SPORADIC SIMPLE GROUPS AND QUOTIENT SINGULARITIES

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to Igor Rostislavovich Shafarevich with deep respect

Abstract. We show that the only sporadic simple group such that some of its faithful representations or some faithful representations of its stem extensions give rise to exceptional (weakly-exceptional but not exceptional, respectively) quotient singularities is the Hall–Janko group (the Suzuki group, respectively).

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1. Introduction

Finite subgroups in $\mathrm{SL}_2(\mathbb{C})$ have been classified long time ago. The corresponding quotients by these groups are $\mathbb{A}$-$\mathbb{D}$-$\mathbb{E}$ singularities, which are also known by other names (Kleinian singularities, Du Val singularities, rational surface double points, two-dimensional canonical singularities etc). Shokurov suggested a higher dimensional generalization of the singularities of type $\mathbb{E}$ and of both types $\mathbb{D}$ and $\mathbb{E}$. He called them exceptional and weakly-exceptional, respectively. The precise definitions of exceptional and weakly-exceptional singularities are quite technical (see, for example, [5, Definitions 1.10] and [8, Definitions 1.4], respectively). Surprisingly, they are connected with a wide range of algebraic and geometric questions.

It turned out that exceptional and weakly-exceptional singularities are related to the Calabi problem for orbifolds with positive first Chern class (see [5]).

Example 1.1. Let $(V \ni O)$ be a germ of three-dimensional isolated quasihomogeneous hypersurface singularity that is given by

$$\phi\big(x, y, z, t\big) = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, y, z, t\big]\Big),$$

where $\phi(x, y, z, t)$ is a quasihomogeneous polynomial of degree d with respect to some weights $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$ such that $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3$ and $\gcd(a_0, a_1, a_2, a_3) = 1$. Let S be a weighted hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d that is given by the same equation $\phi(x, y, z, t) = 0$. Suppose, in addition, that $\sum_{i=0}^{n} a_i > d$, and S is well-formed (see [13, Definition 6.9]). Then S a del Pezzo surfaces with at most quotient singularities. Moreover, if $(V \ni O)$ is either exceptional or weakly-exceptional, then S admits an orbifold Kahler–Einsten metric (this follows, for example, from [23], [18, Theorem 4.9], [14, Theorem 2.1], and [4, Theorem A.3]).

Many old and still open group-theoretic questions have algebro-geometric counterparts related to the exceptionality of quotient singularities (see, for example, [22] and [5, Conjecture 1.25]). It seems that the study of exceptionality and weak-exceptionality of quotient singularities may shed new light on some group-theoretic problems.

Example 1.2. Let G be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ that does not contain reflections[1], and let G' be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ that does not contain reflections such that G' and G has the same image in $\mathrm{PGL}_{n+1}(\mathbb{C})$. Then it follows from [5, Theorem 3.15] and [5, Theorem 3.16] that the singularity $\mathbb{C}^{n+1}/G$ is exceptional (weakly-exceptional, respectively) if and only if the

[1]Recall that an element $g \in G$ is called a *reflection* (or sometimes a *quasi-reflection*) if it has exactly one eigenvalue that is different from 1.

singularity $\mathbb{C}^{n+1}/G'$ is exceptional (weakly-exceptional, respectively). Moreover, it follows from [5, Theorem 1.30] and [5, Theorem 3.15] that the subgroup $G \subset \mathrm{GL}_{n+1}(\mathbb{C})$ is transitive (i. e. the corresponding $(n+1)$-dimensional representation is irreducible) provided that the singularity $\mathbb{C}^{n+1}/G$ is weakly-exceptional. Similarly, it follows from [5, Theorem 1.29] that G must be primitive (see [2] or, for example, [5, Definition 1.21]) if $\mathbb{C}^{n+1}/G$ is exceptional. Finally, it follows from [5, Theorem 3.16] ([5, Theorem 3.15], respectively) that $\mathbb{C}^{n+1}/G$ is not exceptional if G has a semi-invariant[2] of degree at most $n+1$ (of degree at most n, respectively).

