SOME SPECIAL NETS OF QUADRICS IN FOUR-DIMENSIONAL SPACE.

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Contents.

Introduction.

When the properties of a general net of quadrics in [4] are known it may be of interest to discuss the properties of some particularised nets of quadrics, and this paper is devoted to such a discussion. The discussion makes no claim to be exhaustive; the general net of quadrics is merely particularised in different ways and, using the known properties of the general net as a basis, some properties of the particular nets are obtained.

Some properties of a general net of quadrics in [4] have recently been expounded[1]; they hinge, for the greater part, upon the Jacobian curve ϑ of the

[1] »The geometry of a net of quadrics in four-dimensional space». *Acta mathematica* 64 (1935), 185—242. This paper will be referred to as *G.N.Q.*

net. The more fundamental of these properties may be briefly recapitulated here, as we shall often have to appeal to them. The polar solids, in regard to the ∞^2 quadrics of the net, of any point P of ϑ have in common not merely a line but a plane, and this plane is a *secant plane* of ϑ, meeting it in six points; we say that the secant plane is *conjugate* to P. There is thus a singly-infinite family of secant planes of ϑ: they generate a locus R_3^{15} on which ϑ is a sextuple curve. Every plane which has five intersections with ϑ must have a sixth intersection with ϑ. Any solid passing through a secant plane of ϑ meets ϑ further in four points and these four points, together with the point P of ϑ to which the secant plane is conjugate, form a set of five points on ϑ which are the vertices of five cones of the net all belonging to the same pencil of quadrics (*G.N.Q.* §§ 5—6). Next: ϑ has twenty trisecants; through each trisecant there pass three secant planes of ϑ, and the three points of ϑ to which these three secant planes are conjugate lie on a second trisecant; the three secant planes which pass through this second trisecant are conjugate to those three points of ϑ which lie on the first trisecant. The twenty trisecants thus consist of *ten conjugate pairs:* when two trisecants of ϑ are conjugate any point on either of them, whether on ϑ or not, is conjugate to every point on the other in regard to every quadric of the net (*G.N.Q.* § 10). A secant plane α which passes through a trisecant t of ϑ meets ϑ further in three points which are not on t, and the cones of the net whose vertices are at these three points belong to the same pencil. The vertices of the remaining two cones of this pencil are on the trisecant t' which is conjugate to t; they are those two of the three intersections of t' with ϑ other than that intersection to which the secant plane α is conjugate.

It was shown (*G.N.Q.* § 34) that the equations of three linearly independent quadrics in [4] can, in general, be reduced simultaneously to the canonical form

$$\left.\begin{aligned} a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + c\, Z^2 + b_1 Y_1^2 + b_2 Y_2^2 + b_3 Y_3^2 = 0,\\ a_1' X_1^2 + a_2' X_2^2 + a_3' X_3^2 + c' Z^2 + b_1' Y_1^2 + b_2' Y_2^2 + b_3' Y_3^2 = 0,\\ a_1'' X_1^2 + a_2'' X_2^2 + a_3'' X_3^2 + c'' Z^2 + b_1'' Y_1^2 + b_2'' Y_2^2 + b_3'' Y_3^2 = 0, \end{aligned}\right\} \qquad \mathrm{I}$$

where

$$X_1 + X_2 + X_3 \equiv Z \equiv Y_1 + Y_2 + Y_3.$$

Here a form of specialisation at once leaps to the eye. The seven linear forms which occur are such that six of them may be supposed to represent arbitrary solids while the seventh, Z, represents the solid which joins the line of intersection of $X_1 = 0$, $X_2 = 0$ and $X_3 = 0$ to the line of intersection of $Y_1 = 0$, $Y_2 = 0$ and $Y_3 = 0$. If then we specialise the canonical form I by omitting the terms in Z^2 and consider the net of quadrics

$$\left.\begin{aligned} a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + a_4 X_4^2 + a_5 X_5^2 + a_6 X_6^2 = 0, \\ b_1 X_1^2 + b_2 X_2^2 + b_3 X_3^2 + b_4 X_4^2 + b_5 X_5^2 + b_6 X_6^2 = 0, \\ c_1 X_1^2 + c_2 X_2^2 + c_3 X_3^2 + c_4 X_4^2 + c_5 X_5^2 + c_6 X_6^2 = 0, \end{aligned}\right\} \qquad \text{II}$$

where

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \equiv 0$$

and we have written $-X_4$, $-X_5$, $-X_6$ for Y_1, Y_2, Y_3 respectively, we have a net of quadrics related symmetrically to the six faces of a hexahedron. This however, although an admirable illustration, is somewhat too drastic a specialisation to begin with; there are other specialisations of I which are not so particularised as II. On the other hand there are nets of quadrics whose properties we shall consider that are even more particularised than II.

