_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{B}$	—	$B$	$y$	$y$	

For each singular point the 1-cycle  $\Gamma$  is irreducible due to the monomial  $w^2$ ,  $t^3$ , and  $z^7$ .

<b>No. 89:</b> $X_{42} \subset \mathbb{P}(1, 2, 5, 14, 21)$					$A^3 = 1/70$
$w^2 + t^3 + yz^8 + y^{21} + wf_{21}(x, y, z, t) + f_{42}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_z = \frac{1}{5}(1_x, 4_t, 1_w) \textcircled{B}$	—	$2B$	$y$	$w^2$	
$O_t O_w = 1 \times \frac{1}{7}(1_x, 2_y, 5_z) \textcircled{B}$	0	$2B$	$y$	$y$	
$O_y O_t = 3 \times \frac{1}{2}(1_x, 1_z, 1_w) \textcircled{B}$	—	$5B + 2E$	$z$	$z$	

For each singular point the 1-cycle  $\Gamma$  is irreducible because of the monomials  $w^2$  and  $t^3$ .

<b>No. 90:</b> $X_{42} \subset \mathbb{P}(1, 3, 4, 14, 21)$ <span style="float: right;"><math>A^3 = 1/84</math></span>					
$(w - \alpha_1 y^7)(w - \alpha_2 y^7) + t^3 + z^7 t + w f_{21}(x, y, z, t) + x f_{41}(x, y, z, t) + f_{42}(y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_z = \frac{1}{4}(1_x, 3_y, 1_w) \textcircled{D}$	–	$3B$	$y$	$y$	
$O_t O_w = 1 \times \frac{1}{7}(1_x, 3_y, 4_z) \textcircled{D}$	0	$3B$	$y$	$y$	
$O_y O_w = 2 \times \frac{1}{3}(1_x, 1_z, 2_t) \textcircled{D}$	–	$4B + E$	$z$	$z$	
$O_z O_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{D}$	–	$3B + E$	$y$	$y$	

For the singular points other than those of type  $\frac{1}{3}(1, 1, 2)$ , the 1-cycle  $\Gamma$  is irreducible since we have monomials  $w^2$ ,  $t^3$ , and  $z^7 t$ .

For the singular points of type  $\frac{1}{3}(1, 1, 2)$ , consider the linear system generated by  $xy$  and  $z$  on  $X_{42}$ . Its base curves are defined by  $x = z = 0$  and  $y = z = 0$ . The curve defined by  $y = z = 0$  passes through no singular points of type  $\frac{1}{3}(1, 1, 2)$ . The curve defined by  $x = z = 0$  is irreducible because of the monomials  $w^2$  and  $t^3$ . Moreover, its proper transform is equivalent to the 1-cycle defined by  $(4B + E) \cdot B$ . Consequently, the divisor  $T$  is nef since  $(4B + E)^2 \cdot B > 0$ .

<b>No. 91:</b> $X_{44} \subset \mathbb{P}(1, 4, 5, 13, 22)$ <span style="float: right;"><math>A^3 = 1/130</math></span>					
$w^2 + zt^3 + yz^8 + y^{11} + w f_{22}(x, y, z, t) + f_{44}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t = \frac{1}{13}(1_x, 4_y, 9_w) \textcircled{D}$	+	$5B - E$	$z$	$w^2$	
$O_z = \frac{1}{5}(1_x, 3_t, 2_w) \textcircled{D}$	–	$4B$	$y$	$w^2$	
$O_y O_w = 1 \times \frac{1}{2}(1_x, 1_z, 1_t) \textcircled{D}$	–	$5B + 2E$	$z$	$z$	

For each singular point the 1-cycle  $\Gamma$  is irreducible due to the monomial  $w^2$ ,  $y^{11}$ , and  $zt^3$ .

<b>No. 92:</b> $X_{48} \subset \mathbb{P}(1, 3, 5, 16, 24)$ <span style="float: right;"><math>A^3 = 1/120</math></span>					
$w^2 + t^3 + yz^9 + y^{16} + w f_{24}(x, y, z, t) + f_{48}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_z = \frac{1}{5}(1_x, 1_t, 4_w) \textcircled{D}$	–	$3B$	$y$	$t^3$	
$O_t O_w = 1 \times \frac{1}{8}(1_x, 3_y, 5_z) \textcircled{D}$	0	$3B$	$y$	$y$	
$O_y O_w = 2 \times \frac{1}{3}(1_x, 2_z, 1_t) \textcircled{D}$	–	$5B + E$	$z$	$z$	

For each singular point the 1-cycle  $\Gamma$  is irreducible due to the monomial  $w^2$  and  $t^3$ .

<b>No. 93:</b> $X_{50} \subset \mathbb{P}(1, 7, 8, 10, 25)$ <span style="float: right;"><math>A^3 = 1/280</math></span>					
$w^2 + t^5 + z^5 t + y^6(a_1 z + a_2 xy) + w f_{25}(x, y, z, t) + f_{50}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_z = \frac{1}{8}(1_x, 7_y, 1_w) \textcircled{D}$	–	$7B$	$y$	$y$	

$O_y = \frac{1}{7}(1_x, 3_t, 4_w) \textcircled{\text{B}}$	–	$8B$	$z$	$w^2$	$a_1 \neq 0$
$O_y = \frac{1}{7}(1_z, 3_t, 4_w) \textcircled{\text{B}}$	–	$10B + E$	$t$	$t$	$a_1 = 0$
$O_t O_w = 1 \times \frac{1}{5}(1_x, 2_y, 3_z) \textcircled{\text{B}}$	–	$8B + E$	$z$	$z$	
$O_z O_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{\text{B}}$	–	$7B + 3E$	$y$	$y$	

For each singular point the 1-cycle  $\Gamma$  is always irreducible because of the monomials  $w^2$  and  $t^5$ . In particular, the 1-cycle  $\Gamma$  for the singular point  $O_y$  with  $a_1 = 0$  is irreducible even though it is non-reduced.

No. 94: $X_{54} \subset \mathbb{P}(1, 4, 5, 18, 27)$ <span style="float: right;"><math>A^3 = 1/180</math></span>					
$w^2 + t^3 + yz^{10} + y^9t + wf_{27}(x, y, z, t) + f_{54}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_z = \frac{1}{5}(1_x, 3_t, 2_w) \textcircled{\text{B}}$	–	$4B$	$y$	$w^2$	
$O_y = \frac{1}{4}(1_x, 1_z, 3_w) \textcircled{\text{B}}$	–	$18B + 3E$	$t$	$w^2$	
$O_t O_w = 1 \times \frac{1}{9}(1_x, 4_y, 5_z) \textcircled{\text{B}}$	0	$4B$	$y$	$y$	
$O_y O_t = 1 \times \frac{1}{2}(1_x, 1_z, 1_w) \textcircled{\text{B}}$	–	$5B + 2E$	$z$	$z$	

For each singular point the 1-cycle  $\Gamma$  is irreducible due to the monomials  $w^2$ ,  $t^3$  and  $yz^{10}$ .

No. 95: $X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33)$ <span style="float: right;"><math>A^3 = 1/330</math></span>					
$w^2 + t^3 + z^{11} + y^{12}(a_1z + a_2xy) + wf_{33}(x, y, z, t) + f_{66}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_y = \frac{1}{5}(1_x, 2_t, 3_w) \textcircled{\text{B}}$	–	$6B$	$z$	$w^2$	$a_1 \neq 0$
$O_y = \frac{1}{5}(1_z, 2_t, 3_w) \textcircled{\text{B}}$	–	$6B + E$	$z$	$z$	$a_1 = 0$
$O_t O_w = 1 \times \frac{1}{11}(1_x, 5_y, 6_z) \textcircled{\text{B}}$	0	$5B$	$y$	$y$	
$O_z O_w = 1 \times \frac{1}{3}(1_x, 2_y, 1_t) \textcircled{\text{B}}$	–	$5B + E$	$y$	$y$	
$O_z O_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{\text{B}}$	–	$5B + 2E$	$y$	$y$	

The 1-cycle  $\Gamma$  for each singular point is irreducible because of the monomials  $w^2$ ,  $z^{11}$  and  $t^3$ .

## 5 Untwisting singular points

### 5.1 Untwisting

Let  $p$  be a singular point of  $X$  of type  $\frac{1}{r}(1, a, r - a)$ , where  $r$  and  $a$  are relatively prime positive integers. As before, we denote by  $f: Y \rightarrow X$  the weighted blow up of  $X$  at the point  $p$  with weights  $(1, a, r - a)$ , we denote the  $f$ -exceptional divisor by  $E$ , we denote the anticanonical divisor  $-K_Y$  by  $B$ , we denote by  $\mathcal{M}_Y$  the proper transform of the linear system  $\mathcal{M}$  via  $f$ , we denote by  $A$  the divisor  $f^*(-K_X)$ , and we denote by  $S$  the proper transform of the surface on  $X$  that is cut out on  $X$  by  $x = 0$ .

**Definition 5.1.1.** Let  $\tau$  be a birational map of  $X$ . If  $p$  is a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$ , we say that  $\tau$  *untwists* the point  $p$  (as a center of non-canonical

singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$  if  $\tau$  is not biregular and  $\tau = f \circ \tau_Y \circ f^{-1}$  for some  $\tau_Y \in \text{Bir}(Y)$  such that  $\tau_Y$  is biregular in codimension one.

**Lemma 5.1.2.** *Suppose that  $p$  is a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$ , and there exists  $\tau \in \text{bir}(X)$  that untwists the point  $p$  as a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$ . Then  $\tau(\mathcal{M}) \subset |-n_\tau K_X|$  for some positive integer  $n_\tau < n$ .*

*Proof.* Put  $\tau_Y = f^{-1} \circ \tau \circ f$ . Then  $\tau_Y$  is biregular in codimension one, and  $\tau_Y$  is not biregular. In particular,  $\tau_Y$  acts on  $\text{Pic}(Y)$ . Then  $\tau_Y(B) = B$ , since  $B = -K_Y$ . Moreover, we have  $\tau_Y(E) \neq E$ , since otherwise  $\tau$  is biregular by [21, Proposition 3.5]. On the other hand, we have

$$f^*(\mathcal{M}) = \mathcal{M}_Y + mE,$$

for some positive rational number  $m$ . Furthermore,  $m > \frac{n}{r}$  by Theorem 2.2.1.

Since  $\tau_Y$  acts on  $\text{Pic}(Y)$ , there are rational numbers  $a, b, c, d$  such that  $a, c > 0$  and

$$\begin{cases} \tau_Y(A) = aA - bE, \\ \tau_Y(E) = cA - dE. \end{cases}$$

Since  $\tau_Y(B) = B$ , we obtain

$$A - \frac{1}{r}E = \tau_Y(A - \frac{1}{r}E) = \tau_Y(A) - \frac{1}{r}\tau_Y(E) = (a - \frac{c}{r})A - (b - \frac{d}{r})E,$$

and hence  $a - \frac{c}{r} = 1$ . We then see

$$\tau_Y(\mathcal{M}_Y) = \tau_Y(nA - mE) = n\tau_Y(A) - m\tau_Y(E) = (na - mc)A - (nb - md)E.$$

Since  $na - mc = na - m(ar - r) = na - mr(a - 1) < na - n(a - 1) = n$ , we obtain  $\tau(\mathcal{M}) \subset |-n_\tau K_X|$  with  $n_\tau < n$ . This proves the statement.  $\square$

Thus, to complete the proof of Theorem 1.1.9, it is enough to show that every singular point of  $X$  either is not a center of non-canonical singularities of the log pair  $(X, \frac{1}{n}\mathcal{M})$  or can be untwisted by some appropriate birational automorphism of  $X$ . This follows from Theorem 1.2.2, Lemma 5.1.2 and induction on  $n$ .

As in [23], if a singular point of  $X$  can be untwisted by some birational automorphism of  $X$ , it can be untwisted by a very explicit birational involution. Since  $X$  has only finitely many singular points, there are finitely many such involutions for a given  $X$ . These birational automorphisms generate a subgroup, denoted by  $\Gamma_X$ , in the group  $\text{Bir}(X)$  of birational automorphisms. Using [21, Theorem 4.2] instead of Theorem 1.2.2, we prove

**Theorem 5.1.3.** *Let  $X$  be a quasi-smooth hypersurface of degrees  $d$  with only terminal singularities in weighted projective space  $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ , where  $d = \sum a_i$ . Then the group  $\text{Bir}(X)$  of birational automorphisms of  $X$  is generated by  $\Gamma_X$  and the group  $\text{Aut}(X)$  of biregular automorphisms of  $X$ .*

In the case when  $X$  is a general hypersurface in its family, Theorem 5.1.3 is proved in [23] (see [23, Remark 1.4]).

## 5.2 Easy involutions

**Lemma 5.2.1** ([23, Theorem 4.9]). *Suppose that the hypersurface  $X$  is given by*

$$x_{i_3}x_{i_4}^2 + f_e x_{i_4} + g_d = 0,$$

where  $x_{i_4}, x_{i_3}$  are two of the coordinates and  $f_e, g_d$  are quasi-homogeneous polynomials of degrees  $e$  and  $d$  not involving  $x_{i_4}$ . In addition, suppose that the polynomial  $f_e$  is not divisible by  $x_{i_3}$ . Write  $O_{x_{i_4}}$  for the coordinate point at which  $x_{i_4} \neq 0$  and all the other coordinates vanish. Then interchanging the roots of the equation defines a birational involution  $\tau_{O_{x_{i_4}}}$  of  $X$ . If the polynomial  $f_e$  is nonzero, then the involution  $\tau_{O_{x_{i_4}}}$  untwists  $O_{x_{i_4}}$  as a center of non-canonical singularities of  $(X, \frac{1}{n}\mathcal{M})$ .

**Lemma 5.2.2.** *Suppose that the hypersurface  $X$  is given by*

$$x_{i_3}x_{i_4}^2 + x_{i_3}g_e(x, x_{i_1}, x_{i_2}, x_{i_3}) + h_d(x, x_{i_1}, x_{i_2}) = 0,$$

where  $x_{i_k}$ 's are the coordinates of  $\mathbb{P}(1, a_1, a_2, a_3, a_4)$  different from  $x$  and  $g_e, h_d$  are quasi-homogeneous polynomials of degrees  $e, d$ . Write  $O_{x_{i_4}}$  for the coordinate point at which  $x_{i_4} \neq 0$  and all the other coordinates vanish. If the weights of  $x_{i_1}, x_{i_2}$  are less than the weight of  $x_{i_4}$ , then the singular points  $O_{x_{i_4}}$  cannot be a center of non-canonical singularities of  $(X, \frac{1}{n}\mathcal{M})$ .

*Proof.* Note that the quasi-homogeneous polynomial  $h_d$  must be irreducible: otherwise the hypersurface  $X$  would not be quasi-smooth. The singular point  $O_{x_{i_4}}$  is of type  $\frac{1}{a_{i_4}}(1, a_{i_1}, a_{i_2})$ . Suppose that it is a center of non-canonical singularities of  $(X, \frac{1}{n}\mathcal{M})$ .

Let  $T$  be the proper transform of the surface  $S_{x_{i_3}}$  cut by  $x_{i_3} = 0$ . Since the polynomial  $h_d$  cannot be zero, the surface cut by  $x_{i_3} = 0$  vanishes along  $E$  with multiplicity  $\frac{d}{a_{i_4}}$ . Therefore, the surface  $T$  belongs to  $|a_{i_3}B - 2E|$  since  $a_{i_3} + 2a_{i_4} = d$ .

Let  $C_\lambda$  be a curve on the surface  $S_{x_{i_3}}$  defined by

$$\begin{cases} x_{i_3} = 0, \\ x_{i_2} = \lambda x^{a_{i_2}} \end{cases}$$

for a sufficiently general complex number  $\lambda$ . Then the curve  $C_\lambda$  is a curve of degree  $d$  in  $\mathbb{P}(1, a_{i_1}, a_{i_4})$  defined by equation

$$h_d(x, x_{i_1}, \lambda x^{a_{i_2}}) = 0.$$

Instead of proving that  $C_\lambda$  is irreducible, we claim that every curve on  $T$  intersects  $B$  non-negatively. To this end, we consider the linear system  $\mathcal{L}$  on  $X$  given by the monomials  $x^{a_{i_1}+a_{i_2}}, x_{i_1}x_{i_2}, x^{a_{i_1}}x_{i_2}, x^{a_{i_2}}x_{i_1}$ . The proper transform of a surface in  $\mathcal{L}$  is equivalent to  $(a_{i_1} + a_{i_2})B$ . The base locus of the proper transform of the linear system  $\mathcal{L}$  consists of the proper transform of the curve cut by  $x = x_{i_1} = 0$  and the proper transform of the curve cut by  $x = x_{i_2} = 0$ .

Suppose that we have a curve  $R$  on  $T$  such that  $B \cdot R < 0$ . Then we can conclude from the linear system  $\mathcal{L}$  that the curve  $R$  should be either the proper transform  $\tilde{L}_{24}$  of the curve  $L_{24}$  defined by  $x = x_{i_3} = x_{i_1} = 0$  or the proper transform  $\tilde{L}_{14}$  of the curve  $L_{14}$  defined by  $x = x_{i_3} = x_{i_2} = 0$ . However, since  $E \cdot \tilde{L}_{24} = \frac{1}{a_{i_2}}$  and  $E \cdot \tilde{L}_{14} = \frac{1}{a_{i_1}}$ , we obtain

$$B \cdot \tilde{L}_{24} = (A - \frac{1}{a_{i_4}}E) \cdot \tilde{L}_{24} = \frac{1}{a_{i_2}a_{i_4}} - \frac{1}{a_{i_4}a_{i_2}} = 0;$$

$$B \cdot \tilde{L}_{14} = \left(A - \frac{1}{a_{i_4}}E\right) \cdot \tilde{L}_{14} = \frac{1}{a_{i_1}a_{i_4}} - \frac{1}{a_{i_4}a_{i_1}} = 0.$$

This verifies the claim.

Denote by  $\tilde{C}_\lambda$  the proper transform of the curve  $C_\lambda$  on  $Y$ . Then

$$-K_Y \cdot \tilde{C}_\lambda = a_{i_2}B^2 \cdot (a_{i_3}B - 2E) = 0.$$

We then obtain a contradiction from Lemma 3.1.8.  $\square$

**Theorem 5.2.3.** *Suppose that the weights  $a_3, a_4$  are relatively prime and  $2a_3 + a_4 = d$ . In addition, the equation of the hypersurface  $X$  does not involve the monomial  $wt^2$ . Then the singular point  $O_t$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .*

*Proof.* We first note that the singular point  $O_t$  of the hypersurface  $X$  is of type  $\frac{1}{a_3}(1, a_1, a_2)$ . The hypersurface  $X$  may be assumed to be defined by the equation

$$\begin{aligned} x_i t^3 + t^2 g_{a_4}(x, y, z) + t w g_{a_3}(x, y, z) + t g_{a_3+a_4}(x, y, z) + \\ + w^2 g_{d-2a_4}(x, y, z) + w g_{2a_3}(x, y, z) + g_d(x, y, z) = 0, \end{aligned}$$

where  $x_i$  is either  $y$  or  $z$ , and  $g_i$ 's are quasi-homogeneous polynomials of degrees  $i$ . Note that the polynomial  $f_d$  contains either the monomial  $yw^2$  or the monomial  $zw^2$ . We let  $x_j$  be  $z$  if  $x_i$  is  $y$  and vice versa.

Suppose that the singular point  $O_t$  is a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ . Consider the linear system  $\mathcal{L}$  on  $X$  generated by  $x^e$  and  $x_j$ , where  $e$  is the weight of  $x_j$ . The proper transform of each member of  $\mathcal{L}$  is linearly equivalent to  $eB$ . The base locus of the linear system  $\mathcal{L}$  consists of the curves cut by  $x = x_j = 0$ . They are the curve  $L$  and its residual curve  $R$ . Note that the residual curve  $R$  cannot pass through the point  $O_t$  since we have the monomial  $x_i t^3$ . The proper transform of the curve  $R$  intersect  $B$  positively. Therefore,

$$B \cdot C = eB^3 + K_X \cdot R = \frac{2ea_3 + ea_4}{a_1 a_2 a_3 a_4} - \frac{e}{a_1 a_2 a_3} - \frac{3ea_3}{a_1 a_2 a_3 a_4} = -\frac{e}{a_1 a_2 a_4} < 0.$$

Let  $T$  be the proper transform of the surface on  $X$  cut by the equation  $x_i$ . In addition, let  $S_\lambda$  be the proper transform of the surface on  $X$  cut by the equation  $x_j - \lambda x^e = 0$  for a constant  $\lambda$ . We then see that

$$T \cdot S_\lambda = (2a_3 - a_4)C + R_\lambda,$$

where  $R_\lambda$  is the residual curve. Note that the surface  $T$  is equivalent to  $e'B - E$ , where  $e'$  is the weight of  $x_j$ . Then

$$B \cdot R_\lambda = eB^2 \cdot T - (2a_3 - a_4)B \cdot C = \frac{a_1 a_2 + ea_3 - ea_4}{a_1 a_2 a_4} = 0$$

since  $a_4 = e' + a_3$  and  $ee' = a_1 a_2$ . We then obtain a contradiction from Lemma 3.1.8.  $\square$



### 5.3 Hard involutions

Before we proceed, we put a remark here. In this section we construct the birational involutions  $\tau_t$  (see (5.3.2)) and  $\tau_z$  (see (5.3.6)) along the way provided in [23, Section 4.10]. However we are also able to exclude or untwist the corresponding singular points, following the method given in Section 5.4. We take the former way simply because it looks a bit shorter than the latter.

First, we consider the following five families and their singular point  $O_t$ .

- $N = 23$ ,  $X_{14} \subset \mathbb{P}(1, 2, 3, 4, 5)$ ;
- $N = 40$ ,  $X_{19} \subset \mathbb{P}(1, 3, 4, 5, 7)$ ;
- $N = 44$ ,  $X_{20} \subset \mathbb{P}(1, 2, 5, 6, 7)$ ;
- $N = 61$ ,  $X_{25} \subset \mathbb{P}(1, 4, 5, 7, 9)$ ;
- $N = 76$ ,  $X_{30} \subset \mathbb{P}(1, 5, 6, 8, 11)$ .

For these five families, we may suppose that the hypersurface  $X$  is defined by the equation

$$tw^2 + wg_{d-a_4}(x, y, z) - x_it^3 - t^2g_{d-2a_3}(x, y, z) - tg_{d-a_3}(x, y, z) + g_d(x, y, z) = 0, \quad (5.3.1)$$

where  $x_i$  is either  $y$  or  $z$ , and  $g_i$ 's are quasi-homogeneous polynomials of degrees  $i$ . Then we can define an involution  $\tau_t$  of  $X$  as follows:

$$[x : y : z : t : w] \rightarrow \left[ x : y : z : \frac{g_d^2 - g_{d-a_4}^2(v + g_{d-a_3})}{g_{d-a_4}vw + g_{d-a_4}^2x_it - g_dv} : \frac{g_{d-a_4}v(v + g_{d-a_3}) + g_d(vw + g_{d-a_4}x_it)}{g_{d-a_4}vw + g_{d-a_4}^2x_it - g_dv} \right] \quad (5.3.2)$$

where  $v = w^2 - x_it^2 - tg_{d-2a_3} - g_{d-a_3}$ . To see that it is a birational automorphism, refer to [23, Section 4.10]. However, it can be a biregular automorphism under a certain condition. For example, if the polynomial  $g_{d-a_4}$  is identically zero, then the involution becomes the biregular involution  $[x : y : z : t : w] \rightarrow [x : y : z : t : -w]$ .

**Proposition 5.3.3.** *The involution  $\tau_t$  is biregular if and only if the polynomial  $g_{d-a_4}$  is identically zero.*

*Proof.* It is easy to see that it is biregular if  $g_{d-a_4}$  is identically zero. Suppose that  $g_{d-a_4}$  is not a zero polynomial. Consider the surface cut by the equation  $v = 0$ . It is easy to check that on this surface the involution becomes the map

$$[x : y : z : t : w] \rightarrow \left[ x : y : z : t + \frac{g_{d-2a_3}}{x_i} : -w \right].$$

Therefore, unless the polynomial  $g_{d-2a_3}$  is either identically zero or divisible by  $x_i$ , the involution  $\tau_t$  cannot be biregular.

If the polynomial  $g_{d-2a_3}$  is identically zero, then on the surface cut by  $t = 0$ , the involution becomes the map

$$[x : y : z : t : w] \rightarrow [x : y : z : 0 : 0],$$

and hence the involution  $\tau_t$  cannot be biregular.

Finally, suppose that we can write  $g_{d-2a_3}(x, y, z) = x_i g(x, y, z)$  for some polynomial  $g$ . In this case, we consider the surface cut by the equation  $x_i = 0$ . On this surface the involution  $\tau_t$  becomes

$$[x : y : z : t : w] \rightarrow \left[ x : y : z : -t : \frac{w(wg_{d-a_4} + g_d)}{wg_{d-a_4} - g_d} \right].$$

A general point of the curve cut by  $wg_{d-a_4} - g_d = 0$  on the surface  $x_i = 0$  is mapped to the point  $O_w$ . It shows that the involution  $\tau_t$  cannot be biregular.  $\square$

**Theorem 5.3.4** ([23, Theorem 4.13]). *Suppose  $g_{d-a_4}$  is not identically zero. Then the involution  $\tau_t$  untwists the singular point  $O_t$  as the center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .*

**Theorem 5.3.5.** *Suppose that the hypersurface  $X$  is defined by the equation*

$$tw^2 - x_i t^3 - t^2 g_{d-2a_3}(x, y, z) - t g_{d-a_3}(x, y, z) + g_d(x, y, z) = 0,$$

where  $x_i$  is either  $y$  or  $z$ , and  $g_i$ 's are quasi-homogeneous polynomials of degrees  $i$ . Then the singular point  $O_t$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .

*Proof.* Note that the polynomial  $g_d$  cannot be the zero polynomial since  $X$  is quasi-smooth. Moreover, it is irreducible.

Suppose that the singular point  $O_t$  is a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ . Set

$$v = w^2 - x_i t^2 - t g_{d-2a_3} - g_{d-a_3}$$

and then let  $T$  be the proper transform of the surface cut by the equation  $v = 0$ . Then the surface  $T$  belongs to the linea system  $|2a_4 B|$ .

Choose a general point  $[1 : \mu_1 : \mu_2]$  on the curve defined by the equation  $g_d = 0$  in  $\mathbb{P}(1, a_1, a_2)$ . Then let  $C_{\mu_1, \mu_2}$  be the proper transform of the curve defined by the equations  $v = 0$ ,  $y = \mu_1 x^{a_1}$  and  $z = \mu_2 x^{a_2}$ . We have then

$$B \cdot C_{\mu_1, \mu_2} = \left(A - \frac{1}{a_3} E\right) \cdot C_{\mu_1, \mu_2} = \frac{2}{a_3} - \frac{1}{a_3} E \cdot C_{\mu_1, \mu_2} \leq 0$$

since  $E \cdot C_{\mu_1, \mu_2} \geq 2$ . We then obtain a contradiction from Lemma 3.1.8.  $\square$

Now, we consider the following two families and their singular point  $O_z$ .

- $N = 20$ ,  $X_{13} \subset \mathbb{P}(1, 1, 3, 4, 5)$ ;
- $N = 36$ ,  $X_{18} \subset \mathbb{P}(1, 1, 4, 6, 7)$ .

First, we suppose that the defining equation of  $X_{13}$  contains the monomial  $tz^3$ . Note that the defining equation of  $X_{18}$  always contains the monomial  $tz^3$ .

We may then assume that the hypersurface  $X$  is defined by the equation

$$zw^2 + w f_{d-a_4}(x, y, t) - tz^3 - z^2 f_{d-2a_2}(x, y, t) - z f_{d-a_2}(x, y, t) + f_d(x, y, t) = 0,$$

where  $f_i$ 's are quasi-homogeneous polynomials of degrees  $i$ . We can define an involution  $\tau_z$  of  $X$  as follows:

$$[x : y : z : t : w] \rightarrow \left[ x : y : \frac{f_d^2 - f_{d-a_4}^2(u + f_{d-a_2})}{f_{d-a_4}uw + f_{d-a_4}^2zt - f_d u} : t : \frac{f_{d-a_4}u(u + f_{d-a_2}) + f_d(uw + f_{d-a_4}zt)}{f_{d-a_4}uw + f_{d-a_4}^2zt - f_d u} \right] \quad (5.3.6)$$

where  $u = w^2 - tz^2 - zf_{d-2a_2} - f_{d-a_2}$ .

**Proposition 5.3.7.** *The involution  $\tau_z$  is biregular if and only if the polynomial  $f_{d-a_4}$  is identically zero.*

*Proof.* The proof is similar as that of Proposition 5.3.3.  $\square$

**Theorem 5.3.8** ([23, Theorem 4.13]). *Suppose  $f_{d-a_4}$  is not identically zero. Then the involution  $\tau_z$  untwists the singular point  $O_z$  as a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .*

**Theorem 5.3.9.** *If  $f_{d-a_4}$  is not identically zero, then the singular point  $O_z$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .*

*Proof.* The proof is almost same as that of Theorem 5.3.5.  $\square$

Now we suppose that the defining equation of  $X_{13}$  does not contain the monomial  $tz^3$ .

**Theorem 5.3.10.** *Let  $X$  be a quasi-smooth hypersurface of degree 13 in  $\mathbb{P}(1, 1, 3, 4, 5)$ . Suppose that the hypersurface  $X$  is defined by the equation*

$$zw^2 + w(f_8(x, y, t) + at^2) - yz^4 - z^3f_4(x, y) - z^2f_7(x, y, t) - zf_{10}(x, y, t) + f_{13}(x, y, t) = 0,$$

where  $a$  is a constant,  $f_i$ 's are quasi-homogeneous polynomials of degrees  $i$  and  $f_8$  does not contain the monomial  $t^2$ . Then the singular point  $O_z$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .

*Proof.* Note that the polynomial  $f_{13}$  must contain the monomial  $xt^3$ ; otherwise  $X$  would not be quasi-smooth.

Suppose that the singular point  $O_z$  is a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ . Lets us seek for a contradiction.

Before we proceed, we fix the notations. Let  $L_{zt}$  be the proper transform of the curve  $L_{zt}$  on  $X$  and  $\tilde{L}_{tw}$  be the proper transform of the curve  $L_{tw}$ . Let  $\tilde{S}_y$  be the proper transform of the surface  $S_y$ . Finally we let  $\mathcal{L}$  be the linear system on  $X$  generated by  $x^5$ ,  $xt$  and  $w$ .

First we consider the case where  $a = 0$ . The base locus of the linear system  $|-K_X|$  consists of the curve cut by  $x = y = 0$ . The curve has two irreducible components. One is the curve  $L_{zt}$  and the other is the curve  $L_{tw}$ . We see that

$$S \cdot \tilde{S}_y = \tilde{L}_{tw} + 2\tilde{L}_{zt}.$$

Note that the curve  $\tilde{L}_{tw}$  does not pass through the point  $O_z$ . We obtain

$$B \cdot \tilde{L}_{zt} = \frac{1}{2}B \cdot S \cdot \tilde{S}_y - \frac{1}{2}A \cdot \tilde{L}_{tw} = \frac{1}{2}A^3 - \frac{4}{54}E^3 - \frac{1}{40} = -\frac{1}{4}.$$

For the proper transform  $\tilde{S}_{\lambda,\mu}$  of a general member in  $\mathcal{L}$  cut by  $w = \lambda x^5 + \mu xt$ , we have

$$\tilde{S}_y \cdot \tilde{S}_{\lambda,\mu} = \tilde{L}_{zt} + R_{\lambda,\mu},$$

where  $R_{\lambda,\mu}$  is the residual curve and it sweeps the surface  $\tilde{S}_y$ . Meanwhile, we obtain

$$B \cdot R_{\lambda,\mu} = B \cdot \tilde{S}_y \cdot \tilde{S}_{\lambda,\mu} - B \cdot \tilde{L}_{zt} = 5A^3 - \frac{8}{27}E^3 + \frac{1}{4} = 0.$$

Then the argument in the last part of the proof of Theorem 5.2.3 shows a contradiction.

Now we consider the case where  $a \neq 0$ . By coordinate change we may assume that  $a = 1$ . The base locus of the linear system  $| -K_X |$  consists of the curve cut by  $x = y = 0$ . The curve has two irreducible components. One is  $L_{zt}$  and the other is the curve  $L$  defined by  $x = y = zw + t^2 = 0$ . Let  $\tilde{L}$  be the proper transform of the latter. The curves  $\tilde{L}$  and  $\tilde{L}_{zt}$  intersect the exceptional divisor  $E$  at a smooth point. We have  $S \cdot \tilde{S}_y = \tilde{L}_{zt} + \tilde{L}$  and

$$B \cdot \tilde{L}_{zt} = A \cdot \tilde{L}_{zt} - \frac{1}{3}E \cdot \tilde{L}_{zt} = -\frac{1}{4}, \quad B \cdot \tilde{L} = A \cdot \tilde{L} - \frac{1}{3}E \cdot \tilde{L} = \frac{2}{15} - \frac{1}{3} = -\frac{1}{5}.$$

For the proper transform  $\tilde{S}_{\lambda,\mu}$  of a general member in  $\mathcal{L}$  cut by  $w = \lambda x^5 + \mu xt$ , we have

$$\tilde{S}_y \cdot \tilde{S}_{\lambda,\mu} = \tilde{L}_{zt} + R_{\lambda,\mu},$$

where  $R_{\lambda,\mu}$  is the residual curve and it sweeps the surface  $\tilde{S}_y$ . Note that the curve  $R_{\lambda,\mu}$  does not contain the curve  $\tilde{L}_{zt}$  since the defining polynomial of  $X$  contains either  $xt^3$  or  $wt^2$ .

Meanwhile, we obtain

$$B \cdot R_{\lambda,\mu} = B \cdot \tilde{S}_y \cdot \tilde{S}_{\lambda,\mu} - B \cdot \tilde{L}_{zt} = 5A^3 - \frac{8}{27}E^3 + \frac{1}{4} = 0.$$

We then obtain a contradiction from Lemma 3.1.8.  $\square$

We consider the family  $N = 7$ , a quasi-smooth hypersurface  $X$  of degree 8 in  $\mathbb{P}(1, 1, 2, 2, 3)$ . It has four singular points of type  $\frac{1}{2}(1, 1, 1)$  along the curve  $L_{zt}$ . Let  $P$  be one of the singular points. By a coordinate change, we may assume that  $P$  is the point  $O_t$ . Then the hypersurface  $X$  may be assumed to be defined by an equation of one of the following forms

Type I:

$$tw^2 + wg_5(x, y, z) - zt^3 - t^2g_4(x, y, z) - tg_6(x, y, z) + g_8(x, y, z) = 0; \quad (5.3.11)$$

Type II:

$$(z + f_2(x, y))w^2 + wf_5(x, y, z, t) - zt^3 - t^2f_4(x, y, z) - tf_6(x, y, z) + f_8(x, y, z) = 0,$$

where  $f_i$  and  $g_j$  are quasi-homogenous polynomials of degrees  $i$  and  $j$ . In the latter equation, the quasi-homogeneous polynomial  $f_5$  must contain either  $xt^2$  or  $yt^2$ : otherwise it would not be quasi-smooth at the point  $[0 : 0 : 0 : 1 : 1]$ .

**Theorem 5.3.12.** *Suppose that the hypersurface  $X$  is defined by the equation of Type I. If the polynomial  $g_5$  is not a zero polynomial, then the involution  $\tau_t$  defined in the beginning of this section untwists the singular point  $O_t$ . If the polynomial  $g_5$  is a zero polynomial, the singular point  $O_t$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ .*

*Proof.* The proof is the same as in the first cases.  $\square$

The hypersurface defined by the equation of Type II may have an involution that untwists the singular point  $O_t$ . Since its construction is quite complicated, we explain the method in a separate section.

## 5.4 Invisible involution

In this section we consider a special case for the family No. 23 and a special case (Type II) for the family No. 7.

**Theorem 5.4.1.** *Suppose that the hypersurface  $X$  of degree 14 in  $\mathbb{P}(1, 2, 3, 4, 5)$  is defined by the equation*

$$(t + by^2)w^2 + yt(t - \alpha_1y^2)(t - \alpha_2y^2) + z^4y + xtz^3 + xf_{13}(x, y, z, t, w) + yg_{12}(y, z, t, w) = 0.$$

*Then it has a birational involution that untwists the singular point  $O_z$ .*

*Proof.* The polynomial  $yz + xt$  vanishes at the point  $O_z$  with multiplicity  $\frac{5}{3}$ . Let  $\mathcal{H}$  be the linear subsystem of  $|-5K_X|$  generated by  $x^5, xy^2, x^3y$  and  $yz + xt$ . Let  $\pi : X \dashrightarrow \mathbb{P}(1, 2, 5)$  be the rational map induced by  $[x : y : z : t : w] \rightarrow [x : y : yz + xt]$ . Then  $\pi$  is a morphism outside of the curves  $L_{zt}$  and  $L_{zw}$ . Moreover, the map  $\pi$  is dominant, which implies, in particular, that  $\mathcal{H}$  is not composed from a pencil. Furthermore, its general fiber is an irreducible curve that is birational to an elliptic curve. To see this, we put  $y = \lambda x^2$  and  $yz + xt = \mu x^5$  with sufficiently general complex numbers  $\lambda$  and  $\mu$ . On the hypersurface  $X$ , we take the intersection of the surface defined by  $y = \lambda x^2$  and the surface defined by  $yz + xt = \mu x^5$ . This intersection is the same as the intersection of the surface defined by  $y = \lambda x^2$  and the reducible surface defined by  $x(\lambda xz + t - \mu x^4) = 0$ . Therefore, the intersection is the 1-cycle

$$(L_{zw} + 2L_{zt}) + (L_{zw} + C_{\lambda, \mu}) = 2L_{zw} + 2L_{zt} + C_{\lambda, \mu},$$

where the curve  $C_{\lambda, \mu}$  is defined by the equation

$$\begin{aligned} & (\mu x^3 - \lambda z + b\lambda^2 x^3)w^2 + \lambda x^4(\mu x^3 - \lambda z)(\mu x^3 - \alpha_1 \lambda^2 x^3 - \lambda z)(\mu x^3 - \alpha_2 \lambda^2 x^3 - \lambda z) + \\ & + \mu x^4 z^3 + f_{13}(x, \lambda x^2, z, \mu x^4 - \lambda xz, w) + \lambda x g_{12}(\lambda x^2, z, \mu x^4 - \lambda xz, w) = 0 \end{aligned} \quad (5.4.2)$$

in  $\mathbb{P}(1, 3, 5)$ . The curve  $C_{\lambda, \mu}$  is a general fiber of the map  $\pi$ . Setting  $x = 1$  in Equation (5.4.2), we consider the curve defined by

$$\begin{aligned} & (\mu + b\lambda^2 - \lambda z)w^2 + \lambda(\mu - \lambda z)(\mu - \alpha_1 \lambda^2 - \lambda z)(\mu - \alpha_2 \lambda^2 - \lambda z) + \\ & + \mu z^3 + f_{13}(1, \lambda, z, \mu - \lambda z, w) + \lambda g_{12}(\lambda, z, \mu - \lambda z, w) = 0 \end{aligned} \quad (5.4.3)$$

in  $\mathbb{C}^2$ . It is a smooth affine plane cubic curve. Moreover, for a general complex number  $\lambda$ , the curve (5.4.3) is always irreducible and reduced for every value of  $\mu$ . Indeed, when either  $b \neq 0$  or  $\mu \neq 0$ , it is easy to check. When  $b = 0$  and  $\mu = 0$  it follows from the fact that at least one monomial in variables  $x, y, w$  must be contained in either  $f_{13}$  or  $g_{12}$ .

Let  $\mathcal{H}_Y$  be the proper transform of the linear system  $\mathcal{H}$  by the weighted blow up  $f$ . It is the linear system  $|-5K_Y|$ . Let  $g: W \rightarrow Y$  be the weighted blow up at the point over  $O_w$  with weight  $(1, 2, 3)$  and  $\mathcal{H}_W$  the proper transform of  $\mathcal{H}_Y$  by the morphism  $g$ . Let  $\hat{E}$  be the proper transform of  $E$  by the weighted blow up  $g$  and  $G$  be the exceptional divisor of  $g$ .

The linear system  $\mathcal{H}_W$  coincides with the linear system  $|-5K_W|$ .

The base locus of the linear system  $\mathcal{H}$  is given by the equation  $x = yz + xt = 0$ . Therefore, it consists of  $L_{zw}, L_{zt}$  and the curve  $C$  cut by the equation  $x = z = 0$ . The curve  $C$  may not be irreducible. Indeed,  $C$  is irreducible if and only if  $b \neq 0$ . If  $b = 0$ , then  $C$  consists of two irreducible curves  $L_{yw}$  and  $R$ , where  $R$  is an irreducible curve passing through neither

the point  $O_z$  nor the point  $O_w$ . Let  $\hat{L}_{zw}$ ,  $\hat{L}_{zt}$ ,  $\hat{L}_{yw}$ ,  $\hat{R}$  and  $\hat{C}$  be the proper transforms of the curves  $L_{zw}$ ,  $L_{zt}$ ,  $L_{yw}$ ,  $R$  and  $C$ , respectively, by the morphism  $f \circ g$ . We have

$$\begin{aligned} -K_W \cdot \hat{L}_{zw} &= A \cdot L_{zw} - \frac{1}{3}\hat{E} \cdot \hat{L}_{zw} - \frac{1}{5}G \cdot \hat{L}_{zw} = -\frac{1}{6}; \\ -K_W \cdot \hat{L}_{zt} &= A \cdot L_{zt} - \frac{1}{3}\hat{E} \cdot \hat{L}_{zt} = -\frac{1}{4}; \\ -K_W \cdot \hat{L}_{yw} &= A \cdot L_{yw} - \frac{1}{5}G \cdot \hat{L}_{yw} = \frac{1}{30}; \\ -K_W \cdot \hat{R} &= A \cdot R > 0; \quad -K_W \cdot \hat{C} = A \cdot C > 0. \end{aligned}$$

Therefore, the curves  $\hat{L}_{zw}$  and  $\hat{L}_{zt}$  are the only curves that intersect  $-K_W$  negatively. The log pair  $(W, \frac{1}{5}\mathcal{H}_W)$  is canonical, and hence the log pair  $(W, (\frac{1}{5} + \epsilon)\mathcal{H}_W)$  is Kawamata log terminal for sufficiently small  $\epsilon > 0$ . Since the curves  $\hat{L}_{zw}$  and  $\hat{L}_{zt}$  are the only curves that intersect  $K_W + (\frac{1}{5} + \epsilon)\mathcal{H}_W \sim_{\mathbb{Q}} -\epsilon K_W$  negatively, there is a log flip  $\chi : W \dashrightarrow U$  along the curves  $\hat{L}_{zw}$  and  $\hat{L}_{zt}$  ([39]). Let  $\check{E}$  and  $\check{G}$  be the proper transforms of the divisors  $\hat{E}$  and  $G$ , respectively, by  $\chi$ . The anticanonical divisor  $K_U + (\frac{1}{5} + \epsilon)\mathcal{H}_U$  is nef, where  $\mathcal{H}_U$  is the proper transform of  $\mathcal{H}_W$  by the isomorphism  $\chi$  in codimension 1.

By the Log abundance ([33]), the linear system  $| -mK_U |$  is free for sufficiently large  $m$ . Hence, it induces a dominant morphism  $\eta : U \rightarrow \Sigma$  with connected fibers, where  $\Sigma$  is a normal variety. We claim that  $\Sigma$  is a surface and  $\eta$  is an elliptic fibration. For this claim, let  $\hat{C}_{\lambda, \mu}$  be the proper transform of a general fiber  $C_{\lambda, \mu}$  of the map  $\pi$  on the threefold  $W$  and let  $\check{C}_{\lambda, \mu}$  be its proper transform on  $U$ . Then

$$-K_W \cdot \hat{C}_{\lambda, \mu} = -10K_W^3 - 2(-K_W) \cdot (\hat{L}_{zw} + \hat{L}_{zt}) = 0.$$

In particular, the curve  $\check{C}_{\lambda, \mu}$  is disjoint from the curves  $\hat{L}_{zt}$  and  $\hat{L}_{zw}$  because the base locus of the linear system  $| -5K_W |$  contains the curves  $\hat{L}_{zt}$  and  $\hat{L}_{zw}$ . Therefore,

$$-K_U \cdot \check{C}_{\lambda, \mu} = 0.$$

It implies that  $\eta$  contracts  $\check{C}_{\lambda, \mu}$ . Since we already proved that  $C_{\lambda, \mu}$  is birational to an elliptic curve and  $\mathcal{H}$  is not composed from a pencil, we can see that  $\eta$  is an elliptic fibration. Moreover, we have proved the existence of a commutative diagram

$$\begin{array}{ccc} & W & \xrightarrow{\chi} U \\ & \swarrow g & \searrow \eta \\ Y & & \\ \downarrow f & & \\ X & & \\ & \searrow \pi & \\ & \mathbb{P}(1, 2, 5) & \xleftarrow{\theta} \Sigma \end{array}$$

where  $\theta$  is a birational map.

Equation (5.4.2) shows that the divisor  $\check{G}$  is a section of the elliptic fibration  $\eta$  and  $\check{E}$  is a 2-section of  $\eta$ . Let  $\tau_U$  be a birational involution of the threefold  $U$  that is induced by the reflection of the general fiber of  $\eta$  with respect to the section  $\check{G}$ . Then  $\tau_U$  is biregular in codimension one because  $K_U$  is  $\eta$ -nef by constructions. Put  $\tau_W = \chi^{-1} \circ \tau_U \circ \chi$ ,  $\tau_Y = g \circ \tau_W \circ g^{-1}$  and  $\tau = f \circ \tau_Y \circ f^{-1}$ . Then  $\tau_W$  is also biregular in codimension one since  $\chi$  is a log flip. Moreover, we have  $\tau_W(G) = G$  since  $\tau_U(\check{G}) = \check{G}$  by construction. This implies that  $\tau_Y$  is biregular in codimension one.

We claim that the involution  $\tau$  is not biregular. To see this, let us prove a stronger claim that the restriction of  $\tau$  to a general surface in  $|-2K_X|$  is not biregular.

Let  $S_\lambda$  be the surface on the hypersurface  $X$  cut by the equation  $y = \lambda x^2$ . It follows from the defining equation of the surface  $S_\lambda$  that the surface is normal. Moreover, the surface  $S_\lambda$  is  $\tau$ -invariant by construction. Let  $\tau_\lambda$  be the restriction of  $\tau$  to the surface  $S_\lambda$ . It is a birational involution of the surface  $S_\lambda$  since the surface is  $\tau$ -invariant.

We claim that the birational involution  $\tau_\lambda$  is not biregular. To verify the claim, we suppose that it is biregular.

We have a rational map  $\pi_\lambda: S_\lambda \dashrightarrow \mathbb{P}(1, 5) \cong \mathbb{P}^1$  induced by the rational map  $\pi: X \dashrightarrow \mathbb{P}(1, 2, 5)$ . Note that the curves  $L_{zt}$  and  $L_{zw}$  are contained in  $S_\lambda$ . The rational map  $\pi_\lambda: S_\lambda \dashrightarrow \mathbb{P}^1$  is given by the pencil  $\mathbb{P}$  of curves on the surface  $S_\lambda \subset \mathbb{P}(1, 3, 4, 5)$  cut by the equations

$$\delta x^4 = \epsilon(\lambda xz + t),$$

where  $[\delta : \epsilon] \in \mathbb{P}^1$ . Its base locus is cut out on  $S_\lambda$  by  $x = t = 0$ , which implies that the base locus of the pencil  $\mathcal{P}$  is the curve  $L_{zw}$ .

The map  $\pi_\lambda$  is not defined only at the points  $O_w$  and  $O_z$ . To see this, plug  $t = \frac{\delta}{\epsilon}x^4 - \lambda xz$  in the defining equation of the surface  $S_\lambda$  (with general  $[\delta : \epsilon] \in \mathbb{P}^1$ ), divide the resulting equation by  $x$  (removing the base curve  $L_{zw}$ ), and put  $x = 0$  into the resulting equation in  $x$ ,  $z$ , and  $w$  (we know that the base locus of  $\mathcal{P}$  is  $L_{zw}$ ). This gives the system of equations  $zw^2 = x = t = 0$ , which means that the map  $\pi_\lambda$  is not defined only at the points  $O_w$  and  $O_z$ .

Let  $C_\lambda$  be a general fiber of the map  $\pi_\lambda$ . Then  $C_\lambda$  is given by Equation (5.4.3) with generic  $\mu$ . As shown in the beginning, the fiber  $C_\lambda$  is an irreducible curve birational to a smooth elliptic curve. Let  $\nu: \check{C}_\lambda \rightarrow C_\lambda$  be the normalization of the curve  $C_\lambda$ . It follows from (5.4.3) that  $\nu^{-1}(O_w)$  consists of a single point and  $\nu^{-1}(O_z)$  consists of two distinct points. Note that we can consider the curves  $C_\lambda$  and  $\check{C}_\lambda$  (and the map  $\nu$ ) to be defined over the function field  $\mathbb{C}(\mu)$ . In this case,  $\nu^{-1}(O_z)$  consists of a single point of degree 2, i.e., a point splitting into two points over the algebraic closure of  $\mathbb{C}(\mu)$ .

Resolving the indeterminacy of the rational map  $\pi_\lambda$ , we obtain an elliptic fibration  $\bar{\pi}_\lambda: \bar{S}_\lambda \rightarrow \mathbb{P}^1$ . Thus, we have a commutative diagram

$$\begin{array}{ccc} & \bar{S}_\lambda & \\ \sigma \swarrow & & \searrow \bar{\pi}_\lambda \\ S_\lambda & \dashrightarrow \bar{\pi}_\lambda & \mathbb{P}^1, \end{array}$$

where  $\sigma$  is a birational map. Note that there exist exactly two  $\sigma$ -exceptional prime divisors that do not lie in the fibers of  $\bar{\pi}_\lambda$ . One of the divisors is over the point  $O_z$  and the other is over the point  $O_w$ . Let  $\bar{E}_\lambda$  and  $\bar{G}_\lambda$  be these two exceptional divisors, respectively. Then  $\bar{G}_\lambda$

is a section of  $\bar{\pi}_\lambda$  and  $\bar{E}_\lambda$  is a 2-section of  $\bar{\pi}_\lambda$ . Denote other  $\sigma$ -exceptional curves (if any) by  $F_1, \dots, F_r$ .