Staring from this point, we restrict ourselves to the case of quotient singularities.

In low dimensions, the study of exceptional and weakly-exceptional quotient singularities is closely related to the classification of finite collineation groups (see [2], [3], [16], [24], [25], [10]). Using classical results of Blichfeldt, Brauer, and Lindsey, exceptional quotient singularities of dimensions 3, 4, 5 and 6 have been completely classified by Markushevich, Prokhorov, and the authors (see [17], [5], [6]). Moreover, we used the classification obtained by Wales in [24] and [25] to prove that seven-dimensional exceptional quotient singularities do not exist (see [6]). Sakovics classified weakly-exceptional quotient singularities of dimensions 3 and 4 (see [20]). Higher-dimensional weakly-exceptional quotient singularities were studied in [8]. Unfortunately, we have no clear picture which finite subgroups in $\mathrm{GL}_{n+1}(\mathbb{C})$ give rise to exceptional or weakly-exceptional singularities for $n \gg 0$.

A surprising fact observed in [6] is that among the (very few) groups corresponding to exceptional six-dimensional quotient singularities there appears a central extension $2.\mathrm{J}_2$ of the Hall–Janko sporadic simple group (see [15]). Actually, this property is very rare for the projective representations of sporadic simple groups, so that essentially we have only one more example of this kind of behavior among them. It is related to the Suzuki sporadic simple group (see [21]). In this paper we prove the following

Theorem 1.3 (cf. [7, Theorem 14])**.** Let G be a sporadic simple finite group, or its stem extension.[3] Let $G \hookrightarrow \mathrm{GL}(U)$ be a (faithful) finite dimensional complex representation of G. Then the singularity U/G is exceptional if and only if $G \cong 2.\mathrm{J}_2$, and U is a 6-dimensional irreducible representation of G. The singularity U/G is weakly-exceptional but not exceptional if and only if $G \cong 6.\mathrm{Suz}$ is the central extension of the Suzuki simple group by the cyclic group of order 6, and U is a 12-dimensional irreducible representation of G.

Theorem 1.3 shows that the groups J_2 and Suz are somehow distinguished among the sporadic simple groups from the geometric point of view, and therefore motivates the following

Question 1.4. Is there some group-theoretic property that distinguishes the groups J_2 and Suz among the sporadic simple groups?

As one can see from Appendix A, one of the characterizations of these groups comes from the fact that the groups $2.\mathrm{J}_2$ and 6.Suz have irreducible representations with no semi-invariants of low degrees. Note that this is a priori not equivalent to being weakly-exceptional, and the geometric characterization via weak-exceptionality requires another series of coincidences. On the other hand, it would be interesting to know if there is some intrinsic characterization of the groups J_2 and Suz that goes beyond the observation concerning the semi-invariants — possibly not even involving representation theory at all. One of the goals of this paper as we see it is to attract attention of the experts in group theory to Question 1.4, and more generally to a more

[2]Recall that a *semi-invariant* of degree d of the group G is a one-dimensional subrepresentation in $\mathrm{Sym}^d(\mathbb{C}^{n+1})$.

[3]Recall that a *stem extension* of a perfect group G is a central extension of G that can be obtained as a quotient of the universal central extension of G by its Schur multiplier, see e. g. [19, §9.4] for detailes.

broad range of questions on the possible interplay between the properties of certain groups and geometrical properties of the corresponding quotient singularities.[4]

To study exceptionality and weak-exceptionality of a singularity $\mathbb{C}^{n+1}/G$ for a finite subgroup $G \subset \mathrm{GL}_{n+1}(\mathbb{C})$, one can always assume that the group G does not contain reflections (cf. Example 1.2 and [6, Remark 1.16]). Keeping in mind Example 1.2, we see that to prove Theorem 1.3, we may restrict ourselves to the case of irreducible representations. Similarly, it follows from Example 1.2 that we may exclude from our search the groups that have semi-invariants of low degrees by a straightforward case by case study. The results of the corresponding computations are listed in Appendix A. They were obtained using the GAP software (see [11]) and the classification of all finite simple groups (see [9]) and communicated to us by A. Zavarnitsyn. As a result, we are left with just two candidates: the group $2.\mathrm{J}_2$ acting in $U \cong \mathbb{C}^6$, and the group 6.Suz acting in $U \cong \mathbb{C}^{12}$. The exceptionality of the quotient singularity corresponding to the first case was settled in [6]. Therefore, the only new result of geometric nature we obtain here is the following theorem that is proved in Section 2.