In order to keep the work within reasonable compass we must impose some limit on the extent to which we specialise, and we therefore decide that no consideration will be given here to any net of quadrics whose Jacobian curve breaks up into separate component curves or whose Jacobian curve has a multiple point. All the nets of quadrics that are considered have Jacobian curves that are in birational correspondence with plane quintic curves, of genus 6, without multiple points.

When the properties of a general net of quadrics in [4] were obtained the birational correspondence between the Jacobian curve and a plane quintic was discussed in some detail, and it is of interest to see, when the net of quadrics and its Jacobian curve are specialised in any way, how the plane quintic is specialised correspondingly. For example (cf. *G.N.Q.* § 27 and § 35): the equation of a plane quintic in birational correspondence with the Jacobian curve of I is

$$\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1} + \eta_1^{-1} + \eta_2^{-1} + \eta_3^{-1} + L(\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1})(\eta_1^{-1} + \eta_2^{-1} + \eta_3^{-1}) = 0,$$

where $L \equiv cx + c'y + c''z$. Hence, putting $c = c' = c'' = 0$, it follows that a plane quintic in birational correspondence with the Jacobian curve of II has an equation

$$\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1} + \xi_4^{-1} + \xi_5^{-1} + \xi_6^{-1} = 0,$$

and is therefore circumscribed to the hexagram formed by the six lines $\xi_i = 0$. Thus the particularised quintic curve has an inscribed hexagram, whereas a general quintic curve has not. It will also be shown that the converse of this result is true in the sense that if a plane quintic has an inscribed hexagram it can always be put into birational correspondence with the Jacobian curve of a net of quadrics given by the canonical form II.

If three conics are taken in a plane any one meets any other in four points, so that there are twelve intersections in all; there are ∞^8 quintic curves passing through these twelve points. Such a quintic curve has two further intersections with each conic, and if we suppose that, on each conic, these two further intersections coincide in a single contact we impose three further conditions, thus reducing the aggregate of quintic curves to ∞^5. Conversely: there are ∞^{20} quintic curves and ∞^{15} sets of three conics in a plane, so that it would seem that there is a finite number of such sets of three conics associated with a general plane quintic. A set of three conics which is such that all their intersections lie on the quintic and each conic touches the quintic makes up a contact sextic, i. e. a sextic which has two intersections with the quintic wherever it meets it. If $\Gamma_1 = 0$, $\Gamma_2 = 0$, $\Gamma_3 = 0$ are the equations of the conics the equation of the quintic can be written

$$t_1 \Gamma_1^{-1} + t_2 \Gamma_2^{-1} + t_3 \Gamma_3^{-1} = 0,$$

where $t_1 = 0$, $t_2 = 0$, $t_3 = 0$ are the tangents of the respective conics at the points where the quintic touches them. If this could be established as a canonical form for the ternary quintic the existence of the set of three conics would follow immediately.

Suppose now that we specialise the configuration. Let one of the three conics break up into a line-pair and, as the conic touched the quintic, let the intersection of the two lines be on the quintic. Such a configuration of two conics and a line-pair together constituting a degenerate contact-sextic does not exist for a general plane quintic. But we shall see that when we specialise the net of quadrics in [4] in a certain way its Jacobian curve is such that any plane quintic that is in (1, 1) correspondence with it has a degenerate contact-sextic of this kind; indeed we shall actually obtain the equation of such a plane quintic by equating to zero the discriminant of the special net of quadrics (§ 5).

The plane configuration is specialised further when another of the conics becomes a line-pair whose intersection is on the quintic; we then have a plane quintic which passes through the six vertices of a quadrilateral and is such that its remaining eight intersections with the sides of the quadrilateral are on a conic, this conic also touching the quintic. This type of plane quintic will also be obtained by specialising the net of quadrics in [4] and equating its discriminant to zero (§ 12).

If each of the conics is taken to be a line-pair whose intersection is on the quintic then the quintic has an inscribed hexagram and is in birational correspondence with the Jacobian curve of a net of quadrics whose canonical form is II. The two special plane quintics alluded to above will thus arise from nets of quadrics in [4] intermediate between the general form I and the special form II.