Put  $\bar{\tau}_\lambda = \sigma^{-1} \circ \tau_\lambda \circ \sigma$ . Recall that we assumed that the involution  $\tau_\lambda$  is biregular. Thus, we may assume that  $\bar{\tau}_\lambda$  is biregular as well. Let  $\bar{C}_\lambda$  be the proper transform of the curve  $C_\lambda$  on  $\bar{S}_\lambda$ . Then  $\bar{C}_\lambda \cong \check{C}_\lambda$ , since  $\bar{C}_\lambda$  is smooth. Moreover, the curve  $\bar{C}_\lambda$  is  $\bar{\tau}_\lambda$ -invariant. Furthermore,  $\bar{\tau}_\lambda|_{\bar{C}_\lambda}$  is given by the reflection with respect to the point  $\bar{G}_\lambda \cap \bar{C}_\lambda$ . On the other hand, the divisor  $\bar{E}_\lambda$  must be  $\bar{\tau}_\lambda$ -invariant since  $\tau_\lambda$  is biregular. This means, in particular, that the divisor

$$\left(\bar{E}_\lambda - 2\bar{G}_\lambda\right)\Big|_{\bar{C}_\lambda} \in \text{Pic}^0(\bar{C}_\lambda)$$

is torsion (in fact, this means that this divisor is 2-torsion). Therefore, the divisor  $\bar{E}_\lambda - 2\bar{G}_\lambda$  must be numerically equivalent to a linear combinations of curves on  $\bar{S}_\lambda$  that lie in the fibers of  $\bar{\pi}_\lambda$  (This method originates from [3]).

Let  $\bar{L}_{zt}$  and  $\bar{L}_{zw}$  be the proper transforms of the curves  $L_{zt}$  and  $L_{zw}$  by  $\sigma$ , respectively. Then  $\bar{L}_{zt}$  and  $\bar{L}_{zw}$  lies in the same fiber of the elliptic fibration  $\bar{\pi}_\lambda$ . Moreover, we claim that every other fiber of  $\bar{\pi}_\lambda$  contains exactly one irreducible reduced curve that is not  $\sigma$ -exceptional. Indeed, this follows from the fact (mentioned earlier) that for a general complex number  $\lambda$ , the curve (5.4.3) is always irreducible and reduced for every value of  $\mu$ . Since all fibers of  $\bar{\pi}_\lambda$  (with scheme structure) are numerically equivalent and the divisor  $\bar{E}_\lambda - 2\bar{G}_\lambda$  is numerically equivalent to a linear combinations of curves that lie in the fibers of  $\bar{\pi}_\lambda$ , we obtain

$$\bar{E}_\lambda - 2\bar{G}_\lambda \sim_{\mathbb{Q}} c_{zt}\bar{L}_{zt} + c_{zw}\bar{L}_{zw} + \sum_{i=1}^r c_i F_i$$

for some rational numbers  $c_{zt}, c_{zw}, c_1, \dots, c_r$ . The intersection form of the curves  $\bar{E}_\lambda, \bar{G}_\lambda, F_1, \dots, F_r$  is negative definite since these curves are  $\sigma$ -exceptional. Therefore,  $(c_{zt}, c_{zw}) \neq (0, 0)$ . On the other hand, we have

$$0 \sim_{\mathbb{Q}} c_{zt}L_{zt} + c_{zw}L_{zw}$$

on the surface  $S_\lambda$ . In particular, the intersection form of the curves  $L_{zw}$  and  $L_{zt}$  is degenerate on the surface  $S_\lambda$ .

Meanwhile, from the intersection numbers

$$(2L_{zt} + L_{zw}) \cdot L_{zw} = \frac{1}{15}, \quad (2L_{zt} + L_{zw})^2 = \frac{7}{30}, \quad L_{zw} \cdot L_{zt} = \frac{1}{3}$$

on the surface  $S_\lambda$ , we obtain

$$\begin{pmatrix} L_{zw}^2 & L_{zw} \cdot L_{zt} \\ L_{zw} \cdot L_{zt} & L_{zt}^2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{8} \end{pmatrix}.$$

This is a contradiction. The obtained contradiction shows that  $\tau_\lambda$  is not biregular. In particular, the involution  $\tau$  is not biregular. This shows that  $\tau_Y(E) \neq E$  since  $\tau_Y$  is biregular in codimension one. Indeed, if  $\tau_Y(E) = E$ , then  $\tau$  is also biregular in codimension one, which implies that  $\tau$  is biregular since  $\text{Pic}(X) \cong \mathbb{Z}$ . Since  $\tau_Y(E) \neq E$ , the action of  $\tau_Y$  on  $\text{Pic}(Y)$  is not trivial. The action leaves  $-K_Y$  fixed. Therefore,  $\tau$  untwists the singular point  $O_z$  by Definition 5.1.1.  $\square$



Now we go back to the hypersurfaces of Type II in the family No. 7 described in the previous section.

**Theorem 5.4.4.** *Suppose that the hypersurface  $X$  in the family No. 7 is defined by the equation (5.3.11) of Type II. If the singular point  $O_t$  is a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ , then there is a birational involution that untwists the singular point  $O_t$ .*

*Proof.* Before we proceed the proof, note that  $f_5$  must contain either  $xt^2$  or  $yt^2$ . We write  $f_5(x, y, z, t) = g_5(x, y, z, t) + a_1xt^2 + a_2yt^2$ , where  $g_5$  contains neither  $xt^2$  nor  $yt^2$ .

Let  $\mathcal{H}$  be the linear subsystem of  $|-2K_X|$  generated by  $x^2, xy, y^2$  and  $z$ . Let  $\pi: X \dashrightarrow \mathbb{P}(1, 1, 2)$  be the rational map induced by  $[x : y : z : t : w] \rightarrow [x : y : z]$ . It is a morphism outside of the curve  $L_{tw}$ . Moreover, the map is dominant. Its general fiber is an irreducible curve birational to an elliptic curve. To see this, on the hypersurface  $X$ , consider the surface cut by  $y = \lambda x$  and the surface cut by  $z = \mu x^2$ , where  $\lambda$  and  $\mu$  are sufficiently general complex numbers. Then the intersection of these two surfaces is the 1-cycle  $L_{tw} + C_{\lambda, \mu}$ , where the curve  $C_{\lambda, \mu}$  is a curve defined by the equation

$$\begin{aligned} & (\mu + f_2(1, \lambda))xw^2 + (a_1 + a_2\lambda)wt^2 - \mu xt^3 + \\ & + \frac{wg_5(x, \lambda x, \mu x^2, t) - t^2 f_4(x, \lambda x, \mu x^2) - t f_6(x, \lambda x, \mu x^2) + f_8(x, \lambda x, \mu x^2)}{x} = 0 \end{aligned} \quad (5.4.5)$$

in  $\mathbb{P}(1, 2, 3)$ . From the equation we can easily see that the curve  $C_{\lambda, \mu}$  is irreducible and reduced. Furthermore, plugging  $x = 1$  into Equation (5.4.5), we see that the curve is birational to an elliptic curve. The curve  $C_{\lambda, \mu}$  is a general fiber of the map  $\pi$ .

For a general complex number  $\lambda$ , the curve (5.4.5) is always reduced for every value of  $\mu$ . Moreover, for a general complex number  $\lambda$ , the curve  $L_{tw}$  cannot be irreducible component of the curve (5.4.5) for every value of  $\mu$ . Unfortunately, we failed to prove that for a general complex number  $\lambda$ , the curve (5.4.5) is always irreducible for every value of  $\mu$ . If this is not true, we will prove that  $O_t$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ . If this is true, we will construct a birational involution that untwists the singular point  $O_t$ .

Suppose that for general  $\lambda$  there is always  $\mu$  such that the curve (5.4.5) is reducible. Since the base locus of  $\mathcal{H}$  consists of the curve  $L_{tw}$ , there is a one-dimensional family of reducible curves  $C_{\lambda, \mu}$  given by (5.4.5) with general  $\lambda$  and some  $\mu$  depending on  $\lambda$ . Denote the general curve in this one-dimensional family by  $C$ . From (5.4.5) we can immediately notice that a curve  $C_{\lambda, \mu}$  consists of two reduced and irreducible curves. One is the curve  $C_1$  defined by  $y - \lambda x = z - \mu x^2 = w - h_3(x, t) = 0$  for some polynomial  $h_3$ . The other is the curve  $C_2$  defined by  $y - \lambda x = z - \mu x^2 = t^2 - h_4(x, t, w) = 0$  for some polynomial  $h_4$ . Note that  $C_1$  passes through the point  $O_t$ . Let  $\tilde{C}_1$  be the proper transform of the curve  $C_1$  on  $Y$ . Then  $-K_Y \cdot \tilde{C}_1 = 0$  and  $\tilde{C}_1 \cdot E > 0$ . By Lemma 3.1.8, the point  $O_t$  cannot be a center of non-canonical singularities of the pair  $(X, \frac{1}{n}\mathcal{M})$ . Therefore our condition implies that for a general complex number  $\lambda$ , the curve (5.4.5) is always irreducible for every value of  $\mu$ .

Let  $g: Z \rightarrow Y$  be the weighted blow up at the point over  $O_w$  with weight  $(1, 1, 2)$  and let  $F$  be its exceptional divisor. The divisor  $F$  contains a singular point of  $Z$  that is of type  $\frac{1}{2}(1, 1, 1)$ . Let  $h: W \rightarrow Z$  be the blow up at this singular point with the exceptional divisor  $G$ . Let  $\hat{L}_{tw}$  and  $\check{L}_{tw}$  be the proper transforms of the curve  $L_{tw}$  by the morphism  $f \circ g \circ h$  and by the morphism  $f \circ g$ , respectively. Also, let  $\hat{E}$  and  $\hat{F}$  be the proper transforms of the exceptional divisors  $E$  and  $F$  by the morphism  $g \circ h$  and by the morphism  $h$ , respectively.

Let  $\mathcal{H}_Y$  and  $\mathcal{H}_W$  be the proper transforms of the linear system  $\mathcal{H}$  by the morphism  $f$  and by the morphism  $f \circ g \circ h$ , respectively. We then see that  $\mathcal{H}_Y = |-2K_Y|$  and  $\mathcal{H}_W = |-2K_W|$ . The base locus of the linear system  $\mathcal{H}$  consists of the single curve  $L_{tw}$ . We have

$$-K_W \cdot \hat{L}_{tw} = A \cdot L_{tw} - \frac{1}{2} \hat{E} \cdot \hat{L}_{tw} - \frac{1}{3} \check{F} \cdot \check{L}_{tw} - \frac{1}{2} G \cdot \hat{L}_{tw} = -1.$$

Therefore, the curve  $\hat{L}_{tw}$  is the only curve that intersects  $-K_W$  negatively.

The log pair  $(W, \frac{1}{2}\mathcal{H}_W)$  is canonical, and hence the log pair  $(W, (\frac{1}{2} + \epsilon)\mathcal{H}_W)$  is Kawamata log terminal for sufficiently small  $\epsilon > 0$ . Since the curve  $\hat{L}_{tw}$  is the only curve that intersects  $K_W + (\frac{1}{2} + \epsilon)\mathcal{H}_W \sim_{\mathbb{Q}} -\epsilon K_W$  negatively, there exists a log flip  $\chi : W \dashrightarrow U$  along the curve  $\hat{L}_{tw}$  ([39]). Let  $\check{E}$ ,  $\check{F}$  and  $\check{G}$  be the proper transforms of the divisors  $\hat{E}$ ,  $\hat{F}$  and  $G$ , respectively, by  $\chi$ . The anticanonical divisor  $K_U + (\frac{1}{2} + \epsilon)\mathcal{H}_U$  is nef, where  $\mathcal{H}_U$  is the proper transform of  $\mathcal{H}_W$  by the isomorphism  $\chi$  in codimension 1.

By the Log abundance ([33]), the linear system  $|-mK_U|$  is free for sufficiently large  $m$ . Hence, it induces a dominant morphism  $\eta : U \rightarrow \Sigma$  with connected fibers, where  $\Sigma$  is a normal variety. We claim that  $\Sigma$  is a surface and  $\eta$  is an elliptic fibration. For this claim, let  $\hat{C}_{\lambda,\mu}$  be the proper transform of a general fiber  $C_{\lambda,\mu}$  of the map  $\pi$  on the threefold  $W$  and let  $\check{C}_{\lambda,\mu}$  be its proper transform on  $U$ . Then

$$-K_W \cdot \hat{C}_{\lambda,\mu} = -2K_W^3 - (-K_W) \cdot \hat{L}_{tw} = 0.$$

In particular, the curve  $\hat{C}_{\lambda,\mu}$  is disjoint from the curve  $\hat{L}_{tw}$  because the base locus of the linear system  $|-2K_W|$  contains the curves  $\hat{L}_{tw}$ . Therefore,  $-K_U \cdot \check{C}_{\lambda,\mu} = 0$ . It implies that  $\eta$  contracts  $\check{C}_{\lambda,\mu}$ . Since  $\check{C}_{\lambda,\mu}$  is an elliptic curve and  $\mathcal{H}$  is not composed from a pencil, we can see that  $\eta$  is an elliptic fibration. Moreover, we have proved the existence of a commutative diagram

$$\begin{array}{ccccc}
 & & W & \xrightarrow{\chi} & U \\
 & & \swarrow h & & \searrow \eta \\
 Z & & & & \\
 \downarrow g & & & & \\
 Y & & & & \\
 \downarrow f & & & & \\
 X & & & & \\
 \searrow \pi & & & & \\
 \mathbb{P}(1, 1, 2) & \xleftarrow{\theta} & & & \Sigma
 \end{array}$$

where  $\theta$  is a birational map.

Equation (5.4.5) shows that the divisors  $\check{E}$  and  $\check{G}$  are sections of the elliptic fibration  $\eta$ . Let  $\tau_U$  be a birational involution of the threefold  $U$  that is induced by the reflection of the general fiber of  $\eta$  with respect to the section  $\check{G}$ . Then  $\tau_U$  is biregular in codimension one because  $K_U$  is  $\eta$ -nef by our construction. Put  $\tau_W = \chi^{-1} \circ \tau_U \circ \chi$ ,  $\tau_Y = (g \circ h) \circ \tau_W \circ (g \circ h)^{-1}$  and  $\tau = f \circ \tau_Y \circ f^{-1}$ . Then  $\tau_W$  is also biregular in codimension one since  $\chi$  is a log flip. Moreover, we have  $\tau_W(G) = G$  since  $\tau_U(\check{G}) = \check{G}$  by construction. The divisor  $\check{F}$  lies on the fibers of

the elliptic fibration  $\eta$ . Therefore,  $\tau_U(\check{F}) = \check{F}$ , and hence  $\tau_W(\hat{F}) = \hat{F}$ . Consequently,  $\tau_Y$  is biregular in codimension one.

We claim that the involution  $\tau$  is not biregular. To see this, let us prove a stronger claim that the restriction of  $\tau$  to a general surface in  $|-K_X|$  is not biregular.

Let  $S_\lambda$  be the surface on the hypersurface  $X$  cut by the equation  $y = \lambda x$ . It is a  $K3$  surface with only cyclic du Val singularities. The point  $O_t$  is a  $A_1$  singular point of  $S_\lambda$  and the point  $O_w$  is a  $A_2$  singular point of  $S_\lambda$ .

The surface  $S_\lambda$  is  $\tau$ -invariant by construction. Let  $\tau_\lambda$  be the restriction of  $\tau$  to the surface  $S_\lambda$ . It is a birational involution of the surface  $S_\lambda$  since the surface is  $\tau$ -invariant.

We claim that the birational involution  $\tau_\lambda$  is not biregular. To verify this claim, we suppose that it is biregular. Let us seek for a contradiction.

The projection  $\pi: X \dashrightarrow \mathbb{P}(1, 1, 2)$  induces a rational map  $\pi_\lambda: S_\lambda \rightarrow \mathbb{P}(1, 2) \cong \mathbb{P}^1$ . Note that the curve  $L_{tw}$  is contained in  $S_\lambda$ . The rational map  $\pi_\lambda: S_\lambda \dashrightarrow \mathbb{P}^1$  is given by the pencil  $\mathbb{P}$  of curves on the surface  $S_\lambda \subset \mathbb{P}(1, 2, 2, 3)$  cut by the equations

$$\delta x^2 = \epsilon z,$$

where  $[\delta : \epsilon] \in \mathbb{P}^1$ . Its base locus is cut out on  $S_\lambda$  by  $x = z = 0$ , which implies that the base locus of the pencil  $\mathcal{P}$  is the curve  $L_{tw}$ . We can easily see from Equation (5.4.5) that the map  $\pi_\lambda$  is not defined only at the points  $O_w$  and  $O_t$ .

Resolving the indeterminacy of the rational map  $\pi_\lambda$ , we obtain an elliptic fibration  $\bar{\pi}_\lambda: \bar{S}_\lambda \rightarrow \mathbb{P}^1$ . Thus, we have a commutative diagram

$$\begin{array}{ccc} & \bar{S}_\lambda & \\ \sigma \swarrow & & \searrow \bar{\pi}_\lambda \\ S_\lambda & \dashrightarrow \bar{\pi}_\lambda \dashrightarrow & \mathbb{P}^1, \end{array}$$

where  $\sigma$  is a birational map. Note that there exist exactly two  $\sigma$ -exceptional prime divisors that do not lie in the fibers of  $\bar{\pi}_\lambda$ . One of the divisors is over the point  $O_t$  and the other is over the point  $O_w$ . Let  $\bar{E}_\lambda$  and  $\bar{G}_\lambda$  be these two exceptional divisors, respectively. Then  $\bar{E}_\lambda$  and  $\bar{G}_\lambda$  are sections of  $\bar{\pi}_\lambda$ . Denote other  $\sigma$ -exceptional curves (if any) by  $F_1, \dots, F_r$ .

Recall that we assumed that the involution  $\tau_\lambda$  is biregular. Put  $\bar{\tau}_\lambda = \sigma^{-1} \circ \tau_\lambda \circ \sigma$ . Then  $\bar{\tau}_\lambda$  may not be biregular if  $\sigma^{-1}$  is not  $\tau_\lambda$ -equivariant. However, we have freedom in choosing  $\sigma$ . So we may assume that  $\sigma^{-1}$  is  $\tau_\lambda$ -equivariant, which implies that  $\bar{\tau}_\lambda$  is biregular as well.

Let  $C_\lambda$  be a general fiber of the map  $\pi_\lambda$ . Let  $\bar{C}_\lambda$  be the proper transform of the curve  $C_\lambda$  on  $\bar{S}_\lambda$ . The curve  $\bar{C}_\lambda$  is  $\bar{\tau}_\lambda$ -invariant. Furthermore,  $\bar{\tau}_\lambda|_{\bar{C}_\lambda}$  is given by the reflection with respect to the point  $\bar{G}_\lambda \cap \bar{C}_\lambda$ . On the other hand, the divisor  $\bar{E}_\lambda$  must be  $\bar{\tau}_\lambda$ -invariant since  $\tau_\lambda$  is biregular. This means, in particular, that the divisor

$$\left( \bar{E}_\lambda - \bar{G}_\lambda \right) \Big|_{\bar{C}_\lambda} \in \text{Pic}^0(\bar{C}_\lambda)$$

is torsion. Therefore, the divisor  $\bar{E}_\lambda - \bar{G}_\lambda$  must be numerically equivalent to a linear combination of curves on  $\bar{S}_\lambda$  that lie in the fibers of  $\bar{\pi}_\lambda$  (This method originates from [3]).

Note that the equation  $x = 0$  cuts out  $S_\lambda$  into a curve that splits as a union  $L_{tw} + C_x$ , where  $C_x$  is a curve defined by equations  $x = w^2 - t^3 + azt^2 + bz^2t + cz^3 = 0$  for some constants  $a, b, c$  in  $\mathbb{P}(1, 2, 2, 3)$ . The curve  $C_x$  is irreducible and reduced.

Let  $\bar{L}_{tw}$  and  $\bar{C}_x$  be the proper transforms of the curves  $L_{tw}$  and  $C_x$  by  $\sigma$ , respectively. Then  $\bar{L}_{tw}$  and  $\bar{C}_x$  lies in the same fiber of the elliptic fibration  $\bar{\pi}_\lambda$ . Moreover, we claim that every other fiber of  $\bar{\pi}_\lambda$  contains exactly one irreducible reduced curve that is not  $\sigma$ -exceptional. Indeed, this follows from the fact that for a general complex number  $\lambda$ , the curve (5.4.5) is always irreducible and reduced for every value of  $\mu$ . Since all fibers of  $\bar{\pi}_\lambda$  (with scheme structure) are numerically equivalent and the divisor  $\bar{E}_\lambda - \bar{G}_\lambda$  is numerically equivalent to a linear combinations of curves that lie in the fibers of  $\bar{\pi}_\lambda$ , we obtain

$$\bar{E}_\lambda - \bar{G}_\lambda \sim_{\mathbb{Q}} c_{tw}\bar{L}_{tw} + c_x\bar{C}_x + \sum_{i=1}^r c_i F_i$$

for some rational numbers  $c_{tw}, c_x, c_1, \dots, c_r$ . The intersection form of the curves  $\bar{E}_\lambda, \bar{G}_\lambda, F_1, \dots, F_r$  is negative definite since these curves are  $\sigma$ -exceptional. Therefore,  $(c_{tw}, c_x) \neq (0, 0)$ . On the other hand, we have

$$0 \sim_{\mathbb{Q}} c_{tw}L_{tw} + c_x C_x$$

on the surface  $S_\lambda$ . In particular, the intersection form of the curves  $L_{tw}$  and  $C_x$  is degenerate on the surface  $S_\lambda$ .

Meanwhile, from the intersection numbers

$$(L_{tw} + C_x) \cdot L_{tw} = \frac{1}{6}, \quad (L_{tw} + C_x)^2 = \frac{2}{3}, \quad L_{tw} \cdot C_x = 1$$

on the surface  $S_\lambda$ , we obtain

$$\begin{pmatrix} L_{tw}^2 & L_{tw} \cdot C_x \\ L_{tw} \cdot C_x & C_x^2 \end{pmatrix} = \begin{pmatrix} -\frac{5}{6} & 1 \\ 1 & -\frac{1}{2} \end{pmatrix}.$$

This is a contradiction. The obtained contradiction shows that  $\tau_\lambda$  is not biregular. In particular, the involution  $\tau$  is not biregular. This shows that  $\tau_Y(E) \neq E$  since  $\tau_Y$  is biregular in codimension one. Indeed, if  $\tau_Y(E) = E$ , then  $\tau$  is also biregular in codimension one, which implies that  $\tau$  is biregular since  $\text{Pic}(X) \cong \mathbb{Z}$ . Since  $\tau_Y(E) \neq E$ , the action of  $\tau_Y$  on  $\text{Pic}(Y)$  is not trivial. The action leaves  $-K_Y$  fixed. Therefore,  $\tau$  untwists the singular point  $O_t$  by Definition 5.1.1.  $\square$

## 6 Proof II

### 6.1 How to read the tables II

The table can be read in the same way as before. However, every families considered in this section may have singular points to be untwisted. In such a case, we use the symbols below to explain how to untwist or exclude the corresponding singular point.

$\boxed{\tau}$  : Apply Lemma 5.2.1 and Lemma 5.2.2

The given monomial in the table is the monomial  $x_{i_3}x_{i_4}^2$  in Lemma 5.2.1 and Lemma 5.2.2 that plays a central role in defining the involution. If the hypersurface  $X$  is defined by the

equation as in Lemma 5.2.1, the involution given by the quadratic equation is birational and untwists the given singular point. If the hypersurface  $X$  is defined by the equation as in Lemma 5.2.2, the involution given by the quadratic equation is biregular. In such a case, Lemma 5.2.2 excludes the corresponding singular point. Note that both the cases can always happen.

$\tau_1$  : Apply Lemma 5.2.1, Lemma 5.2.2 and Theorem 5.2.3

This method is basically the same as the method  $\tau$ . The difference is that we may have no  $x_{i_3}x_{i_4}^2$  in the defining equation. Such cases occur only when the corresponding singular point is  $O_t$  and  $x_{i_3}x_{i_4}^2 = wt^2$ . In cases, Theorem 5.2.3 excludes the singular point  $O_t$ . These three cases can always occur.

$\epsilon$  : Apply Theorem 5.3.4 and Theorem 5.3.5

This is for the singular point  $O_t$  of hypersurfaces in the families No. 23, 40, 44, 61 and 76. The given polynomial consisting of two monomials in the table is the monomial  $tw^2 - x_it^3$  in Equation 5.3.1 that plays a central role in defining the involution. If Equation 5.3.1 for  $X$  has  $g_{d-a_4} \neq 0$ , then the involution  $\tau_t$  is birational and untwists the singular point  $O_t$ . If Equation 5.3.1 for  $X$  has  $g_{d-a_4} = 0$ , the involution is biregular. In such a case, Theorem 5.3.5 excludes the singular point  $O_t$ . Note that both the cases can always happen.

$\epsilon_1$  : Apply Theorem 5.3.8 and Theorem 5.3.9

This is for the singular point  $O_z$  of hypersurfaces in the family No. 36.

$\epsilon_2$  : Apply Theorem 5.3.8, Theorem 5.3.9 and Theorem 5.3.10

This is for the singular point  $O_z$  of hypersurfaces in the family No. 20.

$\epsilon_3$  : Apply Theorem 5.3.12

This is for the singular points of type  $\frac{1}{2}(1, 1, 1)$  on quasi-smooth hypersurfaces of a certain type in the family No. 7.

