Lecture 7

1. K = Henselian, $\mathcal{O}, I, k$ as usual.

Let $K \subset L$ be finite extension. Then K is also henselian.

Def: L/K is unramified if

1) $k_L \supset k$ is separable

2) $[L:K] = [k_L : k]$.

Here $\mathcal{O}_L, I_L, k_L := \mathcal{O}_L / I_L$ are defined as usual.

Recall that we have a valuation $|\ |: K \to \mathbb{R}_{\geq 0}$

And it is uniquely extended to $|\ |_L : L \to \mathbb{R}_{\geq 0}$

For exponential valuation v of K, we denote its extension to L by w.

2. Any algebraic extension of K is said to be unramified if it is a union of unramified finite extensions.

3. Ex: $\mathbb{Q}_p \subset \mathbb{Q}_p(\sqrt{5})$ $p = 7$ (midterm).

(4) Th: $K \subset L$, $K \subset K'$ = finite extensions. Put $L' = LK$. Suppose L/K is unramified. Then L'/K' is unramified.

Proof: $\underbrace{O', I', k'}_{K'}$, $\underbrace{O_L, I_L, k_L}_{L}$, $\underbrace{O_{L'}, I_{L'}, k_{L'}}_{L'}$

We have $k_L \supset k$ is separable. Then $k_L = k(\bar{\alpha})$. $\bar{\alpha} \in k_L$

Let $\alpha \in O_L$ s.t. $\alpha \to \bar{\alpha}$ in k_L.

$f(x) \in O$ its minimal polynomial.

Put $\bar{f}(x) \in k[x]$ (its image mod I).

$$[k_L : k] \leq \deg \bar{f} \leq \deg f = [K(\alpha) : K] \leq [L : K] = [k_L : k]$$

$\Rightarrow L = K(\alpha)$, $\bar{f}$ is the min poly of $\bar{\alpha}/k$.

$\Rightarrow L' = K'(\alpha)$.

Take $g(x) \in O'[x]$ the minimal poly of α/K'.

Put $\bar{g}(x) \in k'[x]$ its image mod I'. Then $\bar{g} \mid \bar{f}$.

$\Rightarrow \bar{g}(x)$ = irreducible by Hensel lemma.

$$[k_{L'} : k'] \leq [L' : K'] = \deg g = \deg \bar{g} = [k'(\bar{\alpha}) : k'] \leq [k_{L'} : k']$$

Corollary 6.5 $\deg g = \deg \bar{g} \Leftarrow$ ($\bar{g} \neq$ constant) □

(5) Corollary: $K \subset L'$ is unramified, finite.

Consider $K \subset L'' \subset L'$. Then $K \subset L''$ is unramified.

Proof: L'/L'' is unramified by Th.

$$\begin{array}{ccccc} K & \subset & L'' & \subset & L' \\ \vdots & & \vdots & & \vdots \\ k & \subset & k_{L''} & \subset & k_{L'} \end{array}$$

$$[L':K] = [K:L''] \cdot [L':L'']$$

$$[k_{L'}:k] = [k_{L''}:k] \cdot [k_{L'}:k_{L''}]$$

multiply. divide. $\square$

(6) Corollary: $K \subset L$, $K \subset L'$ unramified. finite.

Then $K \subset LL'$ is unramified.

Proof: $LL' \supset L$ is unramified by Th.

Now as in (5).

(7) $K \subset L$ finite extension.

Def: Let $T \subset L$ be composite of all unramified extensions. Then T is maximal unramified subextension of L/K.

$(K \subset T \subset L)$ Lecture 7. $\overline{IV}$

(8) Th: Let $\mathcal{O}_T, I_T, k_T = \mathcal{O}_T / I_T$ as usual.

Then k_T is separable closure of k in k_L, and $w(T^*) = v(K^*)$.

Proof: Take $\bar{\alpha} \in k_L$ that is separable$/k$.

Let $\bar{f}(x) \in k[x]$ be its min poly (monic).

Take monic $f(x) \in \mathcal{O}[x]$ that maps to $\bar{f}$.

Then $f(x)$ is IRREDUCIBLE.

separability By Hensel lemma $f(x)$ has a root $\alpha \in L$

s.t. $\alpha \to \bar{\alpha} \bmod I_L$.

$$\Rightarrow \underbrace{[K(\alpha):K] = [k(\bar{\alpha}):k]}_{\text{UNRAMIFIED}} \Rightarrow \underset{\text{by Def.}}{\bar{\alpha} \in k_T}$$

Now $w(T^*) = v(K^*)$:

$$[T:K] \geq (w(T^*):v(K^*))\,[k_T:k] = (w(T^*):v(K^*)) \times [T:K]$$

$\square$

(9) $K \subset \bar{K}$ Then $K_{nR}|K$ denotes maximal unramified extensions of K in $\bar{K}$.

$\Rightarrow k_{K_{nR}}$ = separable closure of k.

It contains all $\sqrt[n]{1}$ $(n,p)=1$.

(10) L and K as before $K<L$ finite. V
K = henselian.
Suppose $p = \mathrm{char}\, k > 0$.

Def: $L|K$ is tamely ramified if

1) $k_L|k$ is separable

2) $[L:T]$ is prime to p.

Remark: for algebraic extensions similarly.
(for all finite subextensions).

(11) Put $f = [k_L : k]$, $e = [w(L^*) : v(K^*)]$.
T = max unramified subextension.

