Lecture 3

1) Recall $\mathbb{Q} \subseteq \mathbb{Q}_p$ $\quad |x|_p = p^{-v_p(x)}$

$$v_p : \mathbb{Q} \longrightarrow \mathbb{Z} \sqcup \{\infty\} \quad \text{s.t.}$$

i) $v_p(a) = \infty \iff a = 0$

ii) $v_p(ab) = v_p(a) + v_p(b)$ $\quad$ (exp valuation)

iii) $v_p(a+b) \geq \min\{v_p(a), v_p(b)\}$

AND $|x|_p$ satisfies: $\quad$ (valuation)

i) $|a|_p = 0 \iff a = 0$

ii) $|ab|_p = |a|_p$

iii) $|a+b|_p \leq \max\{|a|_p, |b|_p\} \leq |a_p| + |b_p|$

2) K = any field.

Def: a valuation of K is a function

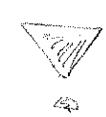
$$|\ | : K \longrightarrow \mathbb{R} \quad \text{s.t.}$$

i) $|a| \geq 0$, $\quad |x| = 0 \iff x = 0$

ii) $|xy| = |x||y|$

iii) $|x+y| \leq |x| + |y|$ $\quad \triangle$

$\downarrow$

No TRIVIAL VALUATION ($|x| = 1\ \forall x \neq 0$, $|0| = 0$) 

Put $d(x,y) = |x-y| \Rightarrow K$ = metric space

$\Downarrow$

K = topological space.

Lecture 3

3) Def: 2 valuations of K are equivalent
$$\Updownarrow$$
they define same topology.

Th: Let $|\cdot|_1$ and $|\ |_2$ be 2 valuations of K.
Then $|\ |_1 \sim |\ |_2 \iff \exists s>0$ s.t. $|x|_1 = |x|_2^s \ \forall x \in K$

Proof: $\Leftarrow$ obvious. $\Rightarrow$

Rem: $|x|<1 \Rightarrow \{x^n\}_{n\in\mathbb{N}}$ converges to 0.

$*$ $\quad |x|_1 < 1 \Rightarrow |x|_2 < 1$.

Pick $y \in K$ s.t. $|y|_1 > 1$. Take $x \in K^*$. Then $|x|_1 = |y|_1^{\alpha}$ for some $\alpha \in \mathbb{R}$

$\frac{m_i}{n_i} \to \alpha^+$ $\quad |x|_1 = |y|_1^{\alpha} < |y_1|^{m_i/n_i}$ $\quad$ $|x|_2 = |y|_2^{\alpha}$ $\Leftarrow$ $|x|_2 \le |y_2|^{\alpha}$

$n_i > 0$
$$\Downarrow$$

(Then $\to \alpha^-$) $\quad \left|\frac{x^{n_i}}{y^{m_i}}\right| < 1 \Rightarrow \left|\frac{x^{n_i}}{y^{m_i}}\right|_2 < 1.$ $\quad \Rightarrow |x|_2 \le |y|_2^{m_i/n_i}$

$$\begin{cases} \log|x|_1 = \alpha \log|y|_1 \\ \log|x|_2 = \alpha \log|y|_2 \end{cases} \Rightarrow \frac{\log|x|_1}{\log|x|_2} = \frac{\log|y|_1}{\log|y|_2}$$
$$\text{co} \qquad \overset{\shortparallel}{s}$$

CALL IT s. Then $|x|_1 = |x|_2^s \ \forall x, y$.
$s>0$

Lecture 3

III

4) Corollary: $\underbrace{|x|_1 < 1 \Rightarrow |x|_2 < 1 \quad \forall x}$
(from the proof)

$$\Downarrow$$

$$|x|_1 \sim |x|_2.$$

5) "Chinese Remainder Theorem"
or "Approximation Theorem"

$\underline{\underline{K}}$

Th $|x|_1, \ldots, |\ |_n$ be pairwise non-eq. valuations.
Let $a_1 \ldots a_n \in K$ some elements.
Then $\forall \varepsilon > 0 \quad \exists x \in K$ s.t. $|x - a_i|_i < \varepsilon \quad \forall i \in \{1..n\}$

$|y|_1 > 1 \quad |y|_n < 1.$

$\Uparrow$

Proof.

Step 1: $\exists \alpha \in K$ s.t. $|\alpha|_1 < 1$ & $|\alpha|_n \geq 1$. Put $y = \beta/\alpha$.
$\exists \beta \in K$ s.t. $|\beta|_n < 1$ & $|\beta| \geq 1$.

