I

(1) Geometry (just example)

$C_d \subset \mathbb{P}^2/\mathbb{C}$ $\quad f_d(x;y;z)=0$

★ Assume smothness: $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$ has only $[0;0;0]$ solution.

Then $\dim_{\mathbb{R}} C_d = 2$ and C_d is compact.

And C_d is oriented.

$\Downarrow$

C_d is homeomorphic to a sphere with g handles.

Th. $g = \frac{(d-1)(d-2)}{2}$.

$\sum a_i = \deg(\text{divisor})$

Def: $\operatorname{Div} C_d$ = free abelian group $\underbrace{\sum_{i=1}^{r} a_i P_i}_{\text{DIVISORS}}$,
where P_i are points.

$\forall f \in K(C_d)$ $\quad \operatorname{div}(f) = (f)_0 - (f)_\infty$ principal divisors.

$K(C_d)$: field of rational functions

$\operatorname{Pic} = \operatorname{Cl}$

$\operatorname{Pic}(C_d) := \operatorname{Div} C_d / \langle \text{principal divisors} \rangle$

$$0 \to \operatorname{Pic}^0(C_d) \to \operatorname{Pic}(C_d) \xrightarrow{\deg} \mathbb{Z} \to 0$$

Th. $\operatorname{Pic}^0(C_d) = \mathbb{C}^g/\Lambda$, Λ = lattice of rank $2g$.

② K = number field. $\mathcal{O}_K \subset K$ = Ring of integers.

Section I.4.&6 II

Th $\mathcal{O}_K$ = noetherian, integrally closed, (non-trivial) and every prime ideal $I \subset \mathcal{O}_K$ is maximal

Th: $\forall$ ideal $\hat{I} \subset \mathcal{O}_K$, one has $\boxed{\hat{I} = I_1 \ldots I_k}$, where I_i are prime ideals.

unique.

③

Def: A fractional ideal is f.g. $\mathcal{O}$-submodule $\subseteq K$. ($\subseteq K$)

Def: $\mathcal{J}_K$ = abelian group of fractional ideals.

Corollary: $\forall$ fractional ideal $= \prod_{i=1}^{k} I_i^{k_i}$ $k_i \in \mathbb{Z}$ $\forall I_i$ = prime.

Here: $I_i^{-1} = \{x \in K \text{ s.t. } x I_i \subseteq \mathcal{O}_K\}$

Def: $Cl_K = \mathcal{J}_K / P_K$ P_K = ideals (fractional) of the form $(a) = a \cdot \mathcal{O}_K$ $a \in K^*$

$$1 \to \mathcal{O}_K^* \to K^* \to \mathcal{J}_K \to Cl_K \to 1$$

Th: Cl_K is a finite group.

Ex: $Cl_{\mathbb{Q}} = 1$.

$|Cl_K|$ is called class number.

point at ∞

Additive vs multiplicative notation.

(4) $K \supset \mathcal{O}_K$ as before

Def: A prime (or place) $\mathfrak{P}$ of K is a class of equivalent valuations of K.

Archimedean primes — INFINITE primes $\mathfrak{P} \mid \infty$

- Real: $K_{\mathfrak{P}} = \mathbb{R}$
- complex: $K_{\mathfrak{P}} = \mathbb{C}$

$\tau : K \to \mathbb{C}$
(all different embeddings)

non-Archimedean primes — FINITE PRIMES $\mathfrak{P} \nmid \infty$

(5) If $\mathfrak{P}$ is a finite prime, then $\mathfrak{P} \mid p$ (prime)

$$\begin{cases} p = \mathrm{char}(k) \\ k = \text{residue field} \end{cases}$$

Remark: If $\mathfrak{P}$ is not finite prime, $K_{\mathfrak{P}}$ can be considered as residue field. $k = K_{\mathfrak{P}}$ of $\mathfrak{P} \mid \infty$.

