BRING'S CURVE

W. L. EDGE

1.

Five planes, no four of them concurrent, in projective space of three dimensions, form a pentahedron P having ten edges and ten vertices; each vertex, common to three faces, is *opposite* to the edge common to the remaining two. Each edge contains three vertices and each vertex is common to three edges. The joins of any vertex to the three on the opposite edge are *diagonals* of P; there are 15 of these. Any face meets the other four in four *edges* forming the sides of a quadrilateral whose three diagonals are among those of P. The plane spanned by any vertex and opposite edge is a *diagonal plane*.

One can appropriate P as pentahedron of reference for five supernumerary homogeneous co-ordinates summing identically to zero:

$$\sum x \equiv x+y+z+t+u \equiv 0 \tag{1.1}$$

at every point of the space. Each vertex of P has its co-ordinates a permutation of $(-1, 1, 0, 0, 0)$; the diagonals have equations such as

$$x = y+z = t+u = 0,$$

these three simultaneous equations being linearly dependent on only two of them because of (1.1). Hence all 15 diagonals lie on Clebsch's *diagonal surface*

$$D : \sum x^3 \equiv x^3+y^3+z^3+t^3+u^3 = 0.$$

The triad of vertices on an edge of P has a Hessian duad: the pair of points each of which completes, with the triad, an equianharmonic tetrad. The vertices V_{12}, V_{23}, V_{13} on $t = u = 0$ are $(1, -1, 0, 0, 0)$, $(0, 1, -1, 0, 0)$, $(-1, 0, 1, 0, 0)$ and are permuted cyclically with x, y, z; this permutation is allowable because it does not violate (1.1). The duad is the pair of points that are fixed under this permutation, namely

$$(1, \omega, \omega^2, 0, 0) \quad \text{and} \quad (1, \omega^2, \omega, 0, 0)$$

with ω a complex cube root of 1; these 20 points, two on each edge of P, clearly all lie on the quadric

$$Q : \sum x^2 \equiv x^2+y^2+z^2+t^2+u^2 = 0.$$

Bring's curve B is the intersection of Q and D; it is a canonical curve of genus 4 [5; p. 166] and is, with Q and D, invariant under the symmetric group $\mathscr{S}_5$ of 120 permutations of the five co-ordinates. As will be observed from what precedes, B can be defined geometrically without any mention of co-ordinates. There are certainly two places where it has been noticed in the literature and some allusion to these should now be made. Concerning Bring himself the footnote on p. 143 of [5] is informative.

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2.

Bring's curve B was encountered by Klein [5; p. 166]. The reason for its appearance lay in the endeavour to remove the terms in X^4, X^3, X^2 from the quintic equation

$$aX^5+bX^4+cX^3+dX^2+eX+f=0$$

and thereby reduce the problem of solving a general quintic equation to that of solving

$$X^5+eX+f=0. \tag{2.1}$$

If (2.1) has roots x, y, z, t, u then $S_1 = S_2 = S_3 = 0$ where

$$S_n \equiv \sum x^n = x^n+y^n+z^n+t^n+u^n.$$

This implies that the point (x, y, z, t, u) in our supernumerary co-ordinate system is on B. It will be convenient, whenever the contraction is unambiguous, to denote the point simply by its parenthesised first co-ordinate (x).

Before acknowledging the other earlier appearance of B it may be noted that the surface $S_n = 0$, should it not contain the whole of B, meets B in a set of points invariant under $\mathscr{S}_5$. The expressions for the S_n are found by using the recurrence relation consequent upon (2.1):

$$S_{n+5} = -eS_{n+1}-fS_n \qquad n = 0, 1, 2, \ldots$$

and they will be polynomials in e and f; moreover f must be a factor whenever n is odd if $S_n \not\equiv 0$. It is only necessary to know S_0, S_1, S_2, S_3, S_4 in order to find all subsequent S_n. But here

$$S_0 = 5, \ \ S_1 = S_2 = S_3 = 0$$

and elementary routine, or indeed the brief footnote on p. 165 of [5], gives $S_4 = -4e$. The consequences are