The consideration of plane quintic curves has shown how we may expect to find nets of quadrics in [4] which are special, and yet not so special as the net II; similar consideration will also show how we may expect to find nets of quadrics in [4] that are still more special than II. The Jacobian curve of the net II is such that any plane quintic which is in birational correspondence with it has an inscribed hexagram; we then enquire as to how a plane quintic with an inscribed hexagram may be further specialised. One mode of specialisation is obvious; we may suppose that the six sides of the hexagram, instead of being any six lines in the plane, are six tangents of a conic. It is then found that there is an infinity of hexagrams inscribed in the quintic curve, and that the sides of these hexagrams all touch this same conic. The condition for the six sides of the hexagram to touch a conic can be written down at once in terms of the coefficients in II; it appears that the Jacobian curve of a net II which is specialised in this way has not twenty but an infinity of trisecants. The properties of this Jacobian curve and of loci that are associated with it are obtained in §§ 16—49. This particular net of quadrics may well be regarded as the analogue, in [4], of the net of polar quadrics of points of a plane in regard to a cubic surface in [3].

We can also consider another method of specialising a plane quintic with an inscribed hexagram. We may always suppose that the equation of such a quintic is

$$\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1} + \xi_4^{-1} + \xi_5^{-1} + \xi_6^{-1} = 0,$$

where the six lines $\xi_i = 0$ are the sides of the hexagram; then the tangent of this curve at the vertex $\xi_i = \xi_j = 0$ of the hexagram is $\xi_i + \xi_j = 0$. Now the six linear forms ξ_i must satisfy three linearly independent linear identities, and we might enquire how the curve is specialised when we assume these linear identities to be of a special kind; it must always however be remembered that, as no three sides of the hexagram are to be concurrent, it must never be possible to obtain any linear identity between less than four of the forms ξ_i. But let us suppose that the identity

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 \equiv 0$$

holds. Then, if the six sides of the hexagram are divided into three pairs in any of the fifteen possible ways, each of the three pairs determines a vertex of the hexagram, and the three tangents of the quintic at these vertices are concurrent; they are, indeed, concurrent in a point of the curve. Thus, when this identity is satisfied, the fifteen tangents of the quintic at the vertices of the hexagram meet by threes in fifteen points of the curve.

We can also enquire whether there are quintic curves with inscribed hexagrams such that some but not all of the fifteen triads of tangents are concurrent in this way. These and many other matters will be treated of in their proper place; enough has been said here to indicate some few of the results to which our investigations may lead.

The Jacobian Curve with two Pairs of Concurrent Trisecants.

1. Let us suppose that through a point O of ϑ there pass two different trisecants OPQ, ORS; call these t_1 and t_2 respectively. Then the plane $OPQRS$ is a secant plane of ϑ, and meets ϑ in a sixth point Z; the trisecants t_1' and t_2' which are conjugate to t_1 and t_2 both pass through the point T of ϑ to which this secant plane is conjugate, and the plane which contains them is the secant plane conjugate to O. Let t_1' meet ϑ again in U and V, and let t_2' meet ϑ again in W and X.

The points Z, R, S are the three further intersections with ϑ of a secant plane passing through the trisecant t_1 and conjugate to T; hence the five cones (Z), (R), (S), (U), (V), belong to a pencil; similarly (Z), (P), (Q), (W), (X) belong to a second pencil. But, if Z' is the sixth intersection of the secant plane $TUVWX$ with ϑ, it follows, in precisely the same way, that each of the two sets of five cones

$$(Z'),\ (U),\ (V),\ (R),\ (S) \text{ and } (Z'),\ (W),\ (X),\ (P),\ (Q)$$

is a set of five cones belonging to a pencil. Thus the two cones (Z) and (Z') must be the same, and the two points Z and Z' must coincide; the point of intersection of the planes $t_1 t_2$ and $t'_1 t'_2$ is on ϑ.

Since the four cones (P), (Q), (W), (X) belong to a pencil of quadrics containing (Z) the solid $PQWX$ must contain that secant plane of ϑ which is conjugate to Z; similarly this same secant plane must lie in the solid $RSUV$: the secant plane β conjugate to Z is therefore the plane of intersection of the two solids $t_1 t'_2$ and $t'_1 t_2$; it meets ϑ in O and T, and in four further points E, F, G, H.