$\iota$  : Apply Theorem 5.4.4

This is for the singular points of type  $\frac{1}{2}(1, 1, 1)$  on quasi-smooth hypersurfaces of a certain type in the family No. 7.

$\iota_1$  : Apply Theorem 5.4.1

This is for the singular point  $O_z$  of a special hypersurface in the family No. 23.

## 6.2 Non-super-rigid families

No. 2: $X_5 \subset \mathbb{P}(1, 1, 1, 1, 2)$					$A^3 = 5/2$
$tw^2 + wf_3(x, y, z, t) + f_5(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{2}(1, 1, 1)$ $\tau$			$tw^2$		

<b>No. 4:</b> $X_6 \subset \mathbb{P}(1, 1, 1, 2, 2)$					$A^3 = 3/2$
$(t - \alpha_1 w)(t - \alpha_2 w)(t - \alpha_3 w) + wf_4(x, y, z, t) + f_6(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t O_w = 3 \times \frac{1}{2}(1, 1, 1)$ $\boxed{\tau}$			$tw^2$		

We may assume that  $\alpha_1 = 0$ . To see how to treat the singular points of type  $\frac{1}{2}(1, 1, 1)$ , we have only to consider the singular point  $O_w$ . The other point can be dealt with in the same way.

<b>No. 5:</b> $X_7 \subset \mathbb{P}(1, 1, 1, 2, 3)$					$A^3 = 7/6$
$zw^2 + wf_4(x, y, z, t) + f_7(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{3}(1, 1, 2)$ $\boxed{\tau}$			$zw^2$		
$O_t = \frac{1}{2}(1, 1, 1)$ $\boxed{\tau_1}$			$wt^2$		

<b>No. 6:</b> $X_8 \subset \mathbb{P}(1, 1, 1, 2, 4)$					$A^3 = 1$
$(w - \alpha_1 t^2)(w - \alpha_2 t^2) + tf_6(x, y, z, w) + wf_4(x, y, z) + f_8(x, y, z) = 0$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t O_w = 2 \times \frac{1}{2}(1, 1, 1)$ $\boxed{\tau}$			$wt^2$		

We may assume that  $\alpha_1 = 0$ . To see how to treat the singular points of type  $\frac{1}{2}(1, 1, 1)$ , we have only to consider the singular point  $O_t$ . The other point can be dealt with in the same way.

<b>No. 7:</b> $X_8 \subset \mathbb{P}(1, 1, 2, 2, 3)$					$A^3 = 2/3$
Type I : $tw^2 + wg_5(x, y, z) - zt^3 - t^2 g_4(x, y, z) - tg_6(x, y, z) + g_8(x, y, z)$					
Type II : $(z + f_2(x, y))w^2 + wf_5(x, y, z, t) - zt^3 - t^2 f_4(x, y, z) - tf_6(x, y, z) + f_8(x, y, z)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{3}(1, 1, 2)$ $\boxed{\tau}$			$tw^2$		
$O_z O_t = 4 \times \frac{1}{2}(1, 1, 1)$ $\boxed{\epsilon_3}$			$tw^2 - zt^3$		Type I
$O_z O_t = 4 \times \frac{1}{2}(1, 1, 1)$ $\boxed{\iota}$					Type II

For the singular points of type  $\frac{1}{2}(1, 1, 1)$  we have only to consider one of them. The others can be untwisted in the same way. The singular point to be considered here may be assumed to be the point  $O_t$  by a suitable coordinate change.

<b>No. 8:</b> $X_9 \subset \mathbb{P}(1, 1, 1, 3, 4)$					$A^3 = 3/4$
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$zw^2 + wf_5(x, y, z, t) + f_9(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{4}(1, 1, 3) \square \tau$	$zw^2$				

<b>No. 9:</b> $X_9 \subset \mathbb{P}(1, 1, 2, 3, 3)$					$A^3 = 1/2$
$(w - \alpha_1 t)(w - \alpha_2 t)(w - \alpha_3 t) + z^3(a_1 w + a_2 yz) + w^2 f_3(x, y, z) + wf_6(x, y, z, t) + f_9(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_z = \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{D}$	0	$B$	$y$	$y$	$a_1 \neq 0$
$O_z = \frac{1}{2}(1_x, 1_t, 1_w) \textcircled{D}$	0	$B - E$	$y$	$w^3$	$a_1 = 0$
$O_t O_w = 3 \times \frac{1}{3}(1, 1, 2) \square \tau$	$tw^2$				

We may assume that the monomial  $z^3 t$  does not appear in  $f_9$ .

If  $a_1 \neq 0$ , then the 1-cycle  $\Gamma$  for the singular point  $O_z$  is irreducible.

Suppose that  $a_1 = 0$ . Then  $a_2 \neq 0$ . Then the 1-cycle  $\Gamma$  consists of three irreducible curves  $\tilde{C}_i$ ,  $i = 1, 2, 3$ , each of which is the proper transform of the curve defined by  $x = y = w - \alpha_i t = 0$ . One can easily check that

$$B \cdot \tilde{C}_i = -\frac{1}{3}, \quad E \cdot \tilde{C}_i = 1$$

for each  $i$ . Therefore, these three curves are numerically equivalent each other.

For the singular points of type  $\frac{1}{3}(1, 1, 2)$  we have only to consider one of them. The others can be untwisted in the same way. The singular point to be considered here may be assumed to be the point  $O_w$  by a suitable coordinate change.

<b>No. 12:</b> $X_{10} \subset \mathbb{P}(1, 1, 2, 3, 4)$					$A^3 = 5/12$
$z(w - \alpha_1 z^2)(w - \alpha_2 z^2) + t^2(a_1 w + a_2 yt) + cz^2 t^2 + wf_6(x, y, z, t) + f_{10}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{4}(1, 1, 3) \square \tau$	$zw^2$				
$O_t = \frac{1}{3}(1, 1, 2) \square \tau_1$	$wt^2$				
$O_z O_w = 2 \times \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{D}$	—	$B$	$y$	$y$	$c \neq 0, a_1 \neq 0$
$O_z O_w = 2 \times \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{S}$	—	$B$	$x, y$	$x, y$	$c \neq 0, a_1 = 0$
$O_z O_w = 2 \times \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{I}$	—	$B$	$x, y$	$x, y$	$c = 0$

By a coordinate change we assume that  $\alpha_1 = 0$ . Furthermore we may assume that the monomials  $z^3 xt$ ,  $z^3 yt$ ,  $z^4 x^2$ ,  $z^4 xy$ ,  $z^4 y^2$  do not appear in the defining equation by changing the coordinate  $w$  in an appropriate way. We may also assume that  $xt^3$  is not contained in  $f_{10}$ .

For the singular points of type  $\frac{1}{2}(1, 1, 1)$  with  $c \neq 0$  and  $a_1 \neq 0$  the 1-cycle  $\Gamma$  is irreducible due to the monomials  $zw^2$ ,  $t^2 w$  and  $z^2 t^2$ .

For the singular points of type  $\frac{1}{2}(1, 1, 1)$  we may assume that  $\alpha_1 = 0$  and we have only to consider the point  $O_z$ . The other singular point can be treated by a suitable coordinate change.

For the singular points of type  $\frac{1}{2}(1, 1, 1)$  with  $c \neq 0$  and  $a_1 = 0$  choose a general surface  $H$  in  $| -K_{X_{10}} |$  and then let  $T$  be the proper transform of the surface  $H$ . It is a K3 surface only with du Val singularities. The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves on the normal surface  $T$ . One is the proper transform  $\tilde{L}_{tw}$  on  $T$  of the curve  $L_{tw}$  on  $H$ . The other is the proper transform  $\tilde{C}$  on  $T$  of the curve  $C$  defined by  $x = y = w^2 - \alpha_2 z^2 w + czt^2 = 0$  in  $\mathbb{P}(1, 1, 2, 3, 4)$ . Since we have

$$\tilde{L}_{tw}^2 = -\frac{7}{12}, \quad \tilde{L}_{tw} \cdot \tilde{C} = \frac{2}{3}, \quad \tilde{C}^2 = -\frac{5}{6}$$

the curves  $\tilde{L}_{tw}$  and  $\tilde{C}$  are negative-definite.

We remark here that the surface obtained from  $T$  by contracting the two curves  $\tilde{L}_{tw}$  and  $\tilde{C}$  is a K3 surface only with one  $E_8$  singular point. Indeed, the surface  $T$  has one  $A_1$  singular point on  $\tilde{C}$ , one  $A_3$  singular point on  $\tilde{L}_{tw}$ , and the curves  $\tilde{C}$ ,  $\tilde{L}_{tw}$  intersect at one  $A_2$  singular point tangentially on an orbifold chart. Therefore, on the minimal resolution of the surface  $T$ , the proper transforms of the curves  $\tilde{C}$ ,  $\tilde{L}_{tw}$  with the exceptional curves over three du Val points form the configuration of the  $-2$ -curves for an  $E_8$  singular point.

Now we consider the singular point  $O_z$  with  $c = 0$ . The quasi-smoothness implies that  $a_1 = 0$  and  $a_2 \neq 0$ . Let  $Z_{\lambda, \mu}$  be a curve cut out on  $X_{10}$  by

$$\begin{cases} y = \lambda x, \\ w = \mu x^4 \end{cases}$$

for some sufficiently general complex numbers  $\lambda$  and  $\mu$ . Then  $Z_{\lambda, \mu} = L_{zt} + C_{\lambda, \mu}$ , where  $C_{\lambda, \mu}$  is an irreducible reduced curve whose normalisation is an elliptic curve. Indeed, the curve  $C_{\lambda, \mu}$  is defined by  $y - \lambda x = w - \mu x^4 = \mu^2 x^7 z - \alpha_2 \mu x^3 z^3 + \lambda a_2 t^3 + \mu x^3 f_6(x, \lambda x, z, t) + f_{10}(x, \lambda x, z, t) = 0$ . Denote by  $\tilde{L}_{zt}$  and  $\tilde{C}_{\lambda, \mu}$  the proper transforms of the curve  $L_{zt}$  and the curve  $C_{\lambda, \mu}$  on  $Y$ , respectively. Then

$$\begin{cases} -K_Y \cdot (\tilde{L}_{zt} + \tilde{C}_{\lambda, \mu}) = 4B^3 = -\frac{1}{3}, \\ -K_Y \cdot \tilde{L}_{zt} = -K_X \cdot L_{zt} - \frac{1}{2}E \cdot \tilde{L}_{zt} = -\frac{1}{3}, \end{cases}$$

and hence  $-K_Y \cdot \tilde{C}_{\lambda, \mu} = 0$ .

No. 13: $X_{11} \subset \mathbb{P}(1, 1, 2, 3, 5)$						$A^3 = 11/30$
$yw^2 + t^2(a_1w + a_2zt) + z^3(b_1w + b_2zt + b_3z^2x + b_4z^2y) + xf_{10}(x, y, z, t, w) + yg_{10}(y, z, t, w)$						
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition	
$O_w = \frac{1}{5}(1, 2, 3)$ $\square$ $\tau$					$yw^2$	
$O_t = \frac{1}{3}(1, 1, 2)$ $\square$ $\tau_1$					$wt^2$	
$O_z = \frac{1}{2}(1, 1, 1)$ $\oplus$	–	$B$	$y$	$y$	$a_1 \neq 0, b_1 \neq 0$ $a_1b_2 - a_2b_1 \neq 0$	
$O_z = \frac{1}{2}(1_x, 1_y, 1_t)$ $\oplus$	–	$B$	$x, y$	$x, y$	$a_1 \neq 0, b_1 \neq 0$ $a_1b_2 - a_2b_1 = 0$	
$O_z = \frac{1}{2}(1_x, 1_y, 1_w)$ $\oplus$	–	$B$	$x, y$	$x, y$	$a_1 \neq 0$ $b_1 = 0, b_2 \neq 0$	



$O_z = \frac{1}{2}(1, 1, 1) \text{ (}\oplus\text{)}$	–	$B$	$y$	$y$	$a_1 \neq 0$ $b_1 = 0, b_2 = 0$
$O_z = \frac{1}{2}(1_x, 1_y, 1_t) \text{ (}\oplus\text{)}$	–	$B$	$x, y$	$x, y$	$a_1 = 0$ $b_1 \neq 0$
$O_z = \frac{1}{2}(1_x, 1_y, 1_w) \text{ (}\oplus\text{)}$	–	$B$	$x, y$	$x, y$	$a_1 = 0$ $b_1 = 0, b_2 \neq 0$
$O_z = \frac{1}{2}(1, 1, 1) \text{ (}\oplus\text{)}$	–	$B$	$y$	$y$	$a_1 = 0$ $b_1 = 0, b_2 = 0$

To exclude the singular point  $O_z$  we first suppose that  $a_1 \neq 0$ . We may then assume that  $a_1 = 1$  and  $a_2 = 0$ .

The conditions  $b_1 \neq 0$  and  $a_1 b_2 - a_2 b_1 \neq 0$  imply that both  $b_1$  and  $b_2$  are non-zero. In such a case the 1-cycle  $\Gamma$  is irreducible since we have the monomials  $t^2 w$ ,  $z^3 w$  and  $z^4 t$ .

The conditions  $b_1 \neq 0$  and  $a_1 b_2 - a_2 b_1 = 0$  imply that  $b_1 \neq 0$  and  $b_2 = 0$ . In such a case we take a general surface  $H$  from the pencil  $|-K_{X_{11}}|$  and then let  $T$  be the proper transform of the surface. It is a K3 surface only with du Val singularities. The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves on the normal surface  $T$ . One is the proper transform  $\tilde{L}_{zt}$  on  $T$  of the curve  $L_{zt}$  on  $H$ . The other is the proper transform  $\tilde{C}$  on  $T$  of the curve  $C$  defined by  $x = y = t^2 + b_1 z^3 = 0$  in  $\mathbb{P}(1, 1, 2, 3, 5)$ . Since

$$\tilde{L}_{zt}^2 = -\frac{4}{3}, \quad \tilde{L}_{zt} \cdot \tilde{C} = 1, \quad \tilde{C}^2 = -\frac{4}{5}$$

the curves  $\tilde{L}_{zt}$  and  $\tilde{C}$  on the normal surface  $T$  are negative-definite.

In the case when  $b_1 = 0$  and  $b_2 \neq 0$  we may assume that  $b_2 = 1$ ,  $b_3 = b_4 = 0$  by a suitable coordinate change. Let  $Z_{\lambda, \mu}$  be a curve cut out on  $X_{11}$  by

$$\begin{cases} y = \lambda x, \\ t = \mu x^3 \end{cases}$$

for some sufficiently general complex numbers  $\lambda$  and  $\mu$ . Then  $Z_{\lambda, \mu} = L_{zw} + C_{\lambda, \mu}$ , where  $C_{\lambda, \mu}$  is an irreducible and reduced curve whose normalisation is an elliptic curve. Denote by  $\tilde{L}_{zw}$  and  $\tilde{C}_{\lambda, \mu}$  the proper transforms of the curve  $L_{zw}$  and the curve  $C_{\lambda, \mu}$ , respectively. Then

$$\begin{cases} -K_Y \cdot (\tilde{L}_{zw} + \tilde{C}_{\lambda, \mu}) = 3B^3 = -\frac{2}{5}, \\ -K_Y \cdot \tilde{L}_{zw} = -K_X \cdot L_{zw} - \frac{1}{2}E \cdot \tilde{L}_{zw} = -\frac{2}{5}, \end{cases}$$

and hence  $-K_Y \cdot \tilde{C}_{\lambda, \mu} = 0$ .

Now suppose that  $b_1 = 0$  and  $b_2 = 0$ . Then  $b_3$  must be non-zero due to the quasi-smoothness of  $X_{11}$ . We may assume that  $b_3 = 1$  and  $b_4 = 0$  by a suitable coordinate change. Let  $Z_\lambda$  be a curve on the surface  $S$  defined by

$$\begin{cases} x = 0, \\ t = \lambda y^3 \end{cases}$$

for a sufficiently general complex number  $\lambda$ . Then  $Z_\lambda = L_{zw} + C_\lambda$ , where  $C_\lambda$  is an irreducible and reduced curve. Denote by  $\tilde{C}_\lambda$  the proper transform of the curve  $C_\lambda$  on  $Y$ . Then

$$\begin{cases} -K_Y \cdot (\tilde{L}_{zw} + \tilde{C}_\lambda) = (B - E)(3B + E)B = -\frac{2}{5}, \\ -K_Y \cdot \tilde{L}_{zw} = -K_X \cdot L_{zw} - \frac{1}{2}E \cdot \tilde{L}_{zw} = -\frac{2}{5}, \end{cases}$$

and hence  $-K_Y \cdot \tilde{C}_\lambda = 0$ .

Now we suppose that  $a_1 = 0$ . Then  $a_2 \neq 0$ , so that we may assume that  $a_2 = 1$ .

Suppose that  $b_1 \neq 0$ . Then by a suitable coordinate change we may assume that  $b_1 = 1$  and  $b_2 = 0$ . We take a general surface  $H$  from the pencil  $|-K_{X_{11}}|$  and then let  $T$  be the proper transform of the surface. It is a K3 surface only with du Val singularities. The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves on the normal surface  $T$ . One is the proper transform  $\tilde{L}_{tw}$  on  $T$  of the curve  $L_{tw}$  on  $H$ . The other is the proper transform  $\tilde{C}$  on  $T$  of the curve  $C$  defined by  $x = y = t^3 + z^2w = 0$  in  $\mathbb{P}(1, 1, 2, 3, 5)$ . Since

$$\tilde{L}_{tw}^2 = -\frac{8}{15}, \quad \tilde{L}_{tw} \cdot \tilde{C} = \frac{3}{5}, \quad \tilde{C}^2 = -\frac{4}{5}$$

the curves  $\tilde{L}_{tw}$  and  $\tilde{C}$  are negative-definite on the normal surface  $T$ .

We suppose that  $b_1 = 0$  and  $b_2 \neq 0$ . We then let  $Z_{\lambda, \mu}$  be a curve cut out on  $X_{11}$  by

$$\begin{cases} y = \lambda x, \\ t = \mu x^3 \end{cases}$$

for some sufficiently general complex numbers  $\lambda$  and  $\mu$ . Then  $Z_{\lambda, \mu} = L_{zw} + C_{\lambda, \mu}$ , where  $C_{\lambda, \mu}$  is an irreducible and reduced curve whose normalisation is an elliptic curve. Denote by  $\tilde{C}_{\lambda, \mu}$  the proper transform of the curve  $C_{\lambda, \mu}$  on  $Y$ . Then

$$\begin{cases} -K_Y \cdot (\tilde{L}_{zw} + \tilde{C}_{\lambda, \mu}) = 3B^3 = -\frac{2}{5}, \\ -K_Y \cdot \tilde{L}_{zw} = -K_X \cdot L_{zw} - \frac{1}{2}E \cdot \tilde{L}_{zw} = -\frac{2}{5}, \end{cases}$$

and hence  $-K_Y \cdot \tilde{C}_{\lambda, \mu} = 0$ .

Finally, we suppose that  $b_1 = 0$  and  $b_2 = 0$ . Then  $b_3$  must be non-zero due to the quasi-smoothness of  $X_{11}$ . We may assume that  $b_3 = 1$  and  $b_4 = 0$  by a suitable coordinate change. Let  $Z_\lambda$  be a curve on the surface  $S$  defined by

$$\begin{cases} x = 0, \\ t = \lambda y^3 \end{cases}$$

for a sufficiently general complex number  $\lambda$ . Then  $Z_\lambda = L_{zw} + C_\lambda$ , where  $C_\lambda$  is an irreducible and reduced curve. Denote by  $\tilde{C}_\lambda$  the proper transform of the curve  $C_\lambda$  on  $Y$ . Then

$$\begin{cases} -K_Y \cdot (\tilde{L}_{zw} + \tilde{C}_\lambda) = (B - E)(3B + E)B = -\frac{2}{5}, \\ -K_Y \cdot \tilde{L}_{zw} = -K_X \cdot L_{zw} - \frac{1}{2}E \cdot \tilde{L}_{zw} = -\frac{2}{5}, \end{cases}$$

and hence  $-K_Y \cdot \tilde{C}_\lambda = 0$ .

<b>No. 15:</b> $X_{12} \subset \mathbb{P}(1, 1, 2, 3, 6)$					$A^3 = 1/3$
$w(w - z^3) + t^2w + az^3t^2 + wf_6(x, y, z, t) + f_{12}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_tO_w = 2 \times \frac{1}{3}(1, 1, 2)$ $\square$	$wt^2$				
$O_zO_w = 2 \times \frac{1}{2}(1_x, 1_y, 1_t)$ $\textcircled{D}$	—	$B$	$y$	$y$	$a \neq 0$
$O_zO_w = 2 \times \frac{1}{2}(1_x, 1_y, 1_t)$ $\textcircled{S}$	—	$B$	$x, y$	$x, y$	$a = 0$

By changing the coordinate  $w$  we may assume that  $t^4$  is not in the polynomial  $f_{12}$ . To see how to deal with the singular points of type  $\frac{1}{3}(1, 1, 2)$  we have only to consider the singular point  $O_t$ . The other point can be treated in the same way.

The 1-cycle  $\Gamma$  for each singular point of type  $\frac{1}{2}(1, 1, 1)$  with  $a \neq 0$  is irreducible due to the monomials  $w^2$  and  $z^3t^2$

For the singular points of type  $\frac{1}{2}(1, 1, 1)$  with  $a = 0$  we take a general surface  $H$  from the pencil  $|-K_{X_{12}}|$  and then let  $T$  be the proper transform of the surface. It is a K3 surface only with du Val singularities. The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves on the normal surface  $T$ . One is the proper transform  $\tilde{L}_{zt}$  on  $T$  of the curve  $L_{zt}$  on  $H$ . The other is the proper transform  $\tilde{C}$  on  $T$  of the curve  $C$  defined by  $x = y = w - z^3 + t^2 = 0$  in  $\mathbb{P}(1, 1, 2, 3, 6)$ . From the intersection numbers

$$(\tilde{L}_{zt} + \tilde{C}) \cdot \tilde{L}_{zt} = -K_Y \cdot \tilde{L}_{zt} = -\frac{1}{3}, \quad (\tilde{L}_{zt} + \tilde{C})^2 = B^3 = -\frac{1}{6}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{zt}^2 = -\frac{1}{3} - \tilde{L}_{zt} \cdot \tilde{C}, \quad \tilde{C}^2 = \frac{1}{6} - \tilde{L}_{zt} \cdot \tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zt}^2 & \tilde{L}_{zt} \cdot \tilde{C} \\ \tilde{L}_{zt} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} - \tilde{L}_{zt} \cdot \tilde{C} & \tilde{L}_{zt} \cdot \tilde{C} \\ \tilde{L}_{zt} \cdot \tilde{C} & \frac{1}{6} - \tilde{L}_{zt} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{zt} \cdot \tilde{C} = 1$ .

<b>No. 16:</b> $X_{12} \subset \mathbb{P}(1, 1, 2, 4, 5)$					$A^3 = 3/10$
$zw^2 + (t - \alpha_1z^2)(t - \alpha_2z^2)(t - \alpha_3z^2) + wf_7(x, y, z, t) + f_{12}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{5}(1, 1, 4)$ $\square$	$zw^2$				
$O_zO_t = 3 \times \frac{1}{2}(1_x, 1_y, 1_w)$ $\textcircled{D}$	—	$B$	$y$	$y$	

The 1-cycle  $\Gamma$  for each singular point of type  $\frac{1}{2}(1, 1, 1)$  is irreducible due to  $zw^2$  and  $t^3$ .

<b>No. 17:</b> $X_{12} \subset \mathbb{P}(1, 1, 3, 4, 4)$					$A^3 = 1/4$
$(t - \alpha_1w)(t - \alpha_2w)(t - \alpha_3w) + z^4 + wf_8(x, y, z, t) + f_{12}(x, y, z, t)$					

Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t O_w = 3 \times \frac{1}{4}(1, 1, 3)$ $\boxed{\tau}$			$tw^2$		

To see how to deal with the singular points of type  $\frac{1}{4}(1, 1, 3)$  we may assume that  $\alpha_1 = 0$  and we have only to consider the singular point  $O_w$ . The other points can be treated in the same way.

No. 18: $X_{12} \subset \mathbb{P}(1, 2, 2, 3, 5)$ <span style="float: right;"><math>A^3 = 1/5</math></span>					
$yw^2 + t^4 + \prod_{i=1}^6 (y - \alpha_i z) + wf_7(x, y, z, t) + f_{12}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{5}(1, 2, 3)$ $\boxed{\tau}$			$yw^2$		
$O_y O_z = 6 \times \frac{1}{2}(1_x, 1_t, 1_w)$ $\textcircled{D}$	—	$2B$	$y - \alpha_i z$	$zw^2$	

The 1-cycle  $\Gamma$  for each singular point of type  $\frac{1}{2}(1, 1, 1)$  is irreducible due to  $yw^2$  and  $t^3$ .