Theorem: $L|K$ is tamely ramified $\Longleftrightarrow$

$$L = T(\sqrt[m_1]{a_1}, \dots \sqrt[m_r]{a_r}) \quad \forall (m_i, p) = 1.$$

Proof: We may assume $K = T$ (so $k_L = k$)

$\Rightarrow$

Suppose $L|K$ is tamely ramified: $k = k_L$
$p \nmid [L:K]$

Claim: $e = 1 \Rightarrow L = K$.

Proof: Take $\alpha \in L \setminus K$. $\alpha = \alpha_1 \dots \alpha_m$ = conjugates

Put $a = \mathrm{Tr}(\alpha) = \sum \alpha_i$. $\Rightarrow \mathrm{Tr}(\alpha - \frac{1}{m} a) = 0$.

$e = 1 \Rightarrow v(K^*) = w(L^*) \Rightarrow \exists b \in K^*$ s.t. $v(b) = w(\alpha - \frac{1}{m} a) = 0$

Put $\varepsilon = \dfrac{\alpha - \frac{1}{m} a}{b} \in$ unit! $\varepsilon = \varepsilon_1, \dots \varepsilon_n$ conjugates.

But $\bar{\varepsilon}_1 = \dots = \bar{\varepsilon}_n$ ($k_L = k$). $\sum \bar{\varepsilon}_i = 0 \Rightarrow m \cdot \bar{\varepsilon} = 0 \Rightarrow m \equiv 0\ (p)$

(12) Continue. Put $n = [L:K]$. VI.

Now $\omega_1 \dots \omega_r \in \omega(L^*)$ generators $/ v(K^*)$.

Let $m_1 \dots m_r$ their orders mod $v(K^*)$.

We have $\omega(L^*) = \frac{1}{n} v(N_{L/K}(L^*)) \subseteq \frac{1}{n} v(K^*)$

Then $m_i \mid n$. $\Rightarrow p \nmid m_i$.

Take $\gamma_i \in L^*$ s.t. $\omega(\gamma_i) = \omega_i$

Then $\omega(\gamma_i^{m_i}) = v(c_i)$ for some $c_i \in K$.

Then $\gamma_i^{m_i} = c_i \varepsilon_i$ for some unit $\varepsilon_i \in L$.

As $k_L = k$, we have $\varepsilon_i = b_i u_i$, $b_i \in K$

$\Downarrow$ Hensel lemma: $\begin{cases} u_i \in L \text{ unit} \\ u_i \to 1 \text{ in } k_L = k. \end{cases}$

[$x^{m_i} - u_i = 0$ has solution in L.

[Call it β_i. Put $\alpha_i = \gamma_i \beta_i^{-1} \in L$. Put $\underbrace{a_i = c_i b_i}_{\text{in } K.}$

Then $\omega(\alpha_i) = \omega_i$ and $\alpha_i^{m_i} = a_i$.

Note $a_i \in K$.

So $K(\sqrt[m_1]{a_1}, \dots \sqrt[m_r]{a_r}) \subseteq L$.

These fields has the same value group & residue field.

$\Rightarrow L = K(\sim)$ by Claim. $\square$ ($\Leftarrow$ in the book

(13) Corollary: $L|K$ is tamely ramified.
$$\Downarrow$$
$$[L:K] = e \cdot f.$$

Proof: We may assume $K = T$. $([T:K] = [\lambda:k] = f)$

We have $L = K(\sqrt[m]{a})$ / by induction.

$$e = (w(L^*) : v(K^*)) \geq m \geq [L:K] \geq ef.$$

(14) Corollary: $L \supseteq K$, $K \subset K'$ 2 extensions $(\subseteq \bar{K})$
Then LK' is tamely ramified over K'
provided L is tamely ramified over K.

(15) Corollary: Every subextension of tamely ramified extention is tamely ramifi.

(17) Corollary: Composite of tamely ramified extensions is tamely ramified.

(16) $L \supset K$. Then the composite of all
Def
tamely ramified subextensions V is called
maximally tamely ramified subextension of $T \subset L$

(18) $\underbrace{K \subset T \subset V \subset L}_{\text{as above}}$

Th: $\omega(V^*) = \omega(L^*)^{(p)}$,
where $\omega(L^*)^{(p)}$ denotes the subgroup in $\omega(L^*)$ s.t. $a \in \omega(L^*)^{(p)}$ $ma \in \omega(K^*)$ $p \nmid m$.

(order of a in $\omega(L^*)/\omega(K^*)$ is coprime to p.

And $k_V = k_T$

(19) Def: $T = K \Rightarrow$ totally ramified $L \supset K$
$V \neq L \Rightarrow$ wildly ramified.

Ex: $\mathbb{Q}_p(\zeta)$ ζ = primitive n-th root of unity.

Then 1) $L = \mathbb{Q}_p(\zeta) \supset \mathbb{Q}_p$ is unramified $n \neq p$. $p \nmid n$. K *.
2) $L = \mathbb{Q}_p(\zeta) \supset \mathbb{Q}_p$ is totally ramified if $n = p^r$
3) $p \nmid n \Rightarrow f \in \mathbb{N}$ smallest $q^f \equiv 1 \mod n$.
$\mathcal{O}_L = \mathcal{O}[\zeta]$.
4) $n = p^r \Rightarrow \mathbb{Z}_p[\zeta] = \mathcal{O}_L$ $\deg = \varphi(p^m)$
$1 - \zeta$ = prime with norm p. $\varphi(p^r) = (p-1)p^{m-1}$