2. The polar lines of a secant plane of ϑ in regard to the quadrics of the net generate a cubic cone whose vertex is that point of ϑ to which the secant plane is conjugate, and this cone contains the six secant planes of ϑ which pass through its vertex ($G.N.Q.$ § 9). In particular the polar lines of the plane $t'_1 t'_2$ generate a cubic cone Π_O whose vertex is O; this cone contains the plane β and the plane $t_1 t_2$; it also contains the two other secant planes, say λ_1 and μ_1, which pass through t_1 and the two other secant planes, say λ_2 and μ_2, which pass through t_2; t_1 and t_2 are nodal lines on Π_O. Since the plane ν common to the solids $\lambda_1\mu_1$ and $\lambda_2\mu_2$ meets Π_O in four lines it must lie entirely on Π_O; moreover ν passes through the line OT since both the solids $\lambda_1\mu_1$ and $\lambda_2\mu_2$ do so. The section of Π_O by an arbitrary solid is a cubic surface Π^3 with two nodes N_1 and N_2; it contains the line $N_1 N_2$, two further lines l_1 and m_1 through N_1, two further lines l_2 and m_2 through N_2, the line n in which the planes $l_1 m_1$ and $l_2 m_2$ intersect and a line b which meets n in a point B on the line OT. The projection of ϑ from O is a curve θ, of order 9 and genus 6, on Π^3; θ has nodes at N_1 and N_2 and meets the line $N_1 N_2$ in one further point, also it meets each of l_1 and m_1 in three points other than its node N_1 and each of l_2 and m_2 in three points other than its node N_2, and it meets b in five points, one of which is B. It can be verified that the canonical series is cut out on θ by the cubic surfaces which contain the three lines b, l_1, m_1 and which have a node at N_1 and pass simply through N_2; the two nodes of θ and its intersections with b, l_1 and m_1 are fixed points, none of which belongs to a general canonical set. Now among these cubic surfaces which cut out canonical sets on θ there are those which consist of the plane $l_1 m_1$ taken together with quadrics containing the line b and the two points N_1 and N_2; thus the quadrics

cut out canonical sets on θ, but all these canonical sets include the point B since B lies not only on each quadric but on the plane $l_1 m_1$. Conversely: all the canonical sets on θ to which B belongs can be obtained as the intersections of θ with quadrics containing the line b and the two points N_1 and N_2, as such quadrics form a system of freedom 4, one less than the freedom of the canonical series on a curve of genus 6. Hence we may state the following:

Those canonical sets of ϑ which contain T are cut out by the quadric cones, vertex O, which contain the plane β and the lines t_1 and t_2; the five intersections of ϑ with β, other than T, and the intersections of ϑ with t_1 and t_2, are not to be reckoned as points belonging to the canonical sets. The twenty intersections of such a quadric cone with ϑ consist of O, counted twice, the eight points P, Q, R, S, E, F, G, H and a canonical set of ten points, of which T is one; conversely any canonical set to which T belongs can be obtained in this way.

We may, in particular, take the quadric cone to consist of the pair of solids $t_1 t_2'$ and $t_1' t_2$; the solid $t_1 t_2'$ meets ϑ in the set of points $(OPQTWXEFGH)$ and the solid $t_1' t_2$ meets ϑ in the set of points $(ORSTUVEFGH)$; subtracting from the sum of these two sets the set $(O^2PQRSEFGH)$ we find that the set $(T^2UVWXEFGH)$ is a canonical set on ϑ. Similarly, since the canonical sets of ϑ which contain O can be cut out by the quadric cones, vertex T, which contain the plane β and the lines t_1' and t_2', it is found that the set $(O^2PQRS EFGH)$ is a canonical set on ϑ. This latter result also follows immediately from the known result ($G.N.Q.$ § 28) that, when a quadric meets ϑ in all the points of a given canonical set, its residual intersections with ϑ also form a canonical set; for we have just seen above that the set $(O^2PQRSEFGH)$ is residual to all the canonical sets which contain T.

3. Suppose now that a birational correspondence is established between the curve ϑ and a plane quintic ζ; then any canonical set on ϑ must correspond to a canonical set on ζ, i. e. to ten points of ζ which lie on a conic. Also if five points of ϑ are the vertices of five cones which belong to the same pencil, the five points of ζ which correspond to them must be collinear. Hence, if we denote corresponding points of ζ and ϑ by the same small and capital letter, we have on ζ two sets of five collinear points $(zuvrs)$ and $(zpqwx)$; a conic through the points p, q, r, s and a conic through the points u, v, w, x are such that their four intersections e, f, g, h are on ζ, while the first of these conics

touches ζ at a point o and the second touches ζ at a point t. A configuration of this kind does not exist for a general plane quintic curve; but it must exist for any plane quintic which is in birational correspondence with a Jacobian curve, of a net of quadrics in [4], that has two pairs of concurrent trisecants.