No. 20: $X_{13} \subset \mathbb{P}(1, 1, 3, 4, 5)$ <span style="float: right;"><math>A^3 = 13/60</math></span>					
$zw^2 + t^2(a_1 w + a_2 y t) + yz^4 + wf_8(x, y, z, t) + f_{13}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{5}(1, 1, 4)$ $\boxed{\tau}$			$zw^2$		
$O_t = \frac{1}{4}(1, 1, 3)$ $\boxed{\tau_1}$			$wt^2$		
$O_z = \frac{1}{3}(1, 1, 2)$ $\boxed{\epsilon_2}$			$zw^2 - tz^3$		

No. 23: $X_{14} \subset \mathbb{P}(1, 2, 3, 4, 5)$ <span style="float: right;"><math>A^3 = 7/60</math></span>					
$(t + by^2)w^2 + y(t - \alpha_1 y^2)(t - \alpha_2 y^2)(t - \alpha_3 y^2) + z^3(a_1 w + a_2 y z) + cz^2 t^2 + xf_{13}(x, y, z, t, w) + yg_{12}(y, z, t, w)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{5}(1, 2, 3)$ $\boxed{\tau}$			$tw^2$		
$O_t = \frac{1}{4}(1, 3, 1)$ $\boxed{\epsilon}$			$tw^2 + yt^3$		
$O_z = \frac{1}{3}(1_x, 2_y, 1_t)$ $\textcircled{B}$	—	$2B$	$y$	$y$	$c \neq 0, a_1 \neq 0$
$O_z = \frac{1}{3}(1_x, 1_t, 2_w)$ $\textcircled{S}$	—	$2B$	$x^2, y$	$x^2, z^2 t^2$	$c \neq 0, a_1 = 0$
$O_z = \frac{1}{3}(1_x, 2_y, 1_t)$ $\textcircled{F}$	—	$2B$	$y$	$y$	$c = 0, a_1 \neq 0$
$O_z = \frac{1}{3}(1_x, 1_t, 2_w)$ $\boxed{t_1}$					$c = 0, a_1 = 0$
$O_y O_t = 3 \times \frac{1}{2}(1_x, 1_z, 1_w)$ $\textcircled{N}$	—	$3B + E$	$z$	$z$	$b \neq 0$
$O_y O_t = 3 \times \frac{1}{2}(1_x, 1_z, 1_w)$ $\textcircled{S}$	—	$3B + E$	$x^3, xy, z$	$xy, z$	$b = 0$

For the singular point  $O_z$  with  $c \neq 0$  and  $a_1 \neq 0$  the 1-cycle  $\Gamma$  is irreducible due to the monomials  $tw^2$ ,  $z^3 w$  and  $z^2 t^2$ .

For the singular point  $O_z$  with  $c \neq 0$  and  $a_1 = 0$  we may assume that  $a_2 = 1$  and  $c = 1$ . We take a general surface  $H$  from the pencil  $|-2K_{X_{14}}|$  and then let  $T$  be the proper transform of

the surface. Note that the surface  $H$  is normal. However, it is not quasi-smooth at the points  $O_z$  and  $O_t$ .

The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves on the normal surface  $T$ . One is the proper transform  $\tilde{L}_{zw}$  on  $T$  of the curve  $L_{zw}$  on  $H$ . The other is the proper transform  $\tilde{C}$  on  $T$  of the curve  $C$  defined by  $x = y = w^2 + z^2t = 0$  in  $\mathbb{P}(1, 2, 3, 4, 5)$ . From the intersection numbers

$$(\tilde{L}_{zw} + \tilde{C}) \cdot \tilde{L}_{zw} = -K_Y \cdot \tilde{L}_{zw} = -\frac{1}{10}, \quad (\tilde{L}_{zw} + \tilde{C})^2 = 2B^3 = -\frac{1}{10}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{zw}^2 = -\frac{1}{10} - \tilde{L}_{zw} \cdot \tilde{C}, \quad \tilde{C}^2 = -\tilde{L}_{zw} \cdot \tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zw}^2 & \tilde{L}_{zw} \cdot \tilde{C} \\ \tilde{L}_{zw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{10} - \tilde{L}_{zw} \cdot \tilde{C} & \tilde{L}_{zw} \cdot \tilde{C} \\ \tilde{L}_{zw} \cdot \tilde{C} & -\tilde{L}_{zw} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{zw} \cdot \tilde{C}$  is positive.

For the singular point  $O_z$  with  $c = 0$  and  $a_1 \neq 0$  we may assume that  $a_1 = 1$  and  $a_2 = 0$ . Furthermore, we may also assume that the quasi-homogeneous polynomial  $f_{13}$  of degree 13 does not contain the monomial  $tz^3$  by changing the coordinate  $w$  in a suitable way. We then consider the surface  $S_w$  cut by the equation  $w = 0$ . Let  $Z_\lambda$  be a curve on the surface  $S_w$  defined by

$$\begin{cases} w = 0 \\ y = \lambda x^2 \end{cases}$$

for a sufficiently general complex number  $\lambda$ . Then  $Z_\lambda = 2L_{zt} + C_\lambda$ , where  $C_\lambda$  is an irreducible and reduced curve. Denote by  $\tilde{L}_{zt}$  and  $\tilde{C}_\lambda$  the proper transforms of the curves  $L_{zw}$  and  $C_\lambda$  on  $Y$ , respectively. Then

$$\begin{cases} -K_Y \cdot (2\tilde{L}_{zt} + \tilde{C}_\lambda) = 10B^3 = -\frac{1}{2}, \\ -K_Y \cdot \tilde{L}_{zt} = -K_X \cdot L_{zt} - \frac{1}{3}E \cdot \tilde{L}_{zt} = -\frac{1}{4}, \end{cases}$$

and hence  $-K_Y \cdot \tilde{C}_\lambda = 0$ .

For the singular point  $O_z$  with  $c = 0$  and  $a_1 = 0$  we observe that the quasi-homogeneous polynomial  $f_{13}$  must contain the monomial  $z^3t$ . We may assume that  $a_2 = 1$  and that the coefficient of  $z^3t$  in  $f_{13}$  is 1. Then Theorem 5.4.1 untwists the singular point  $O_z$ .

For the singular points of type  $\frac{1}{2}(1, 1, 1)$  we may assume that  $\alpha_3 = 0$  and we have only to consider the singular point  $O_y$ . The other singular points can be treated in the same way after suitable coordinate changes.

For the singular point  $O_y$  with  $b \neq 0$  consider the linear system generated by  $xy$  and  $z$  on  $X_{14}$ . Its base curves are defined by  $x = z = 0$  and  $y = z = 0$ . The curve defined by  $y = z = 0$  does not pass through the singular point  $O_y$ . The curve defined by  $x = z = 0$  is irreducible. Indeed, the curve is defined by  $x = z = (t + by^2)w^2 + yt(t - \alpha_1y^2)(t - \alpha_2y^2) = 0$ . Moreover,

its proper transform is equivalent to the 1-cycle defined by  $(3B + E) \cdot B$ . Consequently, the divisor  $T$  is nef since  $(3B + E)^2 \cdot B > 0$ .

For the singular point  $O_y$  with  $b = 0$  we take a general member  $H$  in the linear system generated by  $x^3$ ,  $xy$  and  $z$ . Note that the defining equation of  $X_{14}$  must contain either  $y^3zw$  or  $xy^4w$ . The surface  $H$  is a normal surface of degree 14 in  $\mathbb{P}(1, 2, 4, 5)$  such that is smooth at the point  $x = t = w^2 + \alpha_1\alpha_2y^5 = 0$ . Let  $T$  be the proper transform of the surface  $H$ . The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves on the normal surface  $T$ . One is the proper transform  $\tilde{L}_{yw}$  of the curve  $L_{yw}$  and the other is the proper transform  $\tilde{C}$  of the curve  $C$  defined by  $x = z = w^2 + y(t - \alpha_1y^2)(t - \alpha_2y^2) = 0$ . From the intersection numbers

$$(\tilde{L}_{yw} + \tilde{C}) \cdot \tilde{L}_{yw} = -K_Y \cdot \tilde{L}_{yw} = -\frac{2}{5}, \quad (\tilde{L}_{yw} + \tilde{C})^2 = B^2 \cdot (3B + E) = -\frac{3}{20}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{yw}^2 = -\frac{2}{5} - \tilde{L}_{yw} \cdot \tilde{C}, \quad \tilde{C}^2 = \frac{1}{4} - \tilde{L}_{yw} \cdot \tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yw}^2 & \tilde{L}_{yw} \cdot \tilde{C} \\ \tilde{L}_{yw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} - \tilde{L}_{yw} \cdot \tilde{C} & \tilde{L}_{yw} \cdot \tilde{C} \\ \tilde{L}_{yw} \cdot \tilde{C} & \frac{1}{4} - \tilde{L}_{yw} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since the curves  $L_{yw}$  and the curve  $C$  intersect at the smooth point of  $H$  defined by  $x = z = t = w^2 + \alpha_1\alpha_2y^5 = 0$ .

No. 24: $X_{15} \subset \mathbb{P}(1, 1, 2, 5, 7)$					$A^3 = 3/14$
$yw^2 + t^3 + z^4(a_1w + a_2zt + a_3xz^3 + a_4yz^3) + wf_8(x, y, z, t) + f_{15}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{7}(1, 2, 5) \square$			$yw^2$		
$O_z = \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{B}$	—	$B$	$y$	$y$	$a_1 \neq 0$
$O_z = \frac{1}{2}(1, 1, 1) \textcircled{D}$	—	$B$	$y$	$y$	$a_1 = 0, a_2 = 0$
$O_z = \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{S}$	—	$B$	$x, y$	$x, y$	$a_1 = 0, a_2 \neq 0$

Since  $X_{15}$  is quasi-smooth, one of the constants  $a_1, a_2, a_3$  must be nonzero.

The 1-cycle  $\Gamma$  for the singular point  $O_z$  with  $a_1 \neq 0$  is irreducible since we have  $t^3$  and  $z^4w$ .

The 1-cycle  $\Gamma$  for the singular point  $O_z$  with  $a_1 = a_2 = 0$  is also irreducible even though it is not reduced.

For the singular point  $O_z$  with  $a_1 = 0$  and  $a_2 \neq 0$  we may assume that  $a_2 = 1$  and  $a_3 = a_4 = 0$ . Choose a general member  $H$  in the linear system  $| -K_{15} |$  and then let  $T$  be the proper transform of the surface. It is a K3 surface only with du Val singularities. The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves on the normal surface  $T$ . One is the proper transform  $\tilde{L}_{zw}$  on  $T$  of the curve  $L_{zw}$  on  $H$ . The other is the proper transform  $\tilde{C}$  on  $T$  of the curve  $C$  defined by  $x = y = t^2 + z^5 = 0$  in  $\mathbb{P}(1, 1, 2, 5, 7)$ .

The curves  $L_{zw}$  and  $C$  intersect at the point  $O_w$ . From the intersection numbers

$$(\tilde{L}_{zw} + \tilde{C}) \cdot \tilde{L}_{zw} = -K_Y \cdot \tilde{L}_{zw} = -\frac{3}{7}, \quad (\tilde{L}_{zw} + \tilde{C})^2 = B^3 = -\frac{2}{7}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{zw}^2 = -\frac{3}{7} - \tilde{L}_{zw} \cdot \tilde{C}, \quad \tilde{C}^2 = \frac{1}{7} - \tilde{L}_{zw} \cdot \tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zw}^2 & \tilde{L}_{zw} \cdot \tilde{C} \\ \tilde{L}_{zw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} - \tilde{L}_{zw} \cdot \tilde{C} & \tilde{L}_{zw} \cdot \tilde{C} \\ \tilde{L}_{zw} \cdot \tilde{C} & \frac{1}{7} - \tilde{L}_{zw} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{zw} \cdot \tilde{C} = \frac{5}{7}$ .

<b>No. 25:</b> $X_{15} \subset \mathbb{P}(1, 1, 3, 4, 7)$					$A^3 = 5/28$
$yw^2 + t^2(a_1w + a_2zt) + z^5 + wf_8(x, y, z, t) + f_{15}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{7}(1, 3, 4) \square \tau$	$yw^2$				
$O_t = \frac{1}{4}(1, 1, 3) \square \tau_1$	$wt^2$				

<b>No. 26:</b> $X_{15} \subset \mathbb{P}(1, 1, 3, 5, 6)$					$A^3 = 1/6$
$zw^2 + t^3 + z^5 + wf_9(x, y, z, t) + f_{15}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{6}(1, 1, 5) \square \tau$	$zw^2$				
$O_z O_w = 2 \times \frac{1}{3}(1_x, 1_y, 2_t) \textcircled{D}$	0	$B$	$y$	$y$	

The 1-cycle  $\Gamma$  for each singular point of type  $\frac{1}{3}(1, 1, 2)$  is irreducible since we have the monomials  $zw^2$  and  $t^3$ .

<b>No. 27:</b> $X_{15} \subset \mathbb{P}(1, 2, 3, 5, 5)$					$A^3 = 1/10$
$y^5(a_1w + a_2yz + a_3xy^2) + h_{15}(t, w) + w^2 f_5(x, y, z, t) + wf_{10}(x, y, z, t) + f_{15}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t O_w = 3 \times \frac{1}{5}(1, 2, 3) \square \tau$	$tw^2$				
$O_y = \frac{1}{2}(1, 1, 1) \textcircled{N}$	-	$5B + 2E$	$t$	$t$	

Here the polynomial  $h_{15}(t, w)$  is reduced, so that it defines three distinct singular points of type  $\frac{1}{5}(1, 2, 3)$ . We may assume that  $h_{15}$  is divisible by  $t$ , i.e., the hypersurface  $X_{15}$  has a singular point of type  $\frac{1}{5}(1, 2, 3)$  at the point  $O_w$ . To see how to treat the singular points of type  $\frac{1}{5}(1, 2, 3)$  we have only to consider the singular point  $O_w$ . The others can be dealt with

in the same way.

For the singular point  $O_y$  we consider the linear system  $|-5K_{X_{15}}|$ . Every member in the linear system passes through the point  $O_y$ . It has no base curve. Since the proper transform of a general member in  $|-5K_{X_{15}}|$  belongs to the linear system  $|5B + 2E|$ , the divisor  $T$  is nef.

<b>No. 30:</b> $X_{16} \subset \mathbb{P}(1, 1, 3, 4, 8)$					$A^3 = 1/6$
$(w - \alpha_1 t^2)(w - \alpha_2 t^2) + z^4(a_1 t + a_2 y z) + w f_8(x, y, z, t) + f_{16}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t O_w = 2 \times \frac{1}{4}(1, 1, 3) \boxed{\tau}$	$w t^2$				
$O_z = \frac{1}{3}(1_x, 1_y, 2_w) \textcircled{B}$	0	$B$	$y$	$y$	$a_1 \neq 0$
$O_z = \frac{1}{3}(1_x, 1_t, 2_w) \textcircled{B}$	0	$B - E$	$y$	$w^2$	$a_1 = 0$

We may assume that  $\alpha_1 = 0$ . To see how to treat the singular points of type  $\frac{1}{4}(1, 1, 3)$ , we have only to consider the singular point  $O_t$ . The other point can be dealt with in the same way.

The 1-cycle  $\Gamma$  for the singular point  $O_z$  with  $a_1 \neq 0$  is irreducible due to  $w^2$  and  $z^4 t$ .

The 1-cycle  $\Gamma$  for the singular point  $O_z$  with  $a_1 = 0$  consists of the proper transforms of the curves defined by  $x = y = w - \alpha_1 t^2 = 0$  and  $x = y = w - \alpha_2 t^2 = 0$ . These two irreducible components are symmetric with respect to the biregular involution of  $X_{16}$ . Consequently, the components of  $\Gamma$  are numerically equivalent to each other.

<b>No. 31:</b> $X_{16} \subset \mathbb{P}(1, 1, 4, 5, 6)$					$A^3 = 2/15$
$z(w^2 + z^3) + t^2(a_1 w + a_2 y t) + w f_{10}(x, y, z, t) + f_{16}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{6}(1, 1, 5) \boxed{\tau}$	$z w^2$				
$O_t = \frac{1}{5}(1, 1, 4) \boxed{\tau_1}$	$w t^2$				
$O_z O_w = 1 \times \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{B}$	–	$B$	$y$	$y$	$a_1 \neq 0$
$O_z O_w = 1 \times \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{S}$	–	$B$	$x, y$	$x, y$	$a_1 = 0$

If  $a_1 \neq 0$ , the 1-cycle  $\Gamma$  for the singular point of type  $\frac{1}{2}(1, 1, 1)$  is irreducible due to the monomials  $z^3$  and  $t^2 w$ .

Suppose  $a_1 = 0$ . Choose a general member  $H$  in the linear system  $|-K_X|$ . Then it is a normal K3 surface of degree 16 in  $\mathbb{P}(1, 4, 5, 6)$ . Let  $T$  be the proper transform of the surface  $H$ .

The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves  $\tilde{L}_{tw}$  and  $\tilde{C}$  on the normal surface  $T$ . The curve  $\tilde{L}_{tw}$  is the proper transform of the curve  $L_{tw}$  and the curve  $\tilde{C}$  is the proper transform of the curve  $C$  defined by  $x = y = w^2 + z^3 = 0$ . On the surface  $T$ , we have

$$\tilde{L}_{tw} \cdot \tilde{C} = L_{tw} \cdot C = \frac{2}{5}.$$

From the intersections

$$(\tilde{L}_{tw} + \tilde{C}) \cdot \tilde{L}_{tw} = -K_Y \cdot \tilde{L}_{tw} = \frac{1}{30}, \quad (\tilde{L}_{tw} + \tilde{C})^2 = B^3 = -\frac{11}{30}$$



on the surface  $T$ , we obtain

$$\tilde{L}_{tw}^2 = -\frac{11}{30}, \quad \tilde{C}^2 = -\frac{4}{5}.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{tw}^2 & \tilde{L}_{tw} \cdot \tilde{C} \\ \tilde{L}_{tw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{11}{30} & \frac{2}{5} \\ \frac{2}{5} & -\frac{4}{5} \end{pmatrix}$$

is negative-definite.

No. 32: $X_{16} \subset \mathbb{P}(1, 2, 3, 4, 7)$ <span style="float: right;"><math>A^3 = 2/21</math></span>					
$yw^2 + \prod_{i=1}^4 (t - \alpha_i y^2) + z^3(a_1 w + a_2 t z + a_3 x z^2) + w f_9(x, y, z, t) + f_{16}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{7}(1, 3, 4)$ $\square$ $\tau$			$yw^2$		
$O_z = \frac{1}{3}(1_x, 2_y, 1_t)$ $\textcircled{b}$	–	$2B$	$y$	$y$	$a_1 \neq 0$
$O_z = \frac{1}{3}(1_x, 2_y, 1_w)$ $\textcircled{c}$	–	$2B$	$x^2, y$	$x^2, y$	$a_1 = 0, a_2 \neq 0$
$O_z = \frac{1}{3}(2_y, 1_t, 1_w)$ $\textcircled{d}$	–	$2B$	$y$	$y$	$a_1 = a_2 = 0$
$O_y O_t = 4 \times \frac{1}{2}(1_x, 1_z, 1_w)$ $\textcircled{e}$	–	$3B + E$	$z$	$z$	

The 1-cycle  $\Gamma$  for the singular point  $O_z$  with  $a_1 \neq 0$  is irreducible due to  $t^4$  and  $z^3 w$ .

For the singular point  $O_z$  with  $a_1 = 0$  and  $a_2 \neq 0$  we may assume that  $a_3 = 0$ . The curve  $L_{zw}$  is contained in  $X_{16}$  because  $a_1 = 0$ . Let  $Z_{\lambda, \mu}$  be a curve cut out on  $X_{16}$  by

$$\begin{cases} y = \lambda x^2 \\ t = \mu x^4, \end{cases}$$

for some sufficiently general complex numbers  $\lambda$  and  $\mu$ . Then  $Z_{\lambda, \mu} = 2L_{zw} + C_{\lambda, \mu}$ , where  $C_{\lambda, \mu}$  is an irreducible reduced curve whose normalization is an elliptic curve. Denote by  $\tilde{L}_{zw}$  and  $\tilde{C}_{\lambda, \mu}$  the proper transforms of the curve  $L_{zw}$  and the curve  $C_{\lambda, \mu}$  on  $Y$ , respectively. Then

$$\begin{cases} -K_Y \cdot (2\tilde{L}_{zw} + \tilde{C}_{\lambda, \mu}) = 8B^3 = -\frac{4}{7}, \\ -K_Y \cdot \tilde{L}_{zw} = -K_X \cdot L_{zw} - \frac{1}{3}E \cdot \tilde{L}_{zw} = -\frac{2}{7}, \end{cases}$$

and hence  $-K_Y \cdot \tilde{C}_{\lambda, \mu} = 0$ .

The 1-cycle  $\Gamma$  for the singular point  $O_z$  with  $a_1 = a_2 = 0$  is irreducible even though it is non-reduced.

For the singular point  $O_y$ , consider the linear system generated by  $xy$  and  $z$ . Its base curves are defined by  $x = z = 0$  and  $y = z = 0$ . The curve defined by  $y = z = 0$  does not pass through the point  $O_y$ . The curve defined by  $x = z = 0$  is irreducible because of the monomial  $yw^2$  and  $t^4$ . Since its proper transform is the 1-cycle defined by  $(3B + E) \cdot B$  and  $(3B + E)^2 \cdot B > 0$  the divisor  $T$  is nef.

No. 33: $X_{17} \subset \mathbb{P}(1, 2, 3, 5, 7)$ <span style="float: right;"><math>A^3 = 17/210</math></span>	
$(dx^3 + exy + z)w^2 + t^2(a_1 w + a_2 yt) + z^4(b_1 t + b_2 yz) + y^5(c_1 w + c_2 yt + c_3 y^2 z + c_4 y^3 x) + w f_{10}(x, y, z, t) + f_{17}(x, y, z, t)$	

Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{7}(1, 2, 5) \boxed{\tau}$			$zw^2$		
$O_t = \frac{1}{5}(1, 2, 3) \boxed{\tau_1}$			$wt^2$		
$O_z = \frac{1}{3}(1_x, 2_y, 1_w) \textcircled{\text{D}}$	–	$2B$	$y$	$y$	$a_1 \neq 0, b_1 \neq 0$
$O_z = \frac{1}{3}(1_x, 2_y, 1_w) \textcircled{\text{S}}$	–	$2B$	$x^2, y$	$x^2, y$	$a_1 = 0, b_1 \neq 0$
$O_z = \frac{1}{3}(1_x, 2_t, 1_w) \textcircled{\text{S}}$	–	$2B$	$x^2, y$	$x^2, zw^2$	$b_1 = 0$
$O_y = \frac{1}{2}(1_x, 1_z, 1_t) \textcircled{\text{N}}$	–	$5B + 2E$	$t$	$t$	$c_1 \neq 0$
$O_y = \frac{1}{2}(1_x, 1_z, 1_w) \textcircled{\text{S}}$	–	$5B + E$	$x^5, t$	$x^5, zw^2$	$c_1 = 0, c_2 \neq 0$
$O_y = \frac{1}{2}(1_x, 1_t, 1_w) \textcircled{\text{S}}$	–	$3B$	$x^3, z$	$x^3, yt^3, t^2w$	$c_1 = c_2 = 0$ $c_3 \neq 0$
$O_y = \frac{1}{2}(1_z, 1_t, 1_w) \textcircled{\text{N}}$	–	$7B + 3E$	$w$	$w$	$c_1 = c_2 = c_3 = 0$

The 1-cycle  $\Gamma$  for the singular point  $O_z$  with  $a_1 \neq 0$  and  $b_1 \neq 0$  is irreducible since we have the monomials  $t^2w$ ,  $z^4t$  and  $zw^2$ .

For the singular point  $O_z$  with  $a_1b_1 = 0$  choose a general member  $H$  in the linear system  $|-2K_{X_{17}}|$ . Then it is a normal surface of degree 17 in  $\mathbb{P}(1, 3, 5, 7)$ . Let  $T$  be the proper transform of the divisor  $H$ . Then  $T$  is also normal. Furthermore, the curve  $\tilde{D}$  on  $T$  cut out by the surface  $S$  is the proper transform of the curve cut by the equation  $x = y = 0$ .

Suppose that  $b_1 \neq 0$  and  $a_1 = 0$ . Then  $a_2 \neq 0$ . The curve  $\tilde{D}$  then consists of two irreducible curves  $\tilde{L}_{tw}$  and  $\tilde{C}_1$ . The curve  $\tilde{L}_{tw}$  is the proper transform of the curve  $L_{tw}$  and the curve  $\tilde{C}_1$  is the proper transform of the curve  $C_1$  defined by  $x = y = w^2 + b_1z^3t = 0$ . Note that the curve  $L_{tw}$  and  $C_1$  intersect at the point  $O_t$ . The surface  $H$  is not quasi-smooth at the point  $O_t$ . We also consider the divisor  $D_z$  on  $H$  cut by the equation  $z = 0$ . We easily see that  $D_z = 2L_{tw} + R$ , where  $R$  is a curve whose support does not contain  $L_{tw}$ . The curve  $R$  and  $L_{tw}$  intersects at the point  $O_w$ . The surface  $H$  is quasi-smooth at the point  $O_w$ . Let  $\tilde{R}$  be the proper transform of  $R$ . Then we have  $\tilde{L}_{tw} \cdot \tilde{R} = \frac{3}{7}$ . From the intersection

$$(2\tilde{L}_{tw} + \tilde{R}) \cdot \tilde{L}_{tw} = 3A \cdot \tilde{L}_{tw} = \frac{3}{35}$$

we obtain  $\tilde{L}_{tw}^2 = -\frac{6}{35}$ . From the intersections

$$(\tilde{L}_{tw} + \tilde{C}_1) \cdot \tilde{L}_{tw} = -K_Y \cdot \tilde{L}_{tw} = \frac{1}{35}, \quad (\tilde{L}_{tw} + \tilde{C}_1)^2 = 2B^3 = -\frac{6}{35}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{tw}^2 = -\frac{6}{35}, \quad \tilde{L}_{tw} \cdot \tilde{C}_1 = \frac{1}{5}, \quad \tilde{C}_1^2 = -\frac{2}{5}.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{tw}^2 & \tilde{L}_{tw} \cdot \tilde{C}_1 \\ \tilde{L}_{tw} \cdot \tilde{C}_1 & \tilde{C}_1^2 \end{pmatrix} = \begin{pmatrix} -\frac{6}{35} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{pmatrix}$$

is negative-definite.