$$S_5 = -5f, \ \ S_6 = S_7 = 0, \ \ S_8 = 4e^2, \ldots \tag{2.2}$$

every non-zero S_n being monomial until

$$S_{20} = -4e^5+5f^4 = {S_4}^5/4^4+{S_5}^4/5^3.$$

The pencil of surfaces of order 20, $\lambda {S_4}^5+\mu {S_5}^4 = 0$, cuts a g^1_{20} on B every set of which is invariant under $\mathscr{S}_5$. Since the Jacobian set of a ${g_N}^1$ on a curve of genus p consists of $2N+2p-2$ points [8; p. 75] that of g^1_{20} consists of 246 points. But these, double points of sets of g^1_{20}, include 24 quintuple points on $S_4 = 0$ and 30 quadruple points on $S_5 = 0$; and while the exact attribution of multiplicities can be a subtle matter these present confluences are standard ones [8; p. 77]: each of the quadruple points contributes 3, each of the quintuple points 4, to the Jacobian set, so that there is a residue of

$$246-90-96 = 60.$$

This residue is, as a whole, invariant under $\mathscr{S}_5$ and suggests that a third surface of the pencil may touch B at each of the 60 points. This will be confirmed below.

3.

The other appearance of B is in Wiman's paper [11] where he studies curves of low genus that admit (1, 1) self-transformations; for canonical curves such transformations are projectivities.

The simplest permutation of $\mathscr{S}_5$ is a transposition of two of the co-ordinates, the other three being unpermuted. Now the difference of the two row-vectors

$$(x,\ y,\ z,\ t,\ u) \quad (y,\ x,\ z,\ t,\ u),$$

where x and y are not both zero, is the co-ordinate vector of V_{12} while their sum is that of a point in the plane $x = y$, say the plane π_{12}; hence, as Wiman states, the two points are harmonic inverses in V_{12} and π_{12}. When the points are on B their join generates, to quote a second statement of Wiman's, a cubic cone K_{12} with vertex V_{12}. Thus B is, in ten different ways, in (2, 1) correspondence with a plane curve of genus 1. Incidentally π_{12} is the harmonic conjugate of the diagonal plane $x+y = 0$ with respect to the two faces, $x = 0$ and $y = 0$, of P through the edge opposite to V_{12}. This harmonic inversion may be called h_{12}.

When B is its own harmonic inverse in a point V and a plane π the six tangents to B at its intersections with π all pass through V. Let J_0 be such an intersection, and J a point of B " near " J_0. Then VJ meets B also in J', the transform of J in the inversion, so that the osculating plane of B at J_0, which is the limiting position as $J \to J_0$ along B of the plane joining J to the tangent VJ_0, has 4-point intersection with B at J_0: this, too, is stated by Wiman [**11**; p. 20]. So the 60 intersections of B with the ten planes π_{ij} are *stalls* on B. They are the 60 Weierstrass points on this canonical curve, the number for a canonical curve of genus p being $p(p^2-1)$.

The recognition of these 60 stalls raises the suspicion that they may be the residual members of the Jacobian set of g^1_{120}, and as they form the complete intersection of B with ten planes it might be that some surface $\lambda S_4{}^5 + \mu S_5{}^4 = 0$ is the square of the product of these planes; and this we now prove. A standard procedure expressing the square of the difference-product of the roots of an equation as a polynomial in the S_n gives, when applied to (2.1), by (2.2),

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x & y & z & t & u \\ x^2 & y^2 & z^2 & t^2 & u^2 \\ x^3 & y^3 & z^3 & t^3 & u^3 \\ x^4 & y^4 & z^4 & t^4 & u^4 \end{vmatrix} \begin{vmatrix} 1 & x & x^2 & x^3 & x^4 \\ 1 & y & y^2 & y^3 & y^4 \\ 1 & z & z^2 & z^3 & z^4 \\ 1 & t & t^2 & t^3 & t^4 \\ 1 & u & u^2 & u^3 & u^4 \end{vmatrix} = \begin{vmatrix} 5 & . & . & . & S_4 \\ . & . & . & S_4 & S_5 \\ . & . & S_4 & S_5 & . \\ . & S_4 & S_5 & . & . \\ S_4 & S_5 & . & . & \tfrac{1}{4}S_4{}^2 \end{vmatrix}$$

$$= 5S_5{}^4 - \tfrac{1}{4}S_4{}^5.$$

Hence the surface $S_4{}^5 = 20S_5{}^4$ is the square of the product of the ten planes π_{ij} and the Jacobian set is completed by the anticipated contribution.