Step 2: $\exists z \in K$ s.t. $|z| > 1$, $|z|_2 < 1, \ldots, |z|_n < 1$.
Proof: induction $\Rightarrow$ if $|z|_n \leq 1$, put $z' = z^m y \quad m \gg 0$
if $|z|_n > 1$, put $t_m = z^m/_{1+z^m}$

$\searrow 1 \quad |\ |_1 \quad 1 \quad |\ |_n \qquad \searrow 0 \quad |\ |_2 \ldots |\ |_{n-1}$

$\Rightarrow$ put $z' = t_m y$
$m \gg 0$

Step 3: $\dfrac{z^m}{1+z^m} \longrightarrow \begin{cases} 1 & |\ |_1 \\ 0 & |\ |_2 \ldots |\ |_n \end{cases}$

$\exists z_1$ very close to $1/|\ |_1 \quad 0\ |\ |_2 \ldots |\ |_n$
Similarly $z_2, z_3 \ldots z_n$. (small & close to 1)

Put $x = a_1 z_1 + \ldots + a_n z_n$

Lectures IV

6) Def: $|\ |$ is Archimedean if $|n|$ $n\in\mathbb{N}$ is not bounded

otherwise nonarchimedean.

Th: $|\ |$ is not archimedean $\Leftrightarrow$ $|x+y|\le \max\{|x|,|y|\}$

Proof: $\Leftarrow$ obvious. ($\Rightarrow$)

Suppose $\exists N>0$ s.t. $|n|\le N$ $\forall n\in\mathbb{N}$.

Take $x,y\in K$ s.t. $|x|\ge|y|$.

Then $|x|^a|y|^{n-a}\le|x|^n$ $\forall a\in\{0,..n\}$.

Then $|x+y|^n\le N(n+1)|x|^n$.

Then $|x+y|\le N^{1/n}(n+1)^{1/n}|x|$

$\lim_{n\to\infty}$ $\Rightarrow$ $|x+y|\le|x|=\max\{|x|,|y|\}$.

7) Remark: (nonarchimedean) $|x|\ne|y|\Rightarrow|x+y|=\max\{|x|,|y|\}$

(check)

8) non-archimedean.

$K(t)$. Put $|f(t)|=\max\{|a_i|\}$ $f=\sum^n a_it_i$ (non-Archim valuation)

$|f||g|=|fg|$ Gauss lemma. $|f|$

Lecture 3

9#)

Th: Let $|\ |$ be valuation of $\mathbb{Q}$.
Then either $|\ | \sim |\ |_p$ or $|\ | \sim |\ |_\infty$.

Proof:

Non-Archimedean case.

$\forall n \in \mathbb{N}$ $|n| \leq 1$. But $\exists p$ s.t. $|p| < 1$

(we assume $|\ | \neq$ trivial).
+ FTA

Put $I = \{a \in \mathbb{Z}$ s.t. $|a| < 1\}$.

Then I = ideal & $p \in I$. $\Rightarrow$ $I = p\mathbb{Z}$

$\forall a \in \mathbb{Z}$ $\quad a = bp^m$, $p \nmid b$ $\Rightarrow$ $|b| = 1$

$\Downarrow$

$|a| = |p|^m = |a|_p^s$

Put $s = -\dfrac{\log|\ |}{\log p}$

For $\mathbb{Q}$ we recover $|\ | = |\ |_p^s$
by multiplicativity.

Lecture 3

VI

10)

Archimedean case:

Claim: $\forall m, n \in \mathbb{Z}_{>0}$ $\quad |m|^{\frac{1}{\log m}} = |n|^{\frac{1}{\log n}}$.

Proof: $m = a_0 + a_1 n + \dots + a_r n^r \qquad a_i \in \{0, \dots, n-1\}$

$$\underline{\underline{n^r \le m}}$$

Then $r \le \dfrac{\log m}{\log n}$ & $|a_i| \le a_i |1| = a_i \le n$.

Then $|m| \le \sum |a_i| |n|^i \le \sum |a_i| |n|^r \le n(r+1) |n|^r$

$$\le n\left(1 + \frac{\log m}{\log n}\right) n^{\frac{\log m}{\log n}} \qquad \ge \left(1 + \frac{\log m}{\log n}\right) n |n|^r$$

$$|m|^k \le n\left(1 + k\frac{\log m}{\log n}\right) |n|^{k\frac{\log m}{\log m}}$$

$\sqrt[k]{} \Rightarrow \quad |m| \le \sqrt[k]{n}\left(1 + k\dfrac{\log m}{\log n}\right)^{\frac{1}{k}} |n|^{\frac{\log m}{\log m}}$

$\lim_{k \to +\infty} \qquad |m| \le |n|^{\frac{\log m}{\log n}}$

$\square$

Put $c = |n|^{1/\log n}$. Then $|n| = c^{\log n} \quad \forall n$

Put $s = \log c$. Then $|n| = |n|_\infty^s$

+ multiplicativity $\Rightarrow \mathbb{Q}$.

Lecture 3.

From | | to v.