(6) $\mathfrak{P}$ = prime of K. $v_{\mathfrak{P}} : K^* \longrightarrow \mathbb{R}$

$$\begin{cases} \text{e.g. } \mathfrak{P} = \frac{1}{p} v_p(\ ) \\ \mathfrak{P} \nmid \infty \\ v_p(K^*) = \mathbb{Z} \end{cases}$$

$$v_{\mathfrak{P}}(a) = -\log|\tau a| \quad \begin{cases} \tau : K \to \mathbb{C} \\ \mathfrak{P} \mid \infty \end{cases}$$

(7) Put $\begin{cases} I \nmid \infty & f_I = [k(I) : \mathbb{F}_p] \\ I \mid \infty & f_I = \begin{cases} 2 & K_I = \mathbb{C} \\ 1 & K_I = \mathbb{R} \end{cases} \end{cases}$

Put $N(I) = \begin{cases} p^{f_I} & I \nmid \infty \\ e^{f_I} & I \mid \infty \end{cases}$

Remark: "$\mathbb{C} | \mathbb{R}$ is not ramified", inertia deg $= 2$.

Put $|a|_I = (N(I))^{-v_I(a)}$ for $a \neq 0$

$|0|_I = 0$.

Corollary: If $I \mid \infty$ $\begin{cases} |a|_I = |\tau a| & \tau : K \to \mathbb{R} \\ |a|_I = |\tau a|^2 & \tau : K \to \mathbb{C}. \end{cases}$

(8) Let $\begin{array}{ccc} K & \subset & L \\ \cup & & \cup \\ \mathcal{O}_K & \subset & \mathcal{O}_L \end{array}$ be finite extension

I_L = extension of I (some!!!)

For infinite prime I, we put

$f_{I_L/I} = [L_{I_L} : K_I]$ & $e_{I_L/I} = 1$.

e = "ramification order"
f = "inertia degree"

$\underline{V}$

(9) Theorem (Proposition 1.2)

1) $\sum_{I_L|I} e_{I_L|I} \cdot f_{I_L|I} = \sum_{I_L|I} [L_{I_L} : K_I] = [L:K]$

2) $N(I_L) = \left(N(I)\right)^{f_{I_L|I}}$

3) $v_{I_L}(a) = e_{I_L|I} \cdot v_I(a) \quad \forall a \in K^*$ (normalized)

4) $v_I(N_{L_{I_L}|K_I}(a)) = f_{I_L|I} \cdot v_{I_L}(a)$
$\forall a \in L^*$

5) $|a|_{I_L} = |N_{L_{I_L}|K_I}(a)|_I \quad \forall a \in L.$

(10) Theorem (Proposition 1.3)

$\forall a \in K^*$, one has $|a|_I = 1$ for "almost all" I.

Moreover, we have $\prod_I |a|_I = 1$.

Proof: $N_{K|\mathbb{Q}}(a) = \prod_{I|p} N_{K_I|\mathbb{Q}_p}(a) \quad \forall p =$ prime (include $p = \infty$)

$$\prod_I |a|_I = \prod_p \prod_{I|p} |a|_I = \prod_p \prod_{I|p} |N_{K_I|\mathbb{Q}_p}(a)|_p$$

$$= \prod_p |N_{K|\mathbb{Q}}(a)|_p = 1.$$

(11) Recall $\mathrm{Pic}(\mathcal{O}_K) = Cl(K) = \mathcal{I}_K / P_K$

Def: A replete ideal (Arakelov divisor) is an element of the group:
ADDITIVE notn

$$\mathcal{I}(\bar{\mathcal{O}}_K) = \mathcal{I}_K \times \prod_{I|\infty} \mathbb{R}^*_{>0}$$

Put: $I^x = e^x \in \mathbb{R}_{>0}$ $\forall x \in \mathbb{R}$ of $\underline{\underline{I|\infty}}$.