4.

The equation of K_{12} is

$$2\sum x^3 = 3(x+y)\sum x^2. \tag{4.1}$$

For, while this surface clearly contains B and V_{12}, the join of V_{12} to any other point (x) satisfying (4.1) consists of the points obtained by varying λ in $(x+\lambda,\ y-\lambda,\ z,\ t,\ u)$; and this point satisfies (4.1) if

$$2\{3\lambda^2(x+y)+3\lambda(x^2-y^2)\} = 3(x+y)\,\{2\lambda^2+2\lambda(x-y)\}$$

which is an identity in λ: the line lies wholly on the surface.

The geometry of K_{12} is indicated by that of its section by any plane not containing V_{12}, so one may use π_{12} wherein

$$x = y = -\tfrac{1}{2}(z+t+u).$$

After this substitution (4.1) becomes

$$(z+t+u)^3+2(z^3+t^3+u^3)+3(z+t+u)(z^2+t^2+u^2) = 0,$$

or

$$ztu+(z+t+u)(z^2+t^2+u^2) = 0.$$

This is equivalent to

$$5(z+t+u)^3 = (z+2t+2u)(2z+t+2u)(2z+2t+u) \tag{4.2}$$

so that the diagonals

$$z=t+u=0, \quad t=u+z=0, \quad u=z+t=0$$

of P through V_{12} are three of the inflectional generators of K_{12}. The tangent planes

$$z+2t+2u=0, \quad 2z+t+2u=0, \quad 2z+2t+u=0$$

or, alternatively,

$$t+u=x+y, \quad u+z=x+y, \quad z+t=x+y,$$

of K_{12} along these three generators must, by (4.2), each osculate B at both its intersections with the generator; these are the points where this generator meets Q. The pair on $z=t+u=0$ is $(1, -1, 0, \pm i, \mp i)$. These five unequal co-ordinates admit 120 permutations, but only 30 points occur because multiplication of each co-ordinate by any power of i does not produce a different point. So the 30 points, one pair on each diagonal, make up the complete intersection of B with the faces of P or, alternatively, with the surface $S_5 = 0$.

5.

The 24 intersections of B with the quartic surface $S_4 = 0$, since they satisfy

$$S_1 = S_2 = S_3 = S_4 = 0,$$

are the points whose co-ordinates are permutations of $(1, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4)$ with ε a complex fifth root of 1. These five unequal co-ordinates admit 120 permutations, but they provide five sets of co-ordinates for each of the 24 points because multiplication of all its co-ordinates by any of the five powers of ε does not change the position of the point. It will be convenient therefore, in discussing these 24 points, to insist that the the first co-ordinate x is always 1; the remaining four co-ordinates, when undergoing all 24 permutations, yield the 24 distinct points.

Take then, to open the discussion, $(1, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4)$ and associate it with the three other points that are obtained on replacing ε by $\varepsilon^2, \varepsilon^3, \varepsilon^4$. The set of four points is

$$a(1, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4) \qquad \bar{a}(1, \varepsilon^4, \varepsilon^3, \varepsilon^2, \varepsilon)$$
$$b(1, \varepsilon^2, \varepsilon^4, \varepsilon, \varepsilon^3) \qquad \bar{b}(1, \varepsilon^3, \varepsilon, \varepsilon^4, \varepsilon^2).$$

It is then apparent that each of

$$a, b; \qquad a, \bar{b}; \qquad \bar{a}, b; \qquad \bar{a}, \bar{b}$$

is a conjugate pair of points for $\sum x^2 = 0$; the four lines

$$ab, \qquad a\bar{b}, \qquad \bar{a}b, \qquad \overline{ab}$$

are generators of Q. The 24 points are vertices of six quadrilaterals whose sides, two in each regulus, are on Q (*cf.* [**11**; p. 21]). The diagonals of this quadrilateral of generators, $a\bar{a}$ and $b\bar{b}$, are polar lines for Q and are seen to lie on D. For example: $a\bar{a}$ meets $x = 0$ at $(0, \varepsilon-\varepsilon^4, \varepsilon^2-\varepsilon^3, \varepsilon^3-\varepsilon^2, \varepsilon^4-\varepsilon)$ which is on the diagonal $x = y+u = z+t = 0$; similarly each of its intersections with the faces of P is on D so that, having at least seven intersections with D, it lies on the surface.