4. Let the equations of the three solids which contain the pairs of the three secant planes through t_1 be $X_1 = 0$, $X_2 = 0$, $X_3 = 0$, the plane $t_1 t_2$ being the plane of intersection of $X_2 = 0$ and $X_3 = 0$; let $Y_1 = 0$, $Y_2 = 0$, $Y_3 = 0$ be the equations of the solids which contain the pairs of the three secant planes through t_1', the plane $t_1' t_2'$ being the plane of intersection of $Y_2 = 0$ and $Y_3 = 0$. Then we may suppose that any quadric of the net has an equation of the form

$$a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + cZ^2 + b_1 Y_1^2 + b_2 Y_2^2 + b_3 Y_3^2 = 0,$$

where

$$X_1 + X_2 + X_3 \equiv Z \equiv Y_1 + Y_2 + Y_3.$$

This canonical form for the equations ensures that t_1 and t_1' are conjugate trisecants of ϑ; in order that t_2 and t_2' should also be conjugate trisecants the coefficients will have to be restricted in some way. This condition must arise from the fact that every point of t_2 is conjugate to every point of t_2' in regard to every quadric of the net.

The coordinates of O are $(0, 0, 0;\ 0, 1, -1)$; here we put the three X-coordinates before the semicolon and the three Y-coordinates after it, the Z-coordinate being omitted. Suppose that t_2 joins O to the point $(x_1, 0, 0;\ y_1, y_2, y_3)$. Also let t_2' join T, whose coordinates are $(0, 1, -1;\ 0, 0, 0)$ to the point $(x_1', x_2', x_3';\ y_1', 0, 0)$. Any point on t_2 has coordinates $(x_1, 0, 0;\ y_1, y_2 + \lambda, y_3 - \lambda)$ while any point on t_2' has coordinates $(x_1', x_2' + \mu, x_3' - \mu;\ y_1', 0, 0)$; these two points are conjugate if

$$a_1 x_1 x_1' + c x_1 y_1' + b_1 y_1 y_1' = 0$$

or

$$\frac{a_1 x_1'}{y_1'} + c + \frac{b_1 y_1}{x_1} = 0.$$

The coefficients a_1, c, b_1 must therefore be connected by this linear relation.

This relation may also be obtained by finding the condition that the point of intersection of the two planes $X_2 = X_3 = 0$ and $Y_2 = Y_3 = 0$ should lie on ϑ. The coordinates of this point are $(1, 0, 0;\ 1, 0, 0)$ and its polar prime with respect to the quadric is

$$a_1 X_1 + cZ + b_1 Y_1 = 0.$$

Since all these polar primes are to contain the same plane, whichever quadric of the net is taken, there must be a linear relation between the coefficients a_1, c, b_1. This linear relation can actually be found, because the plane through which the polar primes must pass is the plane, previously called β, common to the solids $t_1 t_2'$ and $t_1' t_2$. The solid $t_1 t_2'$ joins the plane $Z = X_1 = 0$ to the point $(x_1', x_2', x_3';\ y_1', 0, 0)$, so that its equation is

$$y_1' X_1 - x_1' Z = 0.$$

Also the solid $t_1' t_2$ joins the plane $Z = Y_1 = 0$ to the point $(x_1, 0, 0;\ y_1, y_2, y_3)$, so that its equation is

$$-y_1 Z + x_1 Y_1 = 0.$$

We therefore have the relation

$$\begin{vmatrix} a_1, & c, & b_1 \\ y_1', & -x_1', & 0 \\ 0, & -y_1, & x_1 \end{vmatrix} = 0,$$

which is the same as that previously found.

Conversely: suppose the quadrics of a net have equations of the canonical form

$$a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + cZ^2 + b_1 Y_1^2 + b_2 Y_2^2 + b_3 Y_3^2 = 0;$$

then the two lines $X_1 = X_2 = X_3 = 0$ and $Y_1 = Y_2 = Y_3 = 0$ are conjugate trisecants of the Jacobian curve ϑ, three secant planes passing through each trisecant. Let us also suppose, further, that the coefficients of the three terms X_1^2, Z^2, Y_1^2 in the equation of every quadric of the net are subjected to a linear relation

$$c = \varrho a_1 + \sigma b_1;$$

here of course ϱ and σ are numerical constants, whereas the coefficients c, a_1, b_1 differ for different quadrics of the net. Then, when this relation is satisfied, the two points

$$(1, 0, 0;\ -\sigma,\ 1 + \lambda,\ \sigma - \lambda) \text{ and } (-\varrho,\ 1 + \mu,\ \varrho - \mu;\ 1, 0, 0)$$

are conjugate, in regard to every quadric of the net, whatsoever values λ and μ

may have. When λ and μ vary these points describe two lines which will be conjugate trisecants of ϑ; the first of them is a trisecant, other than $X_1 = X_2 = X_3 = 0$, passing through $(0, 0, 0; 0, 1, -1)$ and the second is a trisecant, other than $Y_1 = Y_2 = Y_3 = 0$, passing through $(0, 1, -1; 0, 0, 0)$. Thus when the coefficients c, a_1, b_1 satisfy a linear relation the Jacobian curve has two pairs of intersecting trisecants.