Suppose that  $b_1 = 0$  and  $a_1 \neq 0$ . Then  $b_2 \neq 0$ . The curve  $\tilde{D}$  consists of two irreducible curves  $\tilde{L}_{zt}$  and  $\tilde{C}_2$ . The curve  $\tilde{L}_{zt}$  is the proper transform of the curve  $L_{zt}$  and the curve  $\tilde{C}_2$  is the proper transform of the curve  $C_2$  defined by  $x = y = zw + a_1t^2 = 0$ . From the intersections

$$(\tilde{L}_{zt} + \tilde{C}_2) \cdot \tilde{L}_{zt} = -K_Y \cdot \tilde{L}_{zt} = -\frac{1}{10}, \quad (\tilde{L}_{zt} + \tilde{C}_2)^2 = 2B^3 = -\frac{6}{35}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{zt}^2 = -\frac{1}{10} - \tilde{L}_{zt} \cdot \tilde{C}_2, \quad \tilde{C}_2^2 = -\frac{1}{14} - \tilde{L}_{zt} \cdot \tilde{C}_2.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zt}^2 & \tilde{L}_{zt} \cdot \tilde{C}_2 \\ \tilde{L}_{zt} \cdot \tilde{C}_2 & \tilde{C}_2^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{10} - \tilde{L}_{zt} \cdot \tilde{C}_2 & \tilde{L}_{zt} \cdot \tilde{C}_2 \\ \tilde{L}_{zt} \cdot \tilde{C}_2 & -\frac{1}{14} - \tilde{L}_{zt} \cdot \tilde{C}_2 \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{zt} \cdot \tilde{C}_2$  is non-negative.

Suppose that  $b_1 = 0$  and  $a_1 = 0$ . We also have  $b_2 \neq 0$  and  $a_2 \neq 0$ . Furthermore, the defining equation of  $X_{17}$  must contain  $xz^2t$ ; otherwise  $X_{17}$  would not be quasi-smooth at the point  $x = y = w = a_2t^3 + b_2z^5 = 0$ . Note that the presence of  $xz^2t$  implies the normality of the surfaces  $H$  and  $T$ . The curve  $\tilde{D}$  consists of two irreducible curves  $\tilde{L}_{tw}$  and  $\tilde{L}_{zt}$ . Indeed,  $\tilde{D} = \tilde{L}_{tw} + 2\tilde{L}_{zt}$ . Note that the curve  $L_{tw}$  and  $L_{zt}$  intersect at the point  $O_t$ . The surface  $H$  is not quasi-smooth at the point  $O_t$ . We consider the divisor  $D_z$  on  $H$  cut by the equation  $z = 0$ . We easily see that  $D_z = 2L_{tw} + R$ , where  $R$  is a curve whose support does not contain  $L_{tw}$ . The curve  $R$  and  $L_{tw}$  intersects at the point  $O_w$ . The surface  $H$  is quasi-smooth at the point  $O_w$ . Let  $\tilde{R}$  be the proper transform of  $R$ . Then we have  $\tilde{L}_{tw} \cdot \tilde{R} = \frac{3}{7}$ . From the intersection

$$(2\tilde{L}_{tw} + \tilde{R}) \cdot \tilde{L}_{tw} = 3A \cdot \tilde{L}_{tw} = \frac{3}{35}$$

we obtain  $\tilde{L}_{tw}^2 = -\frac{6}{35}$ . From the intersections

$$(\tilde{L}_{tw} + 2\tilde{L}_{zt}) \cdot \tilde{L}_{tw} = -K_Y \cdot \tilde{L}_{tw} = \frac{1}{35}, \quad (\tilde{L}_{tw} + 2\tilde{L}_{zt})^2 = 2B^3 = -\frac{6}{35}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{tw}^2 = -\frac{6}{35}, \quad \tilde{L}_{tw} \cdot \tilde{L}_{zt} = \frac{1}{10}, \quad \tilde{L}_{zt}^2 = -\frac{1}{10}.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{tw}^2 & \tilde{L}_{tw} \cdot \tilde{L}_{zt} \\ \tilde{L}_{tw} \cdot \tilde{L}_{zt} & \tilde{L}_{zt}^2 \end{pmatrix} = \begin{pmatrix} -\frac{6}{35} & \frac{1}{10} \\ \frac{1}{10} & -\frac{1}{10} \end{pmatrix}$$

is negative-definite.

For the singular point  $O_y$  with  $c_1 \neq 0$  we consider the linear system  $|-5K_{X_{17}}|$ . Every member in the linear system passes through the point  $O_y$ . The base locus of  $|-5K_{X_{17}}|$  is the union of the loci defined by  $x = t = y = 0$  and  $x = t = z = 0$ . It is a 0-dimensional locus. Since the proper transform of a general member in  $|-5K_{X_{15}}|$  belongs to the linear system  $|5B + 2E|$ , the divisor  $T$  is nef.

For the singular point  $O_y$  with  $c_1 = 0$  and  $c_2 \neq 0$  we may assume that  $c_2 = 1$  and  $c_3 = c_4 = 0$  by a coordinate change. Choose a general member  $H$  in the linear system generated by  $x^5$  and  $t$ . Then it is a normal surface of degree 17 in  $\mathbb{P}(1, 2, 3, 7)$ . Let  $T$  be the proper transform of the surface  $H$ . The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two curves  $\tilde{L}_{yw}$  and  $\tilde{C}$ . The curve  $\tilde{L}_{yw}$  is the proper transform of the curve  $L_{yw}$  and the curve  $\tilde{C}$  is the proper transform of the curve  $C$  defined by  $x = t = w^2 + b_2yz^4 + awy^2z + by^4z^2 = 0$ , where  $a$  and  $b$  are constants.

Suppose that  $b_2 \neq 0$ . Then the curve  $\tilde{C}$  is irreducible. From the intersection numbers

$$\begin{aligned} (\tilde{L}_{yw} + \tilde{C}) \cdot \tilde{L}_{yw} &= -K_Y \cdot \tilde{L}_{yw} = -\frac{3}{7} \\ (\tilde{L}_{yw} + \tilde{C})^2 &= B^2 \cdot (5B + E) = -\frac{23}{21} \end{aligned}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{yw}^2 = -\frac{3}{7} - \tilde{L}_{yw} \cdot \tilde{C}, \quad \tilde{C}^2 = -\frac{2}{3} - \tilde{L}_{yw} \cdot \tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yw}^2 & \tilde{L}_{yw} \cdot \tilde{C} \\ \tilde{L}_{yw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} - \tilde{L}_{yw} \cdot \tilde{C} & \tilde{L}_{yw} \cdot \tilde{C} \\ \tilde{L}_{yw} \cdot \tilde{C} & -\frac{2}{3} - \tilde{L}_{yw} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{yw} \cdot \tilde{C}$  is non-negative.

Suppose that  $b_2 = 0$ . The curve  $C$  then consists of two irreducible curves  $C_1$  and  $C_2$  defined by  $x = t = w - \alpha_1 y^2 z = 0$  and  $x = t = w - \alpha_2 y^2 z = 0$ , respectively. Therefore, the curve  $\tilde{C}$  consists of their proper transforms  $\tilde{C}_1$  and  $\tilde{C}_2$ . From the intersections

$$\begin{aligned} (\tilde{L}_{yw} + \tilde{C}_1 + \tilde{C}_2) \cdot \tilde{L}_{yw} &= -K_Y \cdot \tilde{L}_{yw} = -\frac{3}{7}, \\ (\tilde{L}_{yw} + \tilde{C}_1 + \tilde{C}_2) \cdot \tilde{C}_1 &= -K_Y \cdot \tilde{C}_1 = -\frac{1}{3}, \quad (\tilde{L}_{yw} + \tilde{C}_1 + \tilde{C}_2) \cdot \tilde{C}_2 = -K_Y \cdot \tilde{C}_2 = -\frac{1}{3} \end{aligned}$$

on the surface  $T$ , we obtain the intersection matrix of the curves  $\tilde{L}_{yw}$ ,  $\tilde{C}_1$  and  $\tilde{C}_2$

$$\begin{pmatrix} -\frac{3}{7} - \tilde{L}_{yw} \cdot \tilde{C}_1 - \tilde{L}_{yw} \cdot \tilde{C}_2 & \tilde{L}_{yw} \cdot \tilde{C}_1 & \tilde{L}_{yw} \cdot \tilde{C}_2 \\ \tilde{L}_{yw} \cdot \tilde{C}_1 & -\frac{1}{3} - \tilde{L}_{yw} \cdot \tilde{C}_1 - \tilde{C}_1 \cdot \tilde{C}_2 & \tilde{C}_1 \cdot \tilde{C}_2 \\ \tilde{L}_{yw} \cdot \tilde{C}_2 & \tilde{C}_1 \cdot \tilde{C}_2 & -\frac{1}{3} - \tilde{L}_{zw} \cdot \tilde{C}_2 - \tilde{C}_1 \cdot \tilde{C}_2 \end{pmatrix}.$$

It is easy to check that it is negative-definite since  $\tilde{L}_{yw} \cdot \tilde{C}_1$ ,  $\tilde{L}_{yw} \cdot \tilde{C}_2$  and  $\tilde{C}_1 \cdot \tilde{C}_2$  are non-negative.

For the singular point  $O_y$  with  $c_1 = c_2 = 0$  and  $c_3 \neq 0$  we may assume that  $c_3 = 1$  and  $c_4 = 0$  by a coordinate change. Note that in such a case, we must have the monomial  $xyw^2$ , i.e.,  $e \neq 0$ : otherwise the hypersurface  $X_{17}$  is not quasi-smooth at the point defined by  $x = z = t = w^2 + y^7 = 0$ .

Choose a general member  $H$  in the linear system generated by  $x^3$  and  $z$ . Then it is a normal surface of degree 17 in  $\mathbb{P}(1, 2, 5, 7)$ . Let  $D$  be the curve on  $H$  that is cut out by the equation  $x = 0$ . Let  $T$  be the proper transform of the surface  $H$  and let  $\tilde{D}$  be the proper transform of the curve  $D$ . Then  $T$  is normal and the curve  $\tilde{D}$  is cut out by the surface  $S$ .

Suppose that  $a_1 \neq 0$ . We may then assume that  $a_1 = 1$  and  $a_2 = 0$  by a coordinate change. The curve  $\tilde{D}$  then consists of two irreducible curves  $\tilde{L}_{yw}$  and  $\tilde{L}_{yt}$ . The curve  $\tilde{L}_{yw}$  is the proper transform of the curve  $L_{yw}$  and the curve  $\tilde{L}_{yt}$  is the proper transform of the curve  $L_{yt}$ . From the intersection numbers

$$(2\tilde{L}_{yw} + \tilde{L}_{yt}) \cdot \tilde{L}_{yw} = -K_Y \cdot \tilde{L}_{yw} = -\frac{3}{7}, \quad (2\tilde{L}_{yw} + \tilde{L}_{yt})^2 = 3B^3 = -\frac{44}{35}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{yw}^2 = -\frac{3}{14} - \frac{1}{2}\tilde{L}_{yw} \cdot \tilde{L}_{yt}, \quad \tilde{L}_{yt}^2 = -\frac{2}{5} - 2\tilde{L}_{yw} \cdot \tilde{L}_{yt}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yw}^2 & \tilde{L}_{yw} \cdot \tilde{L}_{yt} \\ \tilde{L}_{yw} \cdot \tilde{L}_{yt} & \tilde{L}_{yt}^2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{14} - \frac{1}{2}\tilde{L}_{yw} \cdot \tilde{L}_{yt} & \tilde{L}_{yw} \cdot \tilde{L}_{yt} \\ \tilde{L}_{yw} \cdot \tilde{L}_{yt} & -\frac{2}{5} - 2\tilde{L}_{yw} \cdot \tilde{L}_{yt} \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{yw} \cdot \tilde{L}_{yt}$  is non-negative.

Suppose that  $a_1 = 0$ . By changing the coordinate  $y$ , we may assume that the defining equation of  $X_{17}$  does not contain the monomial  $x^2t^3$ . We have  $D = 3L_{yw} + L_{tw}$ , i.e. the curve  $D$  consists of two irreducible curves  $L_{yw}$  and  $L_{tw}$ . In fact, we have  $\tilde{D} = 3\tilde{L}_{yw} + \tilde{L}_{tw}$ , where the curve  $\tilde{L}_{tw}$  is the proper transform of the curve  $L_{tw}$ . Since the curve  $L_{yw}$  passes through the point  $O_y$  but the curve  $L_{tw}$  does not, we have

$$L_{yw} \cdot L_{tw} = \tilde{L}_{yw} \cdot \tilde{L}_{tw}, \quad L_{tw}^2 = \tilde{L}_{tw}^2.$$

We also have

$$(3\tilde{L}_{yw} + \tilde{L}_{tw}) \cdot \tilde{L}_{yw} = -K_Y \cdot \tilde{L}_{yw} = -\frac{3}{7}, \quad (3L_{yw} + L_{tw}) \cdot L_{tw} = -K_{X_{17}} \cdot L_{tw} = \frac{1}{35}.$$

To compute  $L_{yw} \cdot L_{tw}$ , we consider the divisor  $D_y$  on  $H$  given by the equation  $y = 0$ . Since the defining equation of  $X_{17}$  does not contain the monomial  $x^2t^3$ , we have  $D_y = 3L_{tw} + R$ , where  $R$  is a curve whose support does not contain the curve  $L_{tw}$ . Note that  $R$  meets  $L_{tw}$  only at the point  $O_t$ . Moreover, we can easily see that  $L_{tw} \cdot R = \frac{2}{5}$  since  $H$  is quasi-smooth at the point  $O_t$ . Then the intersection

$$(3L_{tw} + R) \cdot L_{tw} = -2K_{X_{17}} \cdot L_{tw} = \frac{2}{35}$$

implies that  $L_{tw}^2 = -\frac{4}{35}$ . This gives

$$\begin{pmatrix} \tilde{L}_{yw}^2 & \tilde{L}_{yw} \cdot \tilde{L}_{tw} \\ \tilde{L}_{yw} \cdot \tilde{L}_{tw} & \tilde{L}_{tw}^2 \end{pmatrix} = \begin{pmatrix} -\frac{10}{63} & \frac{1}{21} \\ \frac{1}{21} & -\frac{4}{35} \end{pmatrix}$$

which is negative-definite.

For the singular point  $O_y$  with  $c_1 = c_2 = c_3 = 0$ , we consider linear system  $|-7K_{X_{17}}|$ . Every member in the linear system passes through the point  $O_y$ . The proper transform of a general member in  $|-7K_{X_{17}}|$  belongs to the linear system  $|7B+3E|$ . The base locus of the linear system  $|-7K_{X_{17}}|$  possibly contains only the curve  $L_{yz}$  and the curve  $L_{zt}$ . If they are contained in  $X_{17}$ , we see  $(7B+3E) \cdot \tilde{L}_{yz} = -7K_{X_{17}} \cdot L_{yz} - \frac{1}{2}E \cdot \tilde{L}_{yz} = \frac{2}{3}$  and  $(7B+3E) \cdot \tilde{L}_{zt} = -7K_{X_{17}} \cdot L_{zt} = \frac{7}{15}$ . Therefore,  $T$  is nef.

<b>No. 36:</b> $X_{18} \subset \mathbb{P}(1, 1, 4, 6, 7)$					$A^3 = 3/28$
$zw^2 + t^3 - z^3t + wf_{11}(x, y, z, t) + f_{18}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{7}(1, 1, 6) \boxed{\tau}$	$zw^2$				
$O_z = \frac{1}{4}(1, 1, 3) \boxed{\epsilon_1}$	$zw^2 - z^3t$				
$O_z O_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{\text{B}}$	–	$B$	$y$	$y$	

The 1-cycle  $\Gamma$  for the singular point of type  $\frac{1}{2}(1, 1, 1)$  is irreducible because of the monomials  $zw^2$  and  $t^3$ .

<b>No. 38:</b> $X_{18} \subset \mathbb{P}(1, 2, 3, 5, 8)$					$A^3 = 3/40$
$yw^2 + t^2(a_1w + a_2zt) + z^6 + y^9 + wf_{10}(x, y, z, t) + f_{18}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{8}(1, 3, 5) \boxed{\tau}$	$yw^2$				
$O_t = \frac{1}{5}(1, 2, 3) \boxed{\tau_1}$	$wt^2$				
$O_y O_w = 2 \times \frac{1}{2}(1_x, 1_z, 1_t) \textcircled{\text{U}}$	–	$5B + 2E$	$t$	$t$	

For the singular points of type  $\frac{1}{2}(1, 1, 1)$  we consider the linear system  $|-5K_{X_{18}}|$ . Every member of the linear system passes through the point the singular points of type  $\frac{1}{2}(1, 1, 1)$  and the base locus of the linear system contains no curves. Since the proper transform of a general member in  $|-5K_{X_{18}}|$  belongs to the linear system  $|5B + 2E|$ , the divisor  $T$  is nef.

<b>No. 40:</b> $X_{19} \subset \mathbb{P}(1, 3, 4, 5, 7)$					$A^3 = 19/420$
$tw^2 - zt^3 + z^3(a_1w + a_2yz) + y^4(b_1w + b_2yz + b_3y^2x) + ay^2z^2t + by^3t^2 + wf_{12}(x, y, z, t) + f_{19}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{7}(1, 3, 4) \boxed{\tau}$	$tw^2$				
$O_t = \frac{1}{5}(1, 3, 2) \boxed{\epsilon}$	$tw^2 - zt^3$				
$O_z = \frac{1}{4}(1_x, 3_y, 1_t) \textcircled{\text{B}}$	–	$3B$	$y$	$zt^3$	$a_1 \neq 0$
$O_z = \frac{1}{4}(1_x, 1_t, 3_w) \textcircled{\text{S}}$	–	$3B$	$x^3, z$	$x^3, zt^3$	$a_1 = 0$
$O_y = \frac{1}{3}(1_x, 1_z, 2_t) \textcircled{\text{B}}$	–	$7B + E$	$w$	$y^3t^2$	$b_1 \neq 0, b \neq 0$ $a_2 \neq 0$
$O_y = \frac{1}{3}(1_x, 1_z, 2_t) \textcircled{\text{S}}$	–	$7B + E$	$w$	$y^3t^2$	$b_1 \neq 0, b \neq 0$ $a_2 = 0$
$O_y = \frac{1}{3}(1_x, 1_z, 2_t) \textcircled{\text{S}}$	–	$7B + E$	$w$	$z^4y$	$b_1 \neq 0, b = 0$ $a_2 \neq 0$
$O_y = \frac{1}{3}(1_x, 1_z, 2_t) \textcircled{\text{S}}$	–	$7B$	$w$	$zt^3$	$b_1 \neq 0, b = 0$ $a_2 = 0$
$O_y = \frac{1}{3}(1_x, 2_t, 1_w) \textcircled{\text{S}}$	–	$4B$	$x^4, z$	$x^4, tw^2$	$b_1 = 0, b_2 \neq 0$
$O_y = \frac{1}{3}(1_z, 2_t, 1_w) \textcircled{\text{U}}$	–	$7B + 2E$	$w$	$w$	$b_1 = 0, b_2 = 0$

The 1-cycle  $\Gamma$  for the singular point  $O_z$  with  $a_1 \neq 0$  is irreducible due to the monomials  $tw^2$ ,  $zt^3$  and  $z^3w$ .

For the singular point  $O_z$  with  $a_1 = 0$  we choose a general member  $H$  in the linear system  $| -3K_{X_{19}} |$ . Then it is a normal surface of degree 19 in  $\mathbb{P}(1, 4, 5, 7)$ . Let  $T$  be the proper transform of the divisor  $H$ .

The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves  $\tilde{L}_{zw}$  and  $\tilde{C}$ . The curve  $\tilde{L}_{zw}$  is the proper transform of the curve  $L_{zw}$  and the curve  $\tilde{C}$  is the proper transform of the curve  $C$  defined by  $x = y = w^2 - zt^2 = 0$ . From the intersection

$$(\tilde{L}_{zw} + \tilde{C}) \cdot \tilde{L}_{zw} = -K_Y \cdot \tilde{L}_{zw} = -\frac{1}{21}, \quad (\tilde{L}_{zw} + \tilde{C})^2 = 3B^3 = -\frac{4}{35}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{zw}^2 = -\frac{1}{21} - \tilde{L}_{zw} \cdot \tilde{C}, \quad \tilde{C}^2 = -\frac{1}{15} - \tilde{L}_{zw} \cdot \tilde{C}.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zw}^2 & \tilde{L}_{zw} \cdot \tilde{C} \\ \tilde{L}_{zw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{21} - \tilde{L}_{zw} \cdot \tilde{C} & \tilde{L}_{zw} \cdot \tilde{C} \\ \tilde{L}_{zw} \cdot \tilde{C} & -\frac{1}{15} - \tilde{L}_{zw} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{zw} \cdot \tilde{C}$  is non-negative number.

For the singular point  $O_y$  with  $b_1 \neq 0$  we may assume that  $b_1 = 1$  and  $b_2 = b_3 = 0$  by a suitable coordinate change.

For the singular point  $O_y$  with  $b_1 \neq 0$ ,  $b \neq 0$  and  $a_2 \neq 0$  the 1-cycle  $\Gamma$  is irreducible because of the monomials  $zt^3$ ,  $yz^4$  and  $y^3t^2$ .

For the singular point  $O_y$  with  $b_1 \neq 0$ ,  $b \neq 0$  and  $a_2 = 0$  we may assume that the monomial  $xy^2z^3$  does not appear in  $f_{19}$  by changing the coordinate  $w$  in a suitable way. We must then have  $a \neq 0$ ; otherwise the hypersurface would not be quasi-smooth at the point defined by  $x = t = w = a_1z^3 + y^4 = 0$ .

Take a general member  $H$  in the linear system generated by  $x^7$  and  $w$ . Then it is a normal surface of degree 19 in  $\mathbb{P}(1, 3, 4, 5)$ . Let  $T$  be the proper transform of the divisor  $H$ . The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves  $\tilde{L}_{yz}$  and  $\tilde{C}_1$ . The curve  $\tilde{L}_{yz}$  is the proper transform of the curve  $L_{yz}$  and the curve  $\tilde{C}_1$  is the proper transform of the curve  $C_1$  defined by  $x = w = -zt^2 + ay^2z^2 + by^3t = 0$ . From the intersection

$$(\tilde{L}_{yz} + \tilde{C}_1) \cdot \tilde{L}_{yz} = -K_Y \cdot \tilde{L}_{yz} = -\frac{1}{4}, \quad (\tilde{L}_{yz} + \tilde{C}_1)^2 = B^2 \cdot (7B + E) = -\frac{7}{20}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{yz}^2 = -\frac{1}{4} - \tilde{L}_{yz} \cdot \tilde{C}_1, \quad \tilde{C}_1^2 = -\frac{1}{10} - \tilde{L}_{yz} \cdot \tilde{C}_1.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yz}^2 & \tilde{L}_{yz} \cdot \tilde{C}_1 \\ \tilde{L}_{yz} \cdot \tilde{C}_1 & \tilde{C}_1^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} - \tilde{L}_{yz} \cdot \tilde{C}_1 & \tilde{L}_{yz} \cdot \tilde{C}_1 \\ \tilde{L}_{yz} \cdot \tilde{C}_1 & -\frac{1}{10} - \tilde{L}_{yz} \cdot \tilde{C}_1 \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{yz} \cdot \tilde{C}_1$  is non-negative number.

For the singular point  $O_y$  with  $b_1 \neq 0$ ,  $b = 0$  and  $a_2 \neq 0$  we take a general member  $H$  in the linear system generated by  $x^7$  and  $w$ . Then it is a normal surface of degree 19 in  $\mathbb{P}(1, 3, 4, 5)$ . Let  $T$  be the proper transform of the divisor  $H$ .

The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves  $\tilde{L}_{yt}$  and  $\tilde{C}_2$ . The curve  $\tilde{L}_{yt}$  is the proper transform of the curve  $L_{yt}$  and the curve  $\tilde{C}_2$  is the proper transform of the curve  $C_2$  defined by  $x = w = -t^3 + a_2yz^3 + ay^2zt = 0$ . From the intersections

$$(\tilde{L}_{yt} + \tilde{C}_2) \cdot \tilde{L}_{yt} = -K_Y \cdot \tilde{L}_{yt} = -\frac{1}{10}, \quad (\tilde{L}_{yt} + \tilde{C}_2)^2 = B^2 \cdot (7B + E) = -\frac{7}{20}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{yt}^2 = -\frac{1}{10} - \tilde{L}_{yt} \cdot \tilde{C}_2, \quad \tilde{C}_2^2 = -\frac{1}{4} - \tilde{L}_{yt} \cdot \tilde{C}_2.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yt}^2 & \tilde{L}_{yt} \cdot \tilde{C}_2 \\ \tilde{L}_{yt} \cdot \tilde{C}_2 & \tilde{C}_2^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{10} - \tilde{L}_{yt} \cdot \tilde{C}_2 & \tilde{L}_{yt} \cdot \tilde{C}_2 \\ \tilde{L}_{yt} \cdot \tilde{C}_2 & -\frac{1}{4} - \tilde{L}_{yt} \cdot \tilde{C}_2 \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{yt} \cdot \tilde{C}_2$  is a non-negative number.