Theorem 1.5. Let $G \cong$ 6.Suz, and let U be a 12-dimensional irreducible representation of G. Then the singularity U/G is weakly-exceptional but not exceptional.

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2. Suzuki simple group

In this section we prove Theorem 1.5 using the method we first applied in [7] and the following

Theorem 2.1 ([8, Theorem 1.12])**.** Let G be a finite group in $\mathrm{GL}_{n+1}(\mathbb{C})$ that does not contain reflections, and let $\bar{G}$ be the image of the group G in $\mathrm{PGL}_{n+1}(\mathbb{C})$. If $\mathbb{C}^{n+1}/G$ is not weakly-exceptional, then there is a $\bar{G}$-invariant, irreducible, normal, Fano type[5] projectively normal subvariety $V \subset \mathbb{P}^n$ such that

$$\deg\big(V\big) \leqslant \binom{n}{\dim\big(V\big)},$$

and for every $i \geqslant 1$ and for every $m \geqslant 0$, we have $h^i(\mathcal{O}_{\mathbb{P}^n}(m) \otimes \mathcal{I}_V) = h^i(\mathcal{O}_V(m)) = 0$, and

$$h^0\Big(\mathcal{O}_{\mathbb{P}^n}\Big(\big(\dim(V)+1\big)\Big) \otimes \mathcal{I}_V\Big) \geqslant \binom{n}{\dim\big(V\big)+1},$$

where $\mathcal{I}_V$ is the ideal sheaf of the subvariety $V \subset \mathbb{P}^n$. Let Π be a general linear subspace in $\mathbb{P}^n$ of codimension $k \leqslant \dim(V)$. Put $X = V \cap \Pi$. Then $h^i(\mathcal{O}_\Pi(m) \otimes \mathcal{I}_X) = 0$ for every $i \geqslant 1$ and $m \geqslant k$, where $\mathcal{I}_X$ is the ideal sheaf of the subvariety $X \subset \Pi$. Moreover, if $k = 1$ and $\dim(V) \geqslant 2$, then X is irreducible, projectively normal and $h^i(\mathcal{O}_X(m)) = 0$ for every $i \geqslant 1$ and $m \geqslant 1$.

Let $G \cong$ 6.Suz be the central extension of the Suzuki sporadic simple group. Then there is an embedding $G \hookrightarrow \mathrm{SL}_{12}(\mathbb{C})$ that is given by an irreducible nine-dimensional G-representation U.

Denote by Δ_k the collection of dimensions of irreducible subrepresentations of $\mathrm{Sym}^k(U^\vee)$. We will use the following notation: writing $\Delta_k = [\ldots, r\times m, \ldots]$, we mean that among the irreducible subrepresentations of $\mathrm{Sym}^k(U^\vee)$ there are exactly r subrepresentations of dimension m (not

[4] Note that there is an interesting characterization of the groups $2.\mathrm{J}_2$ and 6.Suz together with the representations of these groups arising in Theorem 1.3 via irreducibility of symmetric powers obtained in [12, Theorem 1.1], that has geometric implications concerning stable vector bundles, cf. [12, Corollary 1.3] and [1].

[5] An irreducible normal variety V is said to be of *Fano type* if there exists an effective $\mathbb{Q}$-divisor Δ_V on V such that $-(K_V + \Delta_V)$ is a $\mathbb{Q}$-Cartier ample divisor, and the log pair (V, Δ_V) has at most Kawamata log terminal singularities.

necessarily isomorphic to each other). Furthermore, denote by Σ_k the set of partial sums of Δ_k, i.e. the set of all numbers $s=\sum r_i' m_i$, where $\Delta_k=[r_1\times m_1, r_2\times m_2,\ldots,r_i\times m_i,\ldots]$ and $0\leqslant r_i'\leqslant r_i$ for all i. We use the abbreviation m_i for $1\times m_i$.