Thus $a\bar{a}$ and $b\bar{b}$ are a pair of the double-six δ on D residual to the 15 diagonals of P, and each of the six quadrilaterals provides such a pair: Q is the Schur quadric for δ [7; p. 12 and footnote].

A further item of information can be added to Schur's footnote. He remarks that one among the 36 double-sixes on D is isolated: the addition is that the other 35 are partitioned as $15+20$ in the sense that no member of either set can be transformed into one of the other by an operation of $\mathscr{S}_5$. On D the isolated double-six δ, as already mentioned, includes no diagonal of P. But there are 15 double-sixes δ_1 which include 8 diagonals forming four opposite pairs of δ_1, and 20 double-sixes δ_2 that include only 6 diagonals, three in each regulus on a quadric. These are specialisations on D of the geometry on a general cubic surface and can perhaps best be perceived from the fact that the diagonals can be labelled consistently by Schläfli's 15 symbols c_{ij} and this indicates an alternative way of phrasing the geometrical definition of B: *take P, the diagonal surface D defined thereby, the residual double-six on D and its Schur quadric Q.*

The double-six δ, being invariant under $\mathscr{S}_5$, is a Burnside double-six ([**2**; p. 418] and [**1**; p. 168]).

6.

The tangent plane to $\sum x^3 = 0$ at a is

$$x+\varepsilon^2 y+\varepsilon^4 z+\varepsilon t+\varepsilon^3 u = 0,$$

and this plane is $a\bar{a}b$; hence ab, common to the tangent planes of Q and D at a, is the tangent of B at A. Similarly $b\bar{a}$ is the tangent at b, $\overline{ab}$ that at $\bar{a}$, and $\bar{b}a$ that at $\bar{b}$. That there are 12 lines in each regulus on Q tangent to B is in accord with the Jacobian set of the $g_3{}^1$ cut on B by either regulus consisting of $2.3+2.4-2 = 12$ points.

The osculating plane of B at a is the limiting position, as $a' \to a$ along B, of the plane joining the tangent ab to the point a' on B " near " a. This plane meets Q in ab and the generator of the opposite regulus through a'; its limiting position therefore joins ab to the generator $a\bar{b}$ of the opposite regulus through a: *the osculating plane of B at a is the tangent plane of Q at a. This same coincidence occurs at all* 24 *points.*

7.

The mention of an osculating plane is a reminder that Hesse [**4**; p. 283] obtained the equation of the osculating plane of any irreducible curve that is, as B is, the complete intersection of two surfaces. His arguments were shortened and clarified by Clebsch [**3**; p. 2] and the equation Hesse obtained is reproduced in Salmon's treatise [**6**; p. 328]. When Hesse's equation is worked out the osculating plane of B

at (x) is found to be

$$S_4 \sum x^2 X = -2S_5 \sum xX \tag{7.1}$$

in current co-ordinates (X). It follows that the osculating plane is the tangent plane of Q at those points of B for which $S_4 = 0$, and is the tangent plane of D at those points of B for which $S_5 = 0$. Both these facts have been anticipated: the former at the end of §6, the latter in §4 when one observes that the tangent plane of D at $(1, -1, 0, i, -i)$ is $x+y = t+u$. Neither of these facts affords any check on the constant -2 in (7.1), but such a check could be provided by using the osculating plane at a stall.