We can always find a quadric belonging to the net in whose equation the coefficients of any two of the seven squares vanish. Suppose then we take the quadric for which the coefficients of X_1^2 and Z^2 both vanish; then, in virtue of the linear relation $c = \varrho a_1 + \sigma b_1$, the coefficient of Y_1^2 must also vanish. Hence the left-hand side of the equation of this quadric is the sum of only four squares, so that the quadric is a cone with vertex $X_2 = X_3 = Y_2 = Y_3 = 0$. Hence, when there is a linear relation $c = \varrho a_1 + \sigma b_1$, the intersection of the two secant planes $X_2 = X_3 = 0$ and $Y_2 = Y_3 = 0$ lies on ϑ.

5. The Jacobian curve ϑ is in birational correspondence with the quintic

$$\xi_1\xi_2\xi_3(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2) + \eta_1\eta_2\eta_3(\xi_2\xi_3 + \xi_3\xi_1 + \xi_1\xi_2) \\ + \zeta(\xi_2\xi_3 + \xi_3\xi_1 + \xi_1\xi_2)(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2) = 0,$$

where ξ, η, ζ are linear functions of three homogeneous coordinates, and where, in addition, there is now a linear identity between ξ_1, ζ, η_1; say

$$\zeta \equiv \varrho\xi_1 + \sigma\eta_1,$$

where ϱ and σ are numerical constants. We have the identity

$$\begin{aligned} &\xi_1\xi_2\xi_3(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2) + \eta_1\eta_2\eta_3(\xi_2\xi_3 + \xi_3\xi_1 + \xi_1\xi_2) \\ &\qquad + (\varrho\xi_1 + \sigma\eta_1)(\xi_2\xi_3 + \xi_3\xi_1 + \xi_1\xi_2)(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2) \\ &\equiv (\varrho\xi_1 + \sigma\eta_1)(\xi_2\xi_3 + \xi_3\xi_1 + \xi_1\xi_2 + \varrho^{-1}\xi_2\xi_3)(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2 + \sigma^{-1}\eta_2\eta_3) \\ &\qquad - \varrho\sigma^{-1}\xi_1\eta_2\eta_3(\xi_2\xi_3 + \xi_3\xi_1 + \xi_1\xi_2 + \varrho^{-1}\xi_2\xi_3) \\ &\qquad - \sigma\varrho^{-1}\eta_1\xi_2\xi_3(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2 + \sigma^{-1}\eta_2\eta_3), \end{aligned}$$

so that, writing

$$\xi_2\xi_3 + \xi_3\xi_1 + \xi_1\xi_2 + \varrho^{-1}\xi_2\xi_3 \equiv \Gamma,$$
$$\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2 + \sigma^{-1}\eta_2\eta_3 \equiv \Delta,$$

the equation to the quintic curve is

$$(\varrho\xi_1 + \sigma\eta_1)\Gamma\Delta = \varrho\sigma^{-1}\xi_1\eta_2\eta_3\Gamma + \sigma\varrho^{-1}\eta_1\xi_2\xi_3\Delta.$$

Writing, in this equation,

$$\sigma^{-1}\eta_2\eta_3 = \frac{\Delta - \eta_1(\eta_2 + \eta_3)}{1 + \sigma}, \quad \varrho^{-1}\xi_2\xi_3 = \frac{\Gamma - \xi_1(\xi_2 + \xi_3)}{1 + \varrho},$$

it becomes

$$\varrho\sigma\{(\varrho + 1)\xi_1 + (\sigma + 1)\eta_1\}\Gamma\Delta + \xi_1\eta_1\{\varrho(\varrho + 1)(\eta_2 + \eta_3)\Gamma + \sigma(\sigma + 1)(\xi_2 + \xi_3)\Delta\} = 0.$$