We will need the following properties of the G-representation U that can be verified by direct computations. We used the GAP software (see [11]) to carry them out.

Lemma 2.2. The representations $\mathrm{Sym}^k(U^\vee)$ are irreducible for $2\leqslant k\leqslant 5$ (and have dimensions 78, 364, 1365 and 4368, respectively). Futhermore, $\Delta_6=[364,12012]$, $\Delta_7=[4368,27456]$,

$$\Delta_8=[1365,4290,27027,42900],\quad \Delta_9=[2\times 364, 2\times 16016, 35100, 100100],$$
$$\Delta_{10}=[78,1365,3003,4290,2\times 27027,2\times 75075,139776],$$
$$\Delta_{11}=[12,924,2\times 4368,2\times 12012,2\times 27456,112320,144144,2\times 180180],$$

and

$$\Delta_{12}=[1,143,2\times 364,1001,2\times 5940,2\times 12012,2\times 14300,2\times 15015,15795,25025,$$
$$2\times 40040,54054,75075,88452,2\times 93555,2\times 100100,163800,168960,197120].$$

Corollary 2.3. The group G does not have semi-invariants of degree $d\leqslant 11$, and does have a semi-invariant of degree $d=12$.

Now we are ready to prove Theorem 1.3. Suppose that the singularity U/G is not weakly-exceptional. Let $\bar{G}$ be the image of the group G in $\mathrm{PGL}_{12}(\mathbb{C})$. Then it follows from Theorem 2.1 that there is a $\bar{G}$-invariant, irreducible, normal, Fano type projectively normal subvariety $V\subset\mathbb{P}^{11}$ such that

$$\deg\big(V\big)\leqslant\binom{11}{\dim\big(V\big)},$$

and for every $i\geqslant 1$ and $m\geqslant 0$, we have $h^i(\mathcal{O}_{\mathbb{P}^{11}}(m)\otimes\mathcal{I}_V)=h^i(\mathcal{O}_V(m))=0$, where $\mathcal{I}_V$ is the ideal sheaf of the subvariety $V\subset\mathbb{P}^{11}$. Put $n=\dim(V)$.

Lemma 2.4. One has $1\leqslant n\leqslant 9$.

Proof. One has $n\neq 0$ since U is an irreducible representation of the group G. On the other hand, if $n=10$ then V is an $\bar{G}$-invariant hypersurface such that $\deg(V)\leqslant 11$, which contradicts Corollary 2.3. $\square$

Put $h_m=h^0(\mathcal{O}_V(m))$ and $q_m=h^0(\mathcal{O}_{\mathbb{P}^{11}}(m)\otimes\mathcal{I}_V)$ for every $m\in\mathbb{Z}$. Then

$$q_m=h^0\Big(\mathcal{O}_{\mathbb{P}^{11}}(m)\Big)-h_m=\binom{11+m}{m}-h_m$$

for every $m\geqslant 1$, since $h^1(\mathcal{O}_{\mathbb{P}^{11}}(m)\otimes\mathcal{I}_V)=0$ for every $m\geqslant 0$.

Let H be a general hyperplane section of V. Put $d=H^n=\deg(V)$ and $H_V(m)=\chi(\mathcal{O}_V(m))$. Then $H_V(m)=h_m$ for every $m\geqslant 1$, since $h^i(\mathcal{O}_V(mH))=0$ for every $i\geqslant 1$ and every $m\geqslant 0$. Recall that $H_V(m)$ is a Hilbert polynomial of the subvariety V, which is a polynomial in m of degree n with leading coefficient $d/n!$.