11)
Let | | be nonarchimedean valuation

Put $v(x) = -\log|x| \quad \forall x \neq 0$

Put $v(0) = +\infty$.

$v : K \to \mathbb{R} \sqcup \{\infty\}$ s.t.
i) $v(x) = \infty \iff x = 0$
ii) $v(xy) = v(x) + v(y)$
iii) $v(x+y) \geq \min(v(x), v(y))$.

exponential valuation

12) Def: $v_1 \sim v_2 \iff v_1 = s v_2 \quad s > 0$.

$v \rightsquigarrow |\ |$ by $|x| = q^{-v(x)}$. (multiplicative valuation)

13) Put
$\mathcal{O} = \{x \in K \text{ s.t. } |x| \leq 1\}$ $\quad v \geq 0$
$\mathcal{O}^* = \{x \in K \text{ s.t. } |x| = 1\}$ $\quad v = 0$
$I = \{x \in K \text{ s.t. } |x| < 1\}$ $\quad v > 0$

Then $\mathcal{O}$ = Ring, $\mathcal{O}^*$ = its group of units.
I = unique maximal ideal.
$K = \mathrm{Frac}(\mathcal{O})$. $\boxed{\mathcal{O} = \text{integrally closed}}$

Def: $\mathcal{O}/I$ = Residue class field.

Lecture 3

XII

14) DVR

Def: v is called <u>discrete</u> if it admits a smallest positive value s.

Then $v(K^*) = s\mathbb{Z}$

$\boxed{\text{Normalized if } s = 1.}$

Then $\mathcal{O}$ is called DVR.

Th: Suppose v is discrete.

Then 1) I = principal ideal. (DVR)

2) All non-zero ideals are I^n

3) $I^n / I^{n+1} \simeq \mathcal{O}/I$

15) $I^n = \{ x \in K \text{ s.t. } |x| < \frac{1}{q^{n-1}} \}$ $\quad q^{-v} = |\,|$

$U^{(n)} = 1 + I^n$ subgroups (n-th higher unit group.)

16) **DVR**

Why DVR is good?

Ex: $\mathbb{C} \rightsquigarrow \mathbb{C}[t] \subset \mathbb{C}(t)$

$$\begin{array}{l} \mathbb{C} \\ \cap \\ \mathbb{P}^1 \rightsquigarrow \mathbb{C}[t] \nearrow \mathbb{C}(t) \\ \qquad\quad \mathbb{C}[t^{-1}] \nearrow \end{array}$$

$$f(t) = \frac{g(t)}{h(t)}$$

$$\downarrow$$

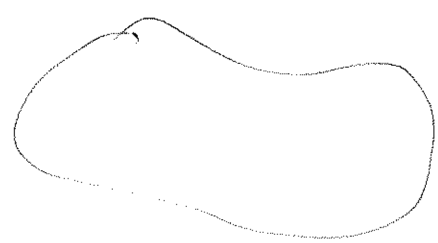
$\sum m_i P_i$ (divisor)

m_i = order of —

$\sum m_i = 0$

Ex: $\mathbb{C}^2$ $f(x,y) = 0$ irreducible + (maybe more)

$R = \mathbb{C}[x,y]/I = \langle f \rangle$



$\mathrm{Frac}(R) \rightsquigarrow h \in \mathrm{Frac}(R)$

$\rightsquigarrow$ poles + zeros

$\sum m_i P_i$

$\sum P_i = 0$

$\mathbb{C}^2 \subset \mathbb{P}^2$

$f(x,y,z) = 0 \rightsquigarrow$ same

Q: How?

A: **DVR**

17) $\mathbb{C}^2 \rightsquigarrow \mathbb{C}[x,y] \rightsquigarrow \mathbb{C}(x,y)$

$$f(x,y) = \frac{h(x,y)}{g(x,y)}$$

$$\begin{cases} h = h_1^{a_1} \cdots h_n^{a_n} \\ g = g_1^{b_1} \cdots g_n^{b_n} \end{cases}$$

$C_i \quad h_i = 0$

$Z_i \quad g_i = 0$

$$\underbrace{\sum a_i C_i - \sum b_i Z_i}$$ DIVISOR of $f(x,y)$

18) $\mathbb{C}^3_{xyz} \supset x = zy \quad S \rightsquigarrow \mathbb{C}[xyz]/\langle x - zy \rangle \rightsquigarrow \text{FRAC}$

$p \downarrow$, $S \xrightarrow{p} \mathbb{C}^2_{xy}$

$\mathbb{C}^2_{xy} \rightsquigarrow \mathbb{C}[x,y] \rightsquigarrow \text{FRAC}$

$\subseteq$, $=$

$z = x/y$

$E = S \cap \left(\begin{matrix} x = 0 \\ y = 0 \end{matrix}\right) \quad p(E) = \text{point.}$

$$\boxed{v_E(f)}$$