So $\forall a \in K^*$ we have (Arakelov principal divisor / replete principal ideal)
ADDitive notatn

$$[a] = \underbrace{(a)}_{\in \mathcal{I}_K} \times \prod_{I|\infty} I^{v_I(a)}$$

Def: $\mathrm{Pic}(\bar{\mathcal{O}}) = \mathcal{I}(\bar{\mathcal{O}}) / \langle \text{principal replete divisors} \rangle$

Def: For a replete ideal $A \in \mathcal{I}(\bar{\mathcal{O}})$, $A = \prod_I I^{v_I}$

$$N(A) = \prod_I N(I)^{v_I}$$

called absolute norm.

(12) The absolute norm is multiplicative.

$$N : \mathcal{I}(\bar{O}) \to \mathbb{R}^*_{>0}$$

AND gives

$$N : \mathrm{Pic}(\bar{O}) \to \mathbb{R}^*_{>0}$$

because $N([a]) = 1 \ \forall \ a \in K^*$.

(13) Def: Arakelov divisor is an additive notation for repleted ideals:

$$\sum_{\mathfrak{I}} \nu_{\mathfrak{I}} \mathfrak{I} \qquad (\text{formal } \underline{\text{finite}} \text{ sums})$$

$\nu_{\mathfrak{I}} \in \mathbb{Z}$ for $\mathfrak{I} \nmid \infty$, $\nu_{\mathfrak{I}} \in \mathbb{R}$ for $\mathfrak{I} \mid \infty$.

They form a group denoted by $\mathrm{Div}(\bar{O})$

(14) $$\mathrm{Div}(\bar{O}) \cong \mathrm{Div}(O) \times \bigoplus_{\mathfrak{I} \mid \infty} \mathbb{R}\,\mathfrak{I}.$$

$\forall \, a \in K^*$ we have $[a] \in \mathrm{Div}(\bar{O})$

$$\mathrm{div}(a) = \sum_{\mathfrak{I}} \nu_{\mathfrak{I}}(f) \mathfrak{I}$$

(15) Th: Kernel of $K^* \to \mathrm{Div}(\bar{O})$

is the group of roots of unity in K.

Image of $K^* \to \mathrm{Div}(\bar{O})$ is a discrete subgroup.

Def: $CH^1(\bar{O}) = \mathrm{Div}(\bar{O}) / \langle$ principal repleted divisors $\rangle$

$$\underset{\text{additive}}{CH^1(\bar{O})} \cong \underset{\text{multiplicative}}{\mathrm{Pic}(\bar{O})}$$

(16) $\deg : \mathrm{Div}(\bar{O}) \to \mathbb{R}$

$$\sum \nu_I I \to \sum \nu_I \log N(I)$$

Remark: $\deg(\mathrm{div}(a)) = \log(\prod_I |a|_I^{-1}) = 0$

$\deg : CH^1(\bar{O}) \to \mathbb{R}$

Kernel: $CH^1(\bar{O})^0$ by defn

or

$\mathrm{Pic}^0(\bar{O})$.

Put $H = \{ (x_I) \in \prod_{I | \infty} \mathbb{R} \mid \sum_{I | \infty} x_I = 0 \}$

Γ = Image of O_K^* via

$$O_K^* \hookrightarrow K^* \xrightarrow{\text{div}} \mathrm{Div}(\bar{O}_K) \xrightarrow{\text{projection}} \prod_{I|\infty} \mathbb{R}$$

Lemma: $\Gamma \subset H$, Γ is a complete lattice in H.

Theorem: $\mathrm{Pic}^0(\bar{O}_K) = H/\Gamma$ (torus of dim $r+s-1$)

r = # real embedd

s = # complex/2

Proof:

P = principal divisors

~~$0 \to$ Image of O^* in $P(\bar{O})$~~ H

$$\begin{array}{ccccccccc}
0 \to & \Gamma & \to & P(\bar{O}_K) & \to & P(O_K) & \to 0 \\
 & \downarrow & & & & \downarrow & \\
0 \to & H & \to & \mathrm{Div}^0(\bar{O}) & \to & \mathrm{Div}(O) & \to 0
\end{array}$$

$\square$