Incidentally, at the six stalls in π_{12}

$$\sum x^2 = \sum x^3 = 0, \qquad x = y = -\tfrac{1}{2}(z+t+u)$$

so that, if $x = y = 1$,

$$z+t+u = z^2+t^2+u^2 = z^3+t^3+u^3 = -2$$

and the stalls occur on permuting the unequal numbers α, β, γ in $(1, 1, \alpha, \beta, \gamma)$ with α, β, γ the roots of $\theta^3+2\theta^2+3\theta+4 = 0$. They form two triangles in triple perspective from V_{34}, V_{35}, V_{45}.

8.

The tangents of a curve of order n and genus p, without cusps, generate a scroll R of order $r = 2n+2p-2$, the number of points in the Jacobian set of a $g_n^{\ 1}$. For B, of order 6 and genus 4, R has order 18. Its intersection with Q includes the 24 tangents of B known to lie on Q; the residue, of order 12, is B, which is cuspidal on R, counted twice. Since each vertex of P is the concurrence of tangents at six stalls of B, an edge of P cannot meet any tangent of B other than those passing through the three vertices on the edge, the ten points V_{ij} are sextuple points of R.

Each tangent of an algebraic curve in [3] meets $r-4$ others and the locus of such intersections, the nodal curve N of R, has order

$$2(n+p-1)(n+p-3).$$

These are among the classical enumerative properties of algebraic curves; the last number is the one Salmon denotes by x [**6**; pp. 293–299], while Semple and Roth label it v [**9**; p. 86]. For B the curve N has order 126 but, as is now to be explained, is reducible with eleven components. Ten of these are plane curves, one in each π_{ij}.

If a tangent t of B meets π_{ij} at 0 its transform t_{ij} in h_{ij} also passes through 0; the plane curve N_{ij} traced by 0 as the contacts (collinear with V_{ij} on a generator of K_{ij}) of t and t_{ij} trace B is the complete intersection of R with π_{ij}. Whenever a line in π_{ij} meets a tangent of B it also (at least if the intersection is not one of the six stalls in π_{ij}) meets a second tangent at the same point; since it meets 18 tangents of B it meets N_{ij} in nine points, so that N_{ij} has order 9. But the edge of P in π_{ij} meets no tangent of B save those through the vertices of P on this edge, which are therefore triple points of N_{ij}. The points of N_{ij} are in (1, 1) correspondence with the generators of K_{ij} so that N_{ij} is an elliptic curve; it has the equivalent of 27 double points, and to these the three triple points contribute. Of the 14 tangents of B meeting any one of its tangents t, 10 are the transforms of t in the inversions h_{ij}; the locus of the remaining four intersections of t with other tangents of B is a curve N_0 of order 126–90 = 36.

The curve N_0 has multiple points at the vertices of P. For through such a vertex, say V_{12}, pass six tangents of B: these, with the two triply perspective triangles of stalls mentioned in §9, are transposed in pairs by h_{34}, h_{35}, h_{45} so that nine of the pairs of tangents through V_{12} are mates in the three inversions, leaving six pairs not so linked; each V_{ij} is a sextuple point of N_0.

If t is among those 24 tangents of B that have been seen to lie on Q, its transforms in the h_{ij}, meeting it at its intersections with the ten planes π_{ij}, also lie on Q and in the opposite regulus ρ. There are however 12 tangents of B in ρ, and those two which belong with t to one of the six quadrilaterals of §5 meet t at points, as there explained, no two of whose co-ordinates are equal and which therefore do not lie in any plane π_{ij}. The vertices of these six quadrilaterals therefore all lie on N_0. Although N_0 has the tangents of B as quadrisecants its only intersections with a tangent t of B that is on Q are the two vertices of the associated quadrilateral which have therefore either both to be counted twice or else one thrice and the other once; for any tangent of B meeting t must be in ρ, and there are only 12 such.

9.

In conclusion, without pursuing the topic, one may allude to the 120 tritangent planes of B. If c is a contact of such a plane it is on a surface $\lambda S_4{}^5 + \mu S_5{}^4 = 0$ whose 120 intersections with B are all contacts of such planes. What is the binary cubic C whose zeros $\lambda : \mu$ account for all 360 contacts? Whatever C may be it has a Hessian quadratic and a cubic covariant; zeros of these also correspond to surfaces meeting B in sets of points having some invariant property.

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Department of Mathematics,
The University,
Edinburgh.

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