For the singular point  $O_y$  with  $b_1 \neq 0$ ,  $b = 0$  and  $a_2 = 0$  we may assume that the monomial  $xy^2z^3$  does not appear in  $f_{19}$  by changing the coordinate  $w$  in a suitable way. We must then have  $a \neq 0$ ; otherwise the hypersurface would not be quasi-smooth at the point defined by  $x = t = w = a_1z^3 + y^4 = 0$ .

Take a general member  $H$  in the linear system generated by  $x^7$  and  $w$ . Then it is a normal surface of degree 19 in  $\mathbb{P}(1, 3, 4, 5)$ . Let  $T$  be the proper transform of the divisor  $H$ . The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of three irreducible curves  $\tilde{L}_{yz}$ ,  $\tilde{L}_{yt}$  and  $\tilde{C}_3$ . The curve  $\tilde{C}_3$  is the proper transform of the curve  $C_3$  defined by  $x = w = -t^2 + ay^2z = 0$ . From the intersections

$$(\tilde{L}_{yz} + \tilde{L}_{yt} + \tilde{C}_3) \cdot \tilde{L}_{yz} = -K_Y \cdot \tilde{L}_{yz} = -\frac{1}{4},$$

$$(\tilde{L}_{yz} + \tilde{L}_{yt} + \tilde{C}_3) \cdot \tilde{L}_{yt} = -K_Y \cdot \tilde{L}_{yt} = -\frac{1}{10},$$

$$(\tilde{L}_{yz} + \tilde{L}_{yt} + \tilde{C}_3) \cdot \tilde{C}_3 = -K_Y \cdot \tilde{C}_3 = -\frac{1}{6}.$$

on the surface  $T$ , we obtain the intersection matrix of the curves  $\tilde{L}_{yz}$ ,  $\tilde{L}_{yt}$  and  $\tilde{C}_3$

$$\begin{pmatrix} -\frac{1}{4} - \tilde{L}_{yz} \cdot \tilde{L}_{yt} - \tilde{L}_{yz} \cdot \tilde{C}_3 & \tilde{L}_{yz} \cdot \tilde{L}_{yt} & \tilde{L}_{yz} \cdot \tilde{C}_3 \\ \tilde{L}_{yz} \cdot \tilde{L}_{yt} & -\frac{1}{10} - \tilde{L}_{yz} \cdot \tilde{L}_{yt} - \tilde{L}_{yt} \cdot \tilde{C}_3 & \tilde{L}_{yt} \cdot \tilde{C}_3 \\ \tilde{L}_{yz} \cdot \tilde{C}_3 & \tilde{L}_{yt} \cdot \tilde{C}_3 & -\frac{1}{6} - \tilde{L}_{yz} \cdot \tilde{C}_3 - \tilde{L}_{yt} \cdot \tilde{C}_3 \end{pmatrix}.$$

It is easy to check that it is negative-definite since  $\tilde{L}_{yz} \cdot \tilde{L}_{yt}$ ,  $\tilde{L}_{yz} \cdot \tilde{C}_3$  and  $\tilde{L}_{yt} \cdot \tilde{C}_3$  are non-negative numbers.

For the singular point  $O_y$  with  $b_1 = 0$  and  $b_2 \neq 0$  we may put  $b_2 = 1$  by a coordinate change. Choose a general member  $H$  in the pencil on  $X_{19}$  generated by  $x^4$  and  $z$ . Then it is a normal



surface of degree 19 in  $\mathbb{P}(1, 3, 5, 7)$ . Let  $T$  be the proper transform of the divisor  $H$ . The surface  $S$  cuts out the surface  $T$  into a divisor  $\tilde{D}$ .

We suppose that  $b \neq 0$ . The divisor  $\tilde{D}$  then consists of two irreducible curves  $\tilde{L}_{yw}$  and  $\tilde{C}$ . The curve  $\tilde{L}_{yw}$  is the proper transform of the curve  $L_{yw}$  and the curve  $\tilde{C}$  is the proper transform of the curve  $C$  defined by  $x = z = w^2 + by^3t = 0$ . From the intersection

$$(\tilde{L}_{yw} + \tilde{C}) \cdot \tilde{L}_{yw} = -K_Y \cdot \tilde{L}_{yw} = -\frac{2}{7}, \quad (\tilde{L}_{yw} + \tilde{C})^2 = 4B^3 = -\frac{17}{35}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{yw}^2 = -\frac{2}{7} - \tilde{L}_{yw} \cdot \tilde{C}, \quad \tilde{C}^2 = -\frac{1}{5} - \tilde{L}_{yw} \cdot \tilde{C}.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yw}^2 & \tilde{L}_{yw} \cdot \tilde{C} \\ \tilde{L}_{yw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{7} - \tilde{L}_{yw} \cdot \tilde{C} & \tilde{L}_{yw} \cdot \tilde{C} \\ \tilde{L}_{yw} \cdot \tilde{C} & -\frac{1}{5} - \tilde{L}_{yw} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{yw} \cdot \tilde{C}$  is non-negative number.

We now suppose that  $b = 0$ . The divisor  $\tilde{D}$  then consists of two irreducible curves  $\tilde{L}_{yw}$  and  $\tilde{L}_{yt}$ . The curve  $\tilde{L}_{yt}$  is the proper transform of the curve  $L_{yt}$ . From the intersection

$$(\tilde{L}_{yw} + 2\tilde{L}_{yt}) \cdot \tilde{L}_{yw} = -K_Y \cdot \tilde{L}_{yw} = -\frac{2}{7}, \quad (\tilde{L}_{yw} + 2\tilde{L}_{yt})^2 = 4B^3 = -\frac{17}{35}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{yw}^2 = -\frac{2}{7} - 2\tilde{L}_{yw} \cdot \tilde{L}_{yt}, \quad \tilde{L}_{yt}^2 = -\frac{1}{20} - \frac{1}{2}\tilde{L}_{yw} \cdot \tilde{L}_{yt}.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yw}^2 & \tilde{L}_{yw} \cdot \tilde{L}_{yt} \\ \tilde{L}_{yw} \cdot \tilde{L}_{yt} & \tilde{L}_{yt}^2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} - 2\tilde{L}_{yw} \cdot \tilde{L}_{yt} & \tilde{L}_{yw} \cdot \tilde{L}_{yt} \\ \tilde{L}_{yw} \cdot \tilde{L}_{yt} & -\frac{1}{20} - \frac{1}{2}\tilde{L}_{yw} \cdot \tilde{L}_{yt} \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{yw} \cdot \tilde{L}_{yt}$  is non-negative number.

For the singular point  $O_y$  with  $b_1 = b_2 = 0$  we consider the linear system generated by  $z^{35}$ ,  $t^{28}$  and  $w^{20}$  on the hypersurface  $X_{19}$ . Its base locus is cut out by  $z = t = w = 0$ . Since we have the monomial  $xy^6$ , the base locus does not contain curves. Therefore the proper transform of a general member in the linear system is nef by Lemma 3.1.6 and it belongs to  $|140B + 40E|$ . Consequently, the surface  $T$  is nef since  $20T \sim_{\mathbb{Q}} 140B + 40E$ .

<b>No. 41:</b> $X_{20} \subset \mathbb{P}(1, 1, 4, 5, 10)$					$A^3 = 1/10$
$(w - \alpha_1 t^2)(w - \alpha_2 t^2) + z^5 + wf_{10}(x, y, z, t) + f_{20}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_t O_w = 2 \times \frac{1}{5}(1, 1, 4) \square \tau$			$wt^2$		
$O_z O_w = 1 \times \frac{1}{2}(1_x, 1_y, 1_t) \oplus$	—	$B$	$y$	$y$	

We may assume that  $\alpha_1 = 0$ . To see how to treat the singular points of type  $\frac{1}{5}(1, 1, 4)$ , we have only to consider the singular point  $O_t$ . The other point can be dealt with in the same way.

The 1-cycle  $\Gamma$  for the singular point of type  $\frac{1}{2}(1, 1, 1)$  is irreducible because of the monomial  $w^2$  and  $z^5$ .

<b>No. 42:</b> $X_{20} \subset \mathbb{P}(1, 2, 3, 5, 10)$					$A^3 = 1/15$
$(w - \alpha_1 y^5)(w - \alpha_2 y^5) + z^5(a_1 t + a_2 y z) + w t^2 + w f_{10}(x, y, z, t) + f_{20}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_z = \frac{1}{3}(1_x, 2_y, 1_w) \textcircled{D}$	–	$2B$	$y$	$y$	$a_1 \neq 0$
$O_z = \frac{1}{3}(1_x, 2_t, 1_w) \textcircled{D}$	–	$2B$	$y$	$w^2$	$a_1 = 0$
$O_t O_w = 2 \times \frac{1}{5}(1, 2, 3) \textcircled{\tau}$	$w t^2$				
$O_y O_w = 2 \times \frac{1}{2}(1, 1, 1) \textcircled{\textcircled{D}}$	–	$5B + 2E$	$t$	$t$	

The 1-cycle  $\Gamma$  for the singular point  $O_z$  with  $a_1 \neq 0$  is irreducible dues to  $w^2$  and  $z^5 t$ .

The 1-cycle  $\Gamma$  for the singular point  $O_z$  with  $a_1 = 0$  consists of the proper transforms of the curve  $L_{zt}$  and the curve defined by  $x = y = w + t^2 = 0$ . These two irreducible components are symmetric with respect to the biregular involution of  $X_{20}$ . Consequently the components of  $\Gamma$  are numerically equivalent to each other.

By changing the coordinate  $w$  we may assume that  $t^4$  is not in the polynomial  $f_{20}$ . To see how to untwist the singular points of type  $\frac{1}{5}(1, 2, 3)$  we have only to consider the singular point  $O_t$ . The other point can be treated in the same way.

For the singular points of type  $\frac{1}{2}(1, 1, 1)$ , we consider the linear system  $|-5K_{X_{20}}|$ . Every member of the linear system passes through the singular points of type  $\frac{1}{2}(1, 1, 1)$  and the base locus of the linear system contains no curves. Since the proper transform of a general member in  $|-5K_{X_{20}}|$  belongs to the linear system  $|5B + 2E|$ , the divisor  $T$  is nef.

<b>No. 43:</b> $X_{20} \subset \mathbb{P}(1, 2, 4, 5, 9)$					$A^3 = 1/18$
$y w^2 + t^4 + \prod_{i=1}^5 (z - \alpha_i y^2) + w f_{11}(x, y, z, t) + f_{20}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{9}(1, 4, 5) \textcircled{\tau}$	$y w^2$				
$O_y O_z = 5 \times \frac{1}{2}(1_x, 1_t, 1_w) \textcircled{D}$	–	$4B + E$	$z - \alpha_i y^2$	$y w^2$	

The 1-cycles  $\Gamma$  for the singular points of type  $\frac{1}{2}(1, 1, 1)$  are irreducible due to the monomials  $y w^2$  and  $t^4$ .

<b>No. 44:</b> $X_{20} \subset \mathbb{P}(1, 2, 5, 6, 7)$					$A^3 = 1/21$
$t w^2 + y(t - \alpha_1 y^3)(t - \alpha_2 y^3)(t - \alpha_3 y^3) + z^4 + w f_{14}(x, y, z, t) + f_{20}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{7}(1, 2, 5) \textcircled{\tau}$	$t w^2$				
$O_t = \frac{1}{6}(1, 5, 1) \textcircled{\epsilon}$	$t w^2 - y t^3$				
$O_y O_t = 3 \times \frac{1}{2}(1_x, 1_z, 1_w) \textcircled{D}$	–	$6B + 2E$	$t - \alpha_i y^3$	$t w^2$	

The 1-cycles  $\Gamma$  for the singular points of type  $\frac{1}{2}(1, 1, 1)$  are irreducible because of the monomials  $tw^2$  and  $z^4$ .

<b>No. 45:</b> $X_{20} \subset \mathbb{P}(1, 3, 4, 5, 8)$					$A^3 = 1/24$
$z(w - \alpha_1 z^2)(w - \alpha_2 z^2) + t^4 + y^4(a_1 w + a_2 yt) + wf_{12}(x, y, z, t) + f_{20}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{8}(1, 3, 5) \boxed{\tau}$	$zw^2$				
$O_y = \frac{1}{3}(1_x, 1_z, 2_t) \textcircled{\text{II}}$	-	$4B + E$	$z$	$z$	
$O_z O_w = 2 \times \frac{1}{4}(1_x, 3_y, 1_t) \textcircled{\text{D}}$	-	$3B$	$y$	$y$	

For the singular point  $O_y$  we consider the linear system generated by  $x^{40}$ ,  $z^{10}$ ,  $t^8$  and  $w^5$  on the hypersurface  $X_{20}$ . Its base locus does not contain curves. Therefore the proper transform of a general member in the linear system is nef by Lemma 3.1.6 and it belongs to  $|40B + 10E|$ . Consequently, the surface  $T$  is nef since  $10T \sim_{\mathbb{Q}} 40B + 10E$ .

The 1-cycles  $\Gamma$  for the singular points of type  $\frac{1}{4}(1, 3, 1)$  are irreducible due to the monomials  $zw^2$  and  $t^4$ .

<b>No. 46:</b> $X_{21} \subset \mathbb{P}(1, 1, 3, 7, 10)$					$A^3 = 1/10$
$yw^2 + t^3 + z^7 + wf_{11}(x, y, z, t) + f_{21}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{10}(1, 3, 7) \boxed{\tau}$	$yw^2$				

<b>No. 47:</b> $X_{21} \subset \mathbb{P}(1, 1, 5, 7, 8)$					$A^3 = 3/40$
$zw^2 + t^3 + yz^4 + wf_{13}(x, y, z, t) + f_{21}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{8}(1, 1, 7) \boxed{\tau}$	$zw^2$				
$O_z = \frac{1}{5}(1_x, 2_t, 3_w) \textcircled{\text{I}}$	+	$B - E$	$y$	$t^3$	

For the singular point  $O_z$ , let  $C_\lambda$  be a curve on the surface  $S_y$  defined by

$$\begin{cases} y = 0, \\ w = \lambda x^8 \end{cases}$$

for a sufficiently general complex number  $\lambda$ . Denote by  $\tilde{C}_\lambda$  the proper transform of the curve  $C_\lambda$  on  $Y$ . Then

$$-K_Y \cdot \tilde{C}_\lambda = (B - E) \cdot (8B + E) \cdot B = 0.$$

Consider the linear system generated by  $x^{72}$ ,  $y^9 t^9$  and  $yw^8$ . Its base curve is defined by  $x = y = 0$ . It is an irreducible curve because we have the monomials  $zw^2$  and  $t^3$ . The proper transform of a general member of the linear system is equivalent to  $72B$ . The only curve that intersects the divisor  $B$  negatively is the proper transform of the irreducible curve defined by

$x = y = 0$ . It is furthermore not on the surface  $T$ . Therefore, if the curve  $\tilde{C}_\lambda$  is reducible, each component of the curve  $\tilde{C}_\lambda$  intersects  $B$  trivially.

<b>No. 48:</b> $X_{21} \subset \mathbb{P}(1, 2, 3, 7, 9)$					$A^3 = 1/18$
$zw^2 + t^3 + z^7 + y^6(a_1w + a_2yt + a_3y^3z + a_4xy^4) + wf_{12}(x, y, z, t) + f_{21}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{9}(1, 2, 7) \boxed{\tau}$			$zw^2$		
$O_y = \frac{1}{2}(1, 1, 1) \textcircled{\text{u}}$	–	$9B + 4E$	$w + yt$	$w$ or $yt$	
$O_zO_w = 2 \times \frac{1}{3}(1_x, 2_y, 1_t) \textcircled{\text{b}}$	–	$2B$	$y$	$y$	

For the singular point  $O_y$  we consider the linear system  $|-9K_{X_{24}}|$ . Every member of the linear system passes through the singular point  $O_y$  and its base locus contains no curves. Since the proper transform of a general member in  $|-9K_{X_{24}}|$  belongs to the linear system  $|9B + 4E|$ , the divisor  $T$  is nef.

The 1-cycles  $\Gamma$  for the singular points of type  $\frac{1}{3}(1, 2, 1)$  are irreducible because of the monomials  $zw^2$  and  $t^3$ .

<b>No. 54:</b> $X_{24} \subset \mathbb{P}(1, 1, 6, 8, 9)$					$A^3 = 1/18$
$zw^2 + t^3 + z^4 + wf_{15}(x, y, z, t) + f_{24}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{9}(1, 1, 8) \boxed{\tau}$			$zw^2$		
$O_zO_w = 1 \times \frac{1}{3}(1_x, 1_y, 2_t) \textcircled{\text{b}}$	–	$B$	$y$	$y$	
$O_zO_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{\text{b}}$	–	$B$	$y$	$y$	

The 1-cycles  $\Gamma$  for the singular points of types  $\frac{1}{3}(1, 1, 2)$  and  $\frac{1}{2}(1, 1, 1)$  are irreducible since we have the monomials  $z^4$  and  $t^3$ .

<b>No. 56:</b> $X_{24} \subset \mathbb{P}(1, 2, 3, 8, 11)$					$A^3 = 1/22$
$yw^2 + t^3 + z^8 + y^{12} + wf_{13}(x, y, z, t) + f_{24}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{11}(1, 3, 8) \boxed{\tau}$			$yw^2$		
$O_yO_t = 3 \times \frac{1}{2}(1_x, 1_z, 1_w) \textcircled{\text{b}}$	–	$3B + E$	$z$	$z$	

The 1-cycle  $\Gamma$  for each singular point of type  $\frac{1}{2}(1, 1, 1)$  is irreducible due to the monomials  $yw^2$  and  $t^3$ .

<b>No. 58:</b> $X_{24} \subset \mathbb{P}(1, 3, 4, 7, 10)$					$A^3 = 1/35$
$zw^2 + t^2(a_1w + a_2yt) + z^6 + y^8 + wf_{14}(x, y, z, t) + f_{24}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{10}(1, 3, 7) \boxed{\tau}$			$zw^2$		

$O_t = \frac{1}{7}(1, 3, 4)$ $\boxed{\tau_1}$	$t^2w$			
$O_zO_w = 1 \times \frac{1}{2}(1_x, 1_y, 1_t)$ $\textcircled{\text{U}}$	–	$7B + 3E$	$t$	$t$

For the singular point of type  $\frac{1}{2}(1, 1, 1)$ , we consider the linear system  $|-7K_{X_{24}}|$ . Every member of the linear system passes through the singular point of type  $\frac{1}{2}(1, 1, 1)$  and its base locus contains no curves. Since the proper transform of a general member in  $|-7K_{X_{24}}|$  belongs to the linear system  $|7B + 3E|$ , the divisor  $T$  is nef.

<b>No. 60:</b> $X_{24} \subset \mathbb{P}(1, 4, 5, 6, 9)$					$A^3 = 1/45$
$tw^2 + (t^2 - \alpha_1y^3)(t^2 - \alpha_2y^3) + z^3(a_1w + a_2yz) + wf_{15}(x, y, z, t) + f_{24}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{9}(1, 4, 5)$ $\boxed{\tau}$	$tw^2$				
$O_z = \frac{1}{5}(1_x, 4_y, 1_t)$ $\textcircled{\text{B}}$	–	$4B$	$y$	$y$	$a_1 \neq 0$
$O_z = \frac{1}{5}(1_x, 1_t, 4_w)$ $\textcircled{\text{B}}$	–	$4B$	$y$	$t^4$	$a_1 = 0$
$O_tO_w = 1 \times \frac{1}{3}(1_x, 1_y, 2_z)$ $\textcircled{\text{B}}$	–	$5B + E$	$z$	$z$	
$O_yO_t = 2 \times \frac{1}{2}(1_x, 1_z, 1_w)$ $\textcircled{\text{U}}$	–	$5B + 2E$	$z$	$z$	

The 1-cycle  $\Gamma$  for the singular point  $O_z$  with  $a_1 \neq 0$  is irreducible due to the monomials  $t^4$  and  $z^3w$ .

The 1-cycle  $\Gamma$  for the singular point  $O_z$  with  $a_1 = 0$  has two irreducible components. One is the proper transform  $\tilde{L}_{zw}$  of the curve  $L_{zw}$  and the other is the proper transform  $\tilde{C}$  of the curve defined by  $x = y = w^2 + t^3 = 0$ . Then we see that

$$E \cdot \tilde{C} = 3E \cdot \tilde{L}_{zw}, \quad B \cdot \tilde{C} = 3B \cdot \tilde{L}_{zw}.$$

Therefore these two components are numerically proportional on  $Y$ .

The 1-cycle  $\Gamma$  for the singular point of type  $\frac{1}{3}(1, 1, 2)$  is irreducible since we have terms  $tw^2$  and  $(t^2 - \alpha_1y^3)(t^2 - \alpha_2y^3)$ . Note that the constants  $\alpha_i$ 's cannot be zero.

For the singular points of type  $\frac{1}{2}(1, 1, 1)$ , we consider the linear system generated by  $xy$  and  $z$  on  $X_{24}$ . Its base curves are defined by  $x = z = 0$  and  $y = z = 0$ . The curve defined by  $y = z = 0$  passes through no singular point of type  $\frac{1}{2}(1, 1, 1)$ . The curve defined by  $x = z = 0$  is irreducible. Moreover, its proper transform is the 1-cycle defined by  $(5B + 2E) \cdot B$ . Consequently, the divisor  $T$  is nef since  $(5B + 2E)^2 \cdot B > 0$ .

<b>No. 61:</b> $X_{25} \subset \mathbb{P}(1, 4, 5, 7, 9)$					$A^3 = 5/252$
$tw^2 - yt^3 + z^5 + y^4(a_1w + a_2yz + a_3xy^2) + wf_{16}(x, y, z, t) + f_{25}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{9}(1, 4, 5)$ $\boxed{\tau}$	$tw^2$				
$O_t = \frac{1}{7}(1, 5, 2)$ $\boxed{\epsilon}$	$tw^2 - yt^3$				
$O_y = \frac{1}{4}(1_x, 1_z, 3_t)$ $\textcircled{\text{B}}$	–	$9B + E$	$w$	$z^5$	$a_1 \neq 0$
$O_y = \frac{1}{4}(1_x, 3_t, 1_w)$ $\textcircled{\text{S}}$	–	$5B$	$x^5, z$	$x^5, tw^2$	$a_1 = 0, a_2 \neq 0$
$O_y = \frac{1}{4}(1_z, 3_t, 1_w)$ $\textcircled{\text{B}}$	–	$7B + E$	$t$	$t$	$a_1 = a_2 = 0$

If  $a_1 \neq 0$ , the 1-cycle  $\Gamma$  for the singular point  $O_y$  is irreducible due to the monomials  $yt^3$  and  $z^5$ .

If  $a_1 = a_2 = 0$ , the 1-cycle  $\Gamma$  for the singular point  $O_y$  is irreducible even though it is not reduced. Indeed, the support of  $\Gamma$  is the locus defined by  $x = z = t = 0$ .

Now we suppose that  $a_1 = 0$  and  $a_2 \neq 0$ . Then we may assume that  $a_2 = 1$  and  $a_3 = 0$ . We take a surface  $H$  cut by an equation  $z = \lambda x^5$  with a general complex number  $\lambda$  and then let  $T$  be the proper transform of the surface. Note that the surface  $H$  is normal. However, it is not quasi-smooth at the point  $O_y$ .

The intersection of  $T$  with the surface  $S$  gives us a divisor consisting of two irreducible curves on the normal surface  $T$ . One is the proper transform  $\tilde{L}_{yw}$  on  $T$  of the curve  $L_{yw}$  on  $H$ . The other is the proper transform  $\tilde{C}$  on  $T$  of the curve  $C$  defined by  $x = z = w^2 - yt^2 = 0$  in  $\mathbb{P}(1, 4, 5, 7, 9)$ . From the intersection numbers

$$(\tilde{L}_{yw} + \tilde{C}) \cdot \tilde{L}_{yw} = -K_Y \cdot \tilde{L}_{yw} = -\frac{2}{9}, \quad (\tilde{L}_{yw} + \tilde{C})^2 = 5B^3 = -\frac{20}{63}$$

on the surface  $T$ , we obtain

$$\tilde{L}_{yw}^2 = -\frac{2}{9} - \tilde{L}_{yw} \cdot \tilde{C}, \quad \tilde{C}^2 = -\frac{2}{21} - \tilde{L}_{yw} \cdot \tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yw}^2 & \tilde{L}_{yw} \cdot \tilde{C} \\ \tilde{L}_{yw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{9} - \tilde{L}_{yw} \cdot \tilde{C} & \tilde{L}_{yw} \cdot \tilde{C} \\ \tilde{L}_{yw} \cdot \tilde{C} & -\frac{2}{21} - \tilde{L}_{yw} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since  $\tilde{L}_{zw} \cdot \tilde{C}$  is positive.