From this equation we can obtain immediately the configuration of the pair of lines and the pair of conics, the existence of which was established by the consideration of certain canonical sets on ϑ; the two lines are in fact $\xi_1 = 0$ and $\eta_1 = 0$, and the two conics are $\Gamma = 0$ and $\Delta = 0$. The intersections of the conic $\Gamma = 0$ with the curve consist of its four intersections with $\Delta = 0$, its two intersections with $\xi_1 = 0$, its two intersections with $\eta_1 = 0$ and its two intersections with $\xi_2 + \xi_3 = 0$; but these last two intersections coincide in a single contact, since $\xi_2 + \xi_3 = 0$ is the tangent of $\Gamma = 0$ at the point $\xi_2 = \xi_3 = 0$; this same line is also the tangent of the quintic at the same point. Similarly for the conic $\Delta = 0$. Finally the intersection of $\xi_1 = 0$ and $\eta_1 = 0$ is on the quintic curve.

That any quintic curve with these properties is in birational correspondence with a Jacobian curve ϑ having two pairs of concurrent trisecants follows at once when the left-hand side of its equation is expressed as a symmetrical determinant of five rows and columns (cf. *G.N.Q.* § 35).

6. We have seen how the Jacobian curve ϑ and the plane quintic ζ are specialised when the coefficients in the canonical form are subjected to a certain type of linear relation, and the question naturally presents itself whether further relations of this type can introduce further specialisations.

Suppose we have a pair of relations such as

$$c = \varrho a_1 + \sigma b_1 = \tau a_1 + \omega b_2,$$

the coefficient a_1 occurring twice. There is a quadric of the net in whose equation the coefficients of X_1^2 and Z^2 both vanish and it follows, from the existence of these linear relations, that the coefficients of Y_1^2 and Y_2^2 also vanish; hence the left-hand side of the equation of this quadric is the sum of only three squares, so that the quadric is a cone with a line for its vertex. Hence, unless the net of quadrics is to contain a line-cone, there cannot exist two such linear relations at the same time.

We might however have a pair of relations such as

$$c = \varrho_1 a_1 + \sigma_1 b_1 = \varrho_2 a_2 + \sigma_2 b_2,$$

in which none of the six coefficients $a_1, a_2, a_3, b_1, b_2, b_3$ occurs twice. In this case the equation of the plane quintic is

$$\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1} + \eta_1^{-1} + \eta_2^{-1} + \eta_3^{-1} + \zeta(\eta_1^{-1} + \eta_2^{-1} + \eta_3^{-1})(\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1}) = 0,$$

where $\zeta = 0$ is the line which joins the point $\xi_1 = \eta_1 = 0$ to the point $\xi_2 = \eta_2 = 0$. The Jacobian curve ϑ now has three pairs of conjugate trisecants t_1, t_1'; t_2, t_2'; t_3, t_3' where t_2 and t_3 meet t_1 and t_2' and t_3' meet t_1'.

We may also go a step further and suppose that the coefficients in the canonical form satisfy the three relations

$$c = \varrho_1 a_1 + \sigma_1 b_1 = \varrho_2 a_2 + \sigma_2 b_2 = \varrho_3 a_3 + \sigma_3 b_3.$$

A plane quintic in birational correspondence with the Jacobian curve of such a net of quadrics is obtained if we take a triangle formed by three lines $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$ and then a second triangle, in perspective with this, formed by three lines $\eta_1 = 0$, $\eta_2 = 0$, $\eta_3 = 0$; we then take the quintic curve

$$\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1} + \eta_1^{-1} + \eta_2^{-1} + \eta_3^{-1} + \zeta(\eta_1^{-1} + \eta_2^{-1} + \eta_3^{-1})(\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1}) = 0,$$

where now $\zeta = 0$ is the axis of perspective. This quintic curve has three contact sextic curves each consisting of two lines and two conics. The curve ϑ now has a pair of conjugate trisecants t and t' such that through any intersection of t or t' with ϑ there passes a second trisecant. The trisecants which pass through the three intersections of ϑ and t are conjugate to those which pass through the three intersections of ϑ and t'.

The Jacobian Curve with four Concurrent Trisecants.

7. Suppose that a net of quadrics in [4] is specialised in such a way that there is one secant plane α whose six intersections with the Jacobian curve ϑ are the vertices of a quadrilateral; the four sides t_1, t_2, t_3, t_4 of the quadrilateral are trisecants of ϑ. Denote the point of intersection of t_1 and t_2 by P_{12}, and

similarly for the other five vertices of the quadrilateral. The four trisecants t_1', t_2', t_3', t_4' which are conjugate to t_1, t_2, t_3, t_4 all pass through that point O of ϑ to which the secant plane α is conjugate and the six secant planes through O, conjugate to the six vertices of the quadrilateral in α, are the six planes which contain the pairs of the four lines t_1' t_2', t_3', t_4'.