It follows from Theorem 2.1 that V has one more property that we need. Let $\Lambda_1,\Lambda_2,\ldots,\Lambda_n$ be general hyperplanes in $\mathbb{P}^{11}$. Put $\Pi_j=\Lambda_1\cap\ldots\cap\Lambda_j$, $V_j=V\cap\Pi_j$, and $H_j=V_j\cap H$ for every $j\in\{1,\ldots,n\}$. Put $V_0=V$, $H_0=H$, $\Pi_0=\mathbb{P}^{11}$. For every $j\in\{0,1,\ldots,n\}$, let $\mathcal{I}_{V_j}$ be the ideal sheaf of the subvariety $V_j\subset\Pi_j$. Then it follows from Theorem 2.1 that $h^i(\mathcal{O}_{\Pi_j}(m)\otimes\mathcal{I}_{V_j})=0$ for every $i\geqslant 1$ and $m\geqslant j$.

Recall that $\Pi_j\cong\mathbb{P}^{11-j}$ and put $q_i(V_j)=h^0(\mathcal{O}_{\Pi_j}(i)\otimes\mathcal{I}_{V_j})$ for every $j\in\{0,1,\ldots,n\}$.

Lemma 2.5. Suppose that $i\geqslant j+1$ and $j\in\{1,\ldots,n\}$. Then

$$q_i(V_j)=q_i-\binom{j}{1}q_{i-1}+\binom{j}{2}q_{i-2}-\ldots+(-1)^jq_{i-j}.$$

Proof. See the proof of [7, Lemma 27]. $\square$

Recall that $q_1 = 0$ since the representation U is irreducible, and $q_i = 0$ for $2 \leqslant i \leqslant 5$ by Lemma 2.2. Therefore, we have

Corollary 2.6. If $n = 9$, one has

$$q_9 - 8q_8 + 28q_7 - 56q_6 = q_9(V_8) \geqslant 0.$$

Playing with the numbers $q_i(V_j)$, we obtain

Lemma 2.7 (cf. [7, Lemma 35])**.** One has

$$\binom{12}{n} - \frac{(n+1)d}{2} > q_n(V_{n-1}) \geqslant \binom{12}{n} - nd - 1.$$

Proof. Recall that the variety $V_{n-1} \subset \Pi_{n-1} \cong \mathbb{P}^{11-n+1}$ is a smooth curve of degree d, since V is normal. Recall also V_{n-1} is irreducible, since V is irreducible. Let g be the genus of the curve V_{n-1}. Then it follows from the adjunction formula that

$$2g - 2 = (K_V + (n-1)) \cdot H^{n-1} = K_V \cdot H^{n-1} + (n-1)d < (n-1)d,$$

since $K_V \cdot H^{n-1} < 0$, because $-K_V$ is big. On the other hand, we have

$$q_m(V_{n-1}) = \binom{11-n+1+m}{m} - h^0\Big(\mathcal{O}_{V_{n-1}}(mH_{n-1})\Big)$$

for every $m \geqslant n$, because $h^1(\mathcal{O}_{\Pi_{n-1}}(m) \otimes \mathcal{I}_{V_{n-1}}) = 0$ for every $m \geqslant n-1$. Since $2g-2 < nd$, the divisor nH_{n-1} is non-special. Therefore, it follows from the Riemann–Roch theorem that

$$q_n(V_{n-1}) = \binom{12}{n} - nd + g - 1,$$

which implies the required inequalities, since $2g-2 < (n-1)d$ and $g \geqslant 0$. $\square$

Combining Lemma 2.7 and Corollary 2.6, we obtain

Corollary 2.8. If $n = 9$, then

$$\max\big(0, 219 - 9d\big) \leqslant q_9 - 8q_8 + 28q_7 - 56q_6 < 220 - 5d.$$

As a by-product of Corollary 2.8, we get

Corollary 2.9. If $n = 9$, then $1 \leqslant d \leqslant 43$.