<b>No. 65:</b> $X_{27} \subset \mathbb{P}(1, 2, 5, 9, 11)$				$A^3 = 3/110$	
$zw^2 + t^3 + yz^5 + y^8(a_1w + a_2yt + a_3y^3z + a_4xy^5) + wf_{16}(x, y, z, t) + f_{27}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{11}(1, 2, 9) \text{ } \boxed{\tau}$			$zw^2$		
$O_z = \frac{1}{5}(1_x, 4_t, 1_w) \text{ } \textcircled{b}$	-	$2B$	$y$	$zw^2$	
$O_y = \frac{1}{2}(1, 1, 1) \text{ } \textcircled{a}$	-	$11B + 5E$	$w + xy^5$	$xy^5$ or $w$	

The 1-cycle  $\Gamma$  for the singular point  $O_z$  are irreducible because of the monomials  $zw^2$  and  $t^3$

For the singular point  $O_y$ , we consider the linear system  $|-11K_{X_{27}}|$ . Note that every member of the linear system passes through the point  $O_y$  and the base locus of the linear system contains no curves. Since the proper transform of a general member in  $|-11K_{X_{24}}|$  belongs to the linear system  $|11B + 5E|$ , the divisor  $T$  is nef.

<b>No. 68:</b> $X_{28} \subset \mathbb{P}(1, 3, 4, 7, 14)$				$A^3 = 1/42$	
$(w - \alpha_1 t^2)(w - \alpha_2 t^2) + z^7 + y^7(a_1 t + a_2 xy^2) + wf_{14}(x, y, z, t) + f_{28}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_y = \frac{1}{3}(1_x, 1_z, 2_w) \text{ } \textcircled{b}$	-	$7B + E$	$t$	$w^2$	$a_1 \neq 0$

$O_y = \frac{1}{3}(1_z, 1_t, 2_w) \textcircled{B}$	–	$4B + E$	$z$	$z$	$a_1 = 0$
$O_t O_w = 2 \times \frac{1}{7}(1, 3, 4) \textcircled{\tau}$	$wt^2$				
$O_z O_w = 1 \times \frac{1}{2}(1_x, 1_y, 1_t) \textcircled{B}$	–	$3B + E$	$y$	$y$	

For the singular point  $O_y$  with  $a_1 \neq 0$  the 1-cycle  $\Gamma$  is irreducible because of the monomials  $w^2$  and  $z^7$ .

The 1-cycle  $\Gamma$  for the singular point  $O_y$  with  $a_1 = 0$  consists of two irreducible curves. These are the proper transforms of the curves defined by  $x = z = w - \alpha_i t^2 = 0$ . Since these two curves on  $X_{28}$  are interchanged by the automorphism defined by  $[x, y, z, t, w] \rightarrow [x, y, z, t, (\alpha_1 + \alpha_2)t^2 - f_{14} - w]$ , their proper transforms are numerically equivalent on  $Y$ .

To see how to deal with the singular points of type  $\frac{1}{7}(1, 3, 4)$  we may assume that  $\alpha_1 = 0$  and we have only to consider the singular point  $O_t$ . The other point can be treated in the same way.

The 1-cycle  $\Gamma$  for the singular point of type  $\frac{1}{2}(1, 1, 1)$  is irreducible due to the monomials  $w^2$  and  $z^7$ .

<b>No. 69:</b> $X_{28} \subset \mathbb{P}(1, 4, 6, 7, 11)$					$A^3 = 1/66$
$zw^2 + t^4 + y(z^2 - \alpha_1 y^3)(z^2 - \alpha_2 y^3) + wf_{17}(x, y, z, t) + f_{28}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{11}(1, 4, 7) \textcircled{\tau}$	$zw^2$				
$O_z = \frac{1}{6}(1_x, 1_t, 5_w) \textcircled{B}$	–	$4B$	$y$	$t^4$	
$O_y O_z = 2 \times \frac{1}{2}(1_x, 1_t, 1_w) \textcircled{B}$	–	$11B + 5E$	$w$	$w$	

The 1-cycle  $\Gamma$  for the singular point  $O_z$  is irreducible due to the monomials  $zw^2$  and  $t^4$ .

For the singular points of type  $\frac{1}{2}(1, 1, 1)$  consider the linear system generated by  $xyz$ ,  $yt$  and  $w$ . Since the base curves of the linear system pass through no singular points of type  $\frac{1}{2}(1, 1, 1)$  the divisor  $T$  is nef.

<b>No. 74:</b> $X_{30} \subset \mathbb{P}(1, 3, 4, 10, 13)$					$A^3 = 1/52$
$zw^2 + t^3 + z^5 t + y^{10} + wf_{17}(x, y, z, t) + f_{30}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{13}(1, 3, 7) \textcircled{\tau}$	$zw^2$				
$O_z = \frac{1}{4}(1_x, 3_y, 1_w) \textcircled{B}$	–	$3B$	$y$	$y$	
$O_z O_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{B}$	–	$3B + E$	$y$	$y$	

The 1-cycles  $\Gamma$  for the singular points of types  $\frac{1}{2}(1, 1, 1)$  and  $\frac{1}{4}(1, 3, 1)$  are irreducible because of the monomials  $zw^2$ ,  $t^3$  and  $z^5 t$ .

<b>No. 76:</b> $X_{30} \subset \mathbb{P}(1, 5, 6, 8, 11)$					$A^3 = 1/88$
$tw^2 + zt^3 + z^5 + y^5 + wf_{19}(x, y, z, t) + f_{30}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition

$O_w = \frac{1}{11}(1, 5, 6) \square \tau$	$tw^2$			
$O_t = \frac{1}{8}(1, 5, 3) \square \epsilon$	$tw^2 - zt^3$			
$O_z O_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w) \textcircled{B}$	-	$5B + 2E$	$y$	$y$

The 1-cycle  $\Gamma$  for the singular point of type  $\frac{1}{2}(1, 1, 1)$  is irreducible because of the monomials  $tw^2$  and  $z^5$ .

<b>No. 79:</b> $X_{33} \subset \mathbb{P}(1, 3, 5, 11, 14)$					$A^3 = 1/70$
$zw^2 + t^3 + yz^6 + y^{11} + wf_{19}(x, y, z, t) + f_{33}(x, y, z, t)$					
Singularity	$B^3$	Linear system	Surface $T$	Vanishing order	Condition
$O_w = \frac{1}{14}(1, 3, 11) \square \tau$	$zw^2$				
$O_z = \frac{1}{5}(1_x, 1_t, 4_w) \textcircled{B}$	-	$3B$	$y$	$t^3$	

The 1-cycle  $\Gamma$  for the singular point  $O_z$  is irreducible because of the monomials  $zw^2$  and  $t^3$ .

## 7 Epilogue

### 7.1 Open problems

Let  $X$  be a quasi-smooth hypersurface of degrees  $d$  with only terminal singularities in weighted projective space  $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ , where  $d = \sum a_i$ . In Theorem 1.1.9, we prove that  $X$  is birationally rigid. In particular,  $X$  is non-rational. Moreover, the proof also explicitly describes the generators of the group of birational automorphisms  $\text{Bir}(X)$  modulo subgroup of biregular automorphisms  $\text{Aut}(X)$  (see Theorem 5.1.3). Furthermore, Theorem 1.1.11 says that  $\text{Bir}(X) = \text{Aut}(X)$  for those families in the list of Fletcher and Reid with entry numbers No. 1, 3, 10, 11, 14, 19, 21, 22, 28, 29, 34, 35, 37, 39, 49, 50, 51, 52, 53, 55, 57, 59, 62, 63, 64, 66, 67, 70, 71, 72, 73, 75, 77, 78, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94 and 95. Of course, some quasi-smooth threefolds in other families may also be birationally super-rigid.

Explicit birational involutions play a key role in the proof of Theorem 1.1.9. In many cases, they arise from a generically 2-to-1 rational map of  $X$  to a suitable 3-dimensional weighted projective space (in [23] such involutions are called *quadratic*). However, in some cases they arise from a rational map  $X \dashrightarrow \mathbb{P}^2$  whose general fiber is birational to a smooth elliptic curve (in [23] such involutions are called *elliptic*). Moreover, we often use such rational maps to *exclude* some singular points of  $X$ , i.e., to prove that some singular points cannot be centers of non-canonical singularities of any log pair  $(X, \frac{1}{n}\mathcal{M})$ , where  $\mathcal{M}$  is a mobile linear subsystem in  $|-nK_X|$ . The latter is done using Corollary 2.2.2 or Lemma 3.1.8. A similar role in the proof of Theorem 1.1.9 is played by so-called *Halphen pencils* on  $X$ , i.e., pencils whose general member is an irreducible surface of Kodaira dimension zero. Implicitly Halphen pencils appear almost every time when we apply Lemmas 3.1.2 and 3.1.7. This leads us to three problems that are closely related to Theorem 1.1.9. They are

- (1) To find relations between generators of the group  $\text{Bir}(X)$ .
- (2) To describe birational transformations of  $X$  into elliptic fibrations.
- (3) To classify Halphen pencils on  $X$ .



While proving Theorem 1.1.9, we noticed many interesting Halphen pencils on  $X$  without naming them so. This gives an evidence for

**Conjecture 7.1.1.** *Every Halphen pencil on  $X$  is a pencil of K3 surfaces.*

We do not know any deep reason why this conjecture should be true. When  $X$  is a general threefold in its family, Conjecture 7.1.1 was proved in [13].

The original proof of Theorem 1.1.3 given by Iskovskikh and Manin in [30] holds in arbitrary characteristic. This also follows from [37]. The short proof of Theorem 1.1.3 given by Corti in [22] holds only in characteristic zero. For some families in the list of Fletcher and Reid, the proof of Theorem 1.1.9 requires vanishing type results and, thus, is valid only in characteristic zero. This suggests the birational rigidity problem of  $X$  and problems (1), (2) and (3) over an algebraically closed field of positive characteristic. For double covers of  $\mathbb{P}^3$  ramified along smooth sextic surfaces, this was done in [12] and [14], which revealed special phenomenon of small characteristics (see [12, Example 1.5]).

## 7.2 General vs. special

The first three problems listed in the previous section are solved in the case when  $X$  is a general hypersurface in its family. This is done in [5], [6], [11] and [13]. In many cases, the same methods can be applied regardless of the assumption that  $X$  is general. For example, we proved in [11] that a general hypersurface in the families No. 3, 60, 75, 83, 87, 93 cannot have a birational transformation to an elliptic fibration. We can prove that it is also true for every quasi-smooth hypersurface in the families No. 3, 75, 83, 87, 93, using the methods given in this paper. However, in the family No. 60, it is no longer true for an arbitrary quasi-smooth hypersurface.

**Example 7.2.1.** Let  $X_{24}$  be a quasi-smooth hypersurface in the family No. 60. Suppose, in addition, that  $X_{24}$  contains the curve  $L_{zw}$ . We may then assume that it is defined by the equation

$$w^2t + w(at^2x^3 + tg_9(x, y, z) + g_{15}(x, y, z)) + t^4 + t^3h_6(x, y, z) + t^2h_{12}(x, y, z) + th_{18}(x, y, z) + h_{24}(x, y, z) = 0$$

in  $\mathbb{P}(1, 4, 5, 6, 9)$ . For the hypersurface  $X_{24}$  to be quasi-smooth at the point  $O_z$ , the polynomial  $h_{24}$  must contain the monomial  $z^4y$ . For the hypersurface  $X_{24}$  to contain the curve  $L_{zw}$ , the polynomial  $g_{15}$  must not contain the monomial  $z^3$ .

Consider the projection  $\pi: X_{24} \dashrightarrow \mathbb{P}(1, 4, 6)$ . Its general fiber is an irreducible curve birational to an elliptic curve. To see this, on the hypersurface  $X_{24}$ , consider the surface cut by  $y = \lambda x^4$  and the surface cut by  $t = \mu x^6$ , where  $\lambda$  and  $\mu$  are sufficiently general complex numbers. Then the intersection of these two surfaces is the 1-cycle  $4L_{zw} + C_{\lambda, \mu}$ , where the curve  $C_{\lambda, \mu}$  is a curve defined by the equation

$$\lambda w^2x^2 + w \left( \mu^2x^{11}g_0 + \mu x^2g_9(x, \lambda x^4, z) + \frac{g_{15}(x, \lambda x^4, z)}{x^4} \right) + \mu^4x^{20} + \mu^3x^{14}h_6(x, \lambda x^4, z) + \mu^2x^8h_{12}(x, \lambda x^4, z) + \mu x^2h_{18}(x, \lambda x^4, z) + \frac{h_{24}(x, \lambda x^4, z)}{x^4} = 0$$

in  $\mathbb{P}(1, 5, 9)$ . Plugging  $x = 1$  into the equation, we see that the curve  $C_{\lambda, \mu}$  is birational to a double cover of  $\mathbb{C}$  ramified at four distinct points.

In some of 95 families of Reid and Fletcher, special quasi-smooth hypersurfaces may have simpler geometry than their general representatives.

**Example 7.2.2.** Let  $X_5$  be a quasi-smooth hypersurface in  $\mathbb{P}(1, 1, 1, 1, 2)$  (family No. 2). The hypersurface  $X_5$  can be given by

$$tw^2 + wf_3(x, y, z, t) + f_5(x, y, z, t) = 0.$$

The natural projection  $X_5 \dashrightarrow \mathbb{P}^3$  is a generically double cover. Therefore, it induces a birational involution of  $X_5$ , denoted by  $\tau$ . By Theorem 5.1.3,  $\text{Bir}(X)$  is generated by  $\text{Aut}(X)$  and  $\tau$ . By Theorem 1.1.9,  $X_5$  is birationally rigid. Moreover, if  $X_5$  is general, then  $X_5$  is not birationally super-rigid, i.e.,  $\text{Bir}(X) \neq \text{Aut}(X)$ . To be precise,  $\tau$  is biregular if and only if  $f_3$  is a zero polynomial. Thus,  $X_5$  is birationally super-rigid if and only if  $f_3$  is a zero polynomial.

However, this is not always the case, i.e., special quasi-smooth hypersurfaces usually have more complicated geometry than their general representatives. Here we provide three illustrating examples.

**Example 7.2.3.** Let  $X_4$  be a smooth quartic threefold in  $\mathbb{P}^4$  (family No. 1). From Theorem 1.1.3 we know that every smooth quartic hypersurface in  $\mathbb{P}^4$  admits no non-biregular birational automorphisms. Moreover, it was proved in [5] that every rational map  $\rho: X_4 \dashrightarrow \mathbb{P}^2$  whose general fiber is birational to a smooth elliptic curve fits a commutative diagram

$$\begin{array}{ccc} & X & \\ \rho \swarrow & & \searrow \pi \\ \mathbb{P}^2 & \dashrightarrow \sigma & \mathbb{P}^2, \end{array}$$

where  $\pi$  is a linear projection from a line and  $\sigma$  is a birational map. Furthermore, it was proved in [13] that every Halphen pencil on  $X_4$  is contained in  $| -K_{X_4} |$  provided that  $X_4$  satisfies some generality assumptions. Earlier, Iskovskikh pointed out in [29] that this is no longer true for an *arbitrary* smooth quartic hypersurface in  $\mathbb{P}^4$ . The complete classification of Halphen pencils on  $X_4$  was obtained recently in [10].

**Example 7.2.4** (For details see the proof of Theorem 5.4.1). Let  $X_{14}$  be a quasi-smooth hypersurface in  $\mathbb{P}(1, 2, 3, 4, 5)$  (family No. 23). If  $X_{14}$  is a general such hypersurface, then there exists an exact sequence of groups

$$1 \longrightarrow \Gamma_{X_{14}} \longrightarrow \text{Bir}(X_{14}) \longrightarrow \text{Aut}(X_{14}) \longrightarrow 1,$$

where  $\Gamma_{X_{14}}$  is a free product of two birational involutions constructed in Section 5.3. This follows from [11, Lemma 4.2] (cf. Theorem 5.1.3). Moreover, let  $\rho: X_{14} \dashrightarrow \mathbb{P}^2$  be a rational map whose general fiber is birational to a smooth elliptic curve. If  $X_{14}$  is general, then there exists a commutative diagram

$$\begin{array}{ccc} & X & \\ \phi \swarrow & & \searrow \rho \\ \mathbb{P}(1, 2, 3) & \dashrightarrow \sigma & \mathbb{P}^2 \end{array}$$

where  $\phi$  is the natural projection, and  $\sigma$  is some birational map. Suppose now that  $X_{14}$  is defined by the equation

$$(t + by^2)w^2 + yt(t - \alpha_1y^2)(t - \alpha_2y^2) + z^4y + xtz^3 + xf_{13}(x, y, z, t, w) + yg_{12}(y, z, t, w) = 0.$$

Then none of these assertions are true. Indeed, let  $\mathcal{H}$  be the linear subsystem of  $| - 5K_{X_{14}} |$  generated by  $x^5, xy^2, x^3y$  and  $yz + xt$ . Let  $\pi: X_{14} \dashrightarrow \mathbb{P}(1, 2, 5)$  be the rational map induced by  $[x : y : z : t : w] \rightarrow [x : y : yz + xt]$ . Then  $\pi$  is dominant and its general fiber is birational to an elliptic curve. Let  $f: Y \rightarrow X_{14}$  be the weighted blow up at the point  $O_z$  with weight  $(1, 1, 2)$ . Denote by  $E$  its exceptional surface. Let  $g: W \rightarrow Y$  be the weighted blow up at the point over  $O_w$  with weight  $(1, 2, 3)$ . Denote by  $G$  be its exceptional divisor. Denote by  $\hat{L}_{zw}, \hat{L}_{zt}$  and  $\hat{L}_{yw}$  the proper transforms of the curves  $L_{zw}, L_{zt}$  and  $L_{yw}$  by the morphism  $f \circ g$ . Then the curves  $\hat{L}_{zw}$  and  $\hat{L}_{zt}$  are the only curves that intersect  $-K_W$  negatively. Moreover, there is an anti-flip  $\chi: W \dashrightarrow U$  along the curves  $\hat{L}_{zw}$  and  $\hat{L}_{zt}$  (see the proof of Theorem 5.4.1). Let  $\check{E}$  and  $\check{G}$  be the proper transforms on  $U$  of the divisors  $E$  and  $G$ , respectively. For  $m \gg 0$ , the linear system  $| - mK_U |$  is free and gives an elliptic fibration  $\eta: U \rightarrow \Sigma$ , where  $\Sigma$  is a normal surface. Furthermore, there exist a commutative diagram

$$\begin{array}{ccccc}
 & & W & \xrightarrow{\chi} & U \\
 & & \swarrow g & & \searrow \eta \\
 Y & & & & \\
 \downarrow f & & & & \\
 X & & & & \\
 \searrow \pi & & & & \\
 & & \mathbb{P}(1, 2, 5) & \xleftarrow{\theta} & \Sigma
 \end{array}$$

where  $\theta$  is a birational map. The divisor  $\check{G}$  is a section of the elliptic fibration  $\eta$  and  $\check{E}$  is a 2-section of  $\eta$ . Let  $\tau_U$  be a birational involution of the threefold  $U$  that is induced by the reflection of the general fiber of  $\eta$  with respect to the section  $\check{G}$ . The involution  $\tau_U$  induces a birational involution of  $X_{14}$ . This new involution is not biregular and not contained in the subgroup in  $\text{Bir}(X_{14})$  generated by two birational involutions constructed in Section 5.3.

**Example 7.2.5.** Let  $X_{17}$  be a quasi-smooth hypersurface in  $\mathbb{P}(1, 2, 3, 5, 7)$  (family No. 33). Then it can be given by the quasi-homogenous polynomial equation

$$\begin{aligned}
 & (dx^3 + exy + z)w^2 + t^2(a_1w + a_2yt) + z^4(b_1t + b_2yz) + \\
 & y^5(c_1w + c_2yt + c_3y^2z + c_4y^3x) + wf_{10}(x, y, z, t) + f_{17}(x, y, z, t) = 0.
 \end{aligned}$$

The pencil  $| - 2K_{X_{17}} |$  is a Halphen pencil. Moreover, if the defining equation of  $X_{17}$  is sufficiently general, then this is the only Halphen pencil on  $X_{17}$  (see [13, Corollary 1.1]). Suppose that  $c_1 = c_2 = 0$  and  $c_3 \neq 0$ . Then we may assume that  $c_3 = 1$  and  $c_4 = 0$  by a coordinate change. Here we can obtain an extra Halphen pencil. Indeed, the pencil on  $X_{17}$  cut out by  $\lambda x^3 + \mu z = 0$ , where  $[\lambda : \mu] \in \mathbb{P}^1$ , is a Halphen pencil different from  $| - 2K_{X_{17}} |$ .

### 7.3 Calabi problem

Let  $X$  be a quasi-smooth hypersurface of degrees  $d$  with only terminal singularities in weighted projective space  $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ , where  $d = \sum a_i$ . In many applications it is useful to *measure* how singular effective  $\mathbb{Q}$ -divisors  $D$  equivalent to  $-K_X$  can be. A possible *measurement* can be given by the so-called  $\alpha$ -invariant of the Fano hypersurface  $X$ . It is defined by the number

$$\alpha(X) = \sup \left\{ \lambda \in \mathbb{Q} \left| \begin{array}{l} \text{the pair } (X, \lambda D) \text{ is Kawamata log terminal} \\ \text{for every effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X. \end{array} \right. \right\}.$$

If  $X$  is a general threefold in its family, then  $\alpha(X) = 1$  by [7, Theorem 1.3] and [8, Theorem 1.15] except the case when  $X$  lies in the families No. 1, 2, 3, 4 and 5. If  $X$  is a general quartic threefold in  $\mathbb{P}^3$  (family No. 1), we have  $\alpha(X) \geq \frac{7}{9}$  by [15, Theorem 1.1.6]. If  $X$  is a double cover of  $\mathbb{P}^3$  ramified along smooth sextic surface (family No. 3), then all possible values of  $\alpha(X)$  are found in [15, Theorem 1.1.5]. For general threefolds in the families No. 2, 4 and 5, the bound  $\alpha(X) > \frac{3}{4}$  proved in [7] and [9]. In particular, we have

**Corollary 7.3.1.** *Let  $X$  be a quasi-smooth hypersurface of degrees  $d$  with only terminal singularities in weighted projective space  $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ , where  $d = \sum a_i$ . Suppose that  $X$  is a general threefold in this family. Then  $\alpha(X) > \frac{3}{4}$ .*

Similarly, we can define the  $\alpha$ -invariant of any Fano variety with at most Kawamata log terminal singularities. This invariant has been studied intensively by many people who used different notation for it. The notation  $\alpha(X)$  is due to Tian who defined the  $\alpha$ -invariant in a different way (see [40]). However, his definition coincides with the one we just gave (see [16, Theorem A.3]).

Tian proved in [40] that smooth Fano variety of dimension  $n$  whose  $\alpha$ -invariant is greater than  $\frac{n}{n+1}$  admits a Kähler–Einstein metric. This result was generalized for Fano varieties with quotient singularities by Demaily and Kollár (see [24, Criterion 6.4]). Thus, Corollary 7.3.1 implies

**Theorem 7.3.2.** *Let  $X$  be a quasi-smooth hypersurface of degrees  $d$  with only terminal singularities in weighted projective space  $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ , where  $d = \sum a_i$ . Suppose that  $X$  is a general threefold in this family. Then  $X$  admits an orbifold Kähler–Einstein metric.*

Recently, Chen, Donaldson and Sun and independently Tian proved that smooth Fano variety admits a Kähler–Einstein metric if and only if it is  $K$ -stable (see [17], [18], [19], [20] and [41]). Earlier Odaka and Okada proved that birationally super-rigid smooth Fano varieties with base-point-free anticanonical linear system must be slope-stable (see [35]). Furthermore, Odaka and Sano proved that Fano varieties with at most log terminal singularities of dimension  $n$  whose  $\alpha$ -invariant is greater than  $\frac{n}{n+1}$  must be  $K$ -stable (see [35]). These results suggest that *every* quasi-smooth hypersurface in the 95 families of Fletcher and Reid should admit an orbifold Kähler–Einstein metric.

Using methods we developed in the proof of Theorem 1.1.9, it is possible to explicitly describe all quasi-smooth threefolds in 95 families of Fletcher and Reid whose  $\alpha$ -invariant is greater than  $\frac{3}{4}$ . All of them must admit orbifold Kähler–Einstein metric by [24, Criterion 6.4]. Moreover, we can apply [7, Theorem 6.5] to those whose  $\alpha$ -invariant is 1.

## 7.4 Arithmetics

As it was pointed out by Pukhlikov and Tschinkel, the solution of problem (1) is closely related to the problem of potential density of rational points on  $X$  in the case when  $X$  is defined over a number field. For example, if  $\text{Bir}(X)$  is infinite, then we are able to show that  $X$  contains infinitely many rational surfaces. It implies the potential density of rational points on  $X$ .

The papers [3], [4], [27] use birational transformations into elliptic fibrations to prove potential density on all smooth Fano threefolds possibly except double covers of  $\mathbb{P}^3$  ramified along smooth sextic surfaces (family No. 3 in the list of Fletcher and Reid).

If  $X$  is defined over a number field, it seems likely that the set of rational points on  $X$  is potentially dense. For every smooth quartic threefold in  $\mathbb{P}^4$  (family No. 1), this was proved by Harris and Tschinkel in [27]. For generic Fano threefolds in the families No. 2, 4, 5, 6, 7, 9, 11, 12, 13, 15, 17, 19, 20, 23, 25, 27, 30, 31, 33, 36, 38, 40, 41, 42, 44, 58, 61, 68 and 76, this was proved in [11] and [14]. Despite many attempts, this problem is still open for double covers of  $\mathbb{P}^3$  ramified along smooth sextic surfaces. These are smooth hypersurfaces in family No. 3.

The methods we use in proof of Theorem 1.1.9 can be applied to prove the potential density of rational points on the quasi-smooth threefolds in some families in the list of Fletcher and Reid. In fact, for some families one can use our methods to prove the *density* of rational points on  $X$ .

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