The above restrictions reduce the problem to a combinatorial question of finding all polynomials H_V of degree n with a leading coefficient $d/n!$, such that $h_m = H_V(m) \in \Sigma_m$ for sufficiently many $m \geqslant 1$, and such that the numbers h_m, $q_m = h^0(\mathcal{O}_{\mathbb{P}^{11}}(m)) - h_m$ and d satisfy the conditions arising from Corollaries 2.8 and 2.9. This can be done in a straighforward way, although the number of cases to be considered is so large that it requires some checks to be done by a computer. Doing this, we get the following facts which we leave without proofs.

Lemma 2.10. There are no polynomials $\mathrm{H}(m)$ of degree $n \leqslant 8$ such that the values $h_m = \mathrm{H}(m)$ are in Σ_m for $1 \leqslant m \leqslant 12$.

Lemma 2.11. There does not exist a polynomial $\mathrm{H}(m)$ of degree $n = 9$ with a leading coefficient $d/n!$ with $d \in \mathbb{Z}$ and $1 \leqslant d \leqslant 43$, such that the values $h_m = \mathrm{H}(m)$ are in Σ_m for $1 \leqslant m \leqslant 12$, and the numbers h_m and $q_m = \binom{11+m}{m} - h_m$ satisfy the bounds of Corollary 2.8.

This completes the proof of Theorem 1.5.

Appendix A. Semi-invariants of low degrees

In this section we list the results of the computations of the low degree semi-invariants of the irreducible representations of the relevant groups (communicated to us by A. Zavarnitsyn). The GAP software (see [11]) was used to carry them out.

The tables below contain the information about the representations of stem extensions of sporadic simple groups with the least possible value of $\mu(U) = d(U)/\dim(U)$ among all irreducible representations U of the corresponding group G, where $d(U)$ is a minimal degree of a semi-invariant[6] of G for the G-representation U. In each case we list the values of $d(U)$ and $\dim(U)$ for which the minimum of $\mu(U)$ is attained (except for the groups $2.J_2$ and 6.Suz where we list slightly different information, see below). It appears a posteriori that for each of our groups the value of $\dim(U)$, and thus also of $d(U)$, is unique.

Mathieu groups.

G	M_{11}	M_{12}	$2.M_{12}$	M_{22}	$2.M_{22}$	$3.M_{22}$	$4.M_{22}$	$6.M_{22}$	$12.M_{22}$	M_{23}	M_{24}
$d(U)$	4	3	6	2	4	3	4	6	12	2	4
$\dim(U)$	10	16	10	21	10	21	56	66	120	22	45

Conway groups.

G	Co_1	$2.Co_1$	Co_2	Co_3
$d(U)$	2	2	2	2
$\dim(U)$	276	24	23	23

Leech lattice groups except Conway groups (for the group $2.J_2$ the 6-dimesional representations are ignored, and for the group 6.Suz the 12-dimensional representations are ignored when computing n and d — these actually lead to weakly-exceptional singularities, and the values of d and n with $d/n \geqslant 1$).

G	HS	2.HS	J_2	$2.J_2$[7]	McL	3.McL	Suz	2.Suz	3.Suz	6.Suz[8]
$d(U)$	2	2	2	4	2	3	2	4	6	6
$\dim(U)$	22	56	14	14	22	126	143	220	78	780

Fischer groups.

G	Fi_{22}	$2.Fi_{22}$	$3.Fi_{22}$	$6.Fi_{22}$	Fi_{23}	Fi'_{24}	$3.Fi'_{24}$
$d(U)$	2	2	3	6	2	2	3
$\dim(U)$	78	352	351	1728	782	8671	783

Monster sections except Fischer groups.

G	He	HN	Th	B	2.B	M
$d(U)$	3	2	2	2	2	2
$\dim(U)$	51	133	248	4371	96256	196883

Tits group and pariahs.

G	T	J_1	O'N	3.O'N	J_3	$3.J_3$	Ru	2.Ru	J_4	Ly
$d(U)$	6	2	4	6	3	6	4	4	4	6
$\dim(U)$	26	56	13376	342	85	18	378	28	1333	2480

[6]Actually, for stem extensions of simple groups all semi-invariants are invariants.
[8]Without 6-dimensional representations.
[8]Without 12-dimensional representations.

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