Consider now the solid which is determined by the two secant planes $t_1' t_2'$ and $t_1' t_3'$, both passing through the trisecant t_1'; since these are the secant planes conjugate to P_{12} and P_{13} the solid must contain that point of ϑ which is the remaining intersection of ϑ with t_1, i. e. P_{14}, and P_{14} is the unique intersection of the solid with ϑ which lies neither on the plane $t_1' t_2'$ nor on the plane $t_1' t_3'$. It therefore follows that P_{14} is that intersection of the secant plane $t_2' t_3'$ with ϑ which lies neither on t_2' nor on t_3'. Hence *the six points of ϑ which lie in α are the intersections of α with the six secant planes through O;* the secant plane which is conjugate to any vertex of the quadrilateral in α meets α in the opposite vertex of the quadrilateral, and the three pairs of opposite vertices of the quadrilateral are pairs of conjugate points in regard to all the quadrics of the net.

8. The polar lines of α in regard to the quadrics of the net generate a cubic cone Π_0, vertex O, containing the six secant planes of ϑ which pass through O and having the four trisecants t_1', t_2', t_3', t_4' as nodal lines. An arbitrary solid, not passing through O, meets Π_0 in a four-nodal cubic surface and ϑ is projected from O into a curve θ, of order 9 and genus 6, lying on this surface and having a node at each node of the surface. The canonical series is cut out on this curve by the quadrics through its four nodes, and therefore *the canonical series on ϑ is cut out by those quadric cones with vertex O which contain the four trisecants t_1', t_2', t_3', t_4',* The twenty intersections of such a cone with ϑ consist of O, counted twice, the remaining eight intersections of ϑ with its four trisecants through O, and ten points forming a canonical set. In particular we may take the quadric cone to be a pair of solids through O and, if O_i, O_i' are the two intersections, other than O, of ϑ with the trisecant t_i', we may note the following six canonical sets on ϑ:

$$(O_2\, O_2'\, O_3\, O_3'\, P_{14}^2\, P_{24}\, P_{34}\, P_{31}\, P_{12}), \qquad (O_1\, O_1'\, O_4\, O_4'\, P_{23}^2\, P_{31}\, P_{12}\, P_{24}\, P_{34}),$$
$$(O_3\, O_3'\, O_1\, O_1'\, P_{24}^2\, P_{34}\, P_{14}\, P_{12}\, P_{23}), \qquad (O_2\, O_2'\, O_4\, O_4'\, P_{31}^2\, P_{12}\, P_{23}\, P_{34}\, P_{14}),$$
$$(O_1\, O_1'\, O_2\, O_2'\, P_{34}^2\, P_{14}\, P_{24}\, P_{23}\, P_{31}), \qquad (O_3\, O_3'\, O_4\, O_4'\, P_{12}^2\, P_{23}\, P_{31}\, P_{14}\, P_{24}).$$

The first of these, for example, is obtained if the quadric cone consists of the two solids $t_1' t_2' t_3'$ and $t_2' t_3' t_4'$; the other five sets are obtained similarly. Also, since any quadric which contains a canonical set of points on ϑ meets ϑ residually in another canonical set, the point O, counted twice, and the eight further intersections of ϑ with its four trisecants through O, make up a canonical set on ϑ; hence we may note the seventh canonical set

$$(O^2\, O_1\, O_1'\, O_2\, O_2'\, O_3\, O_3'\, O_4\, O_4').$$

9. Suppose now that the quadrics of the net are represented by the points of a plane, ϑ being thereby put into birational correspondence with a plane quintic curve ζ; any set of ten points of ζ which corresponds to a canonical set on ϑ must consist of the intersections of ζ with some conic. Now it is seen at once, on referring to the group of six canonical sets on ϑ, that, so long as we do not take a pair of sets written in the same horizontal row, any two of these six sets have in common *six* points of ϑ; the two corresponding canonical sets on ζ therefore also have six points in common and so, since two different conics can only have more than four common points when they are both line-pairs, with a line common to both pairs, it follows that the six conics which cut out the six canonical sets on ζ are the six line-pairs which can be formed from the sides of a quadrilateral. Two canonical sets on ζ which correspond to a pair of canonical sets of ϑ written in the same horizontal row of the group of six are cut out by line pairs without a common line; two such sets have four points in common, and these points are four of the vertices of the quadrilateral; thus the vertices of the quadrilateral all lie on ζ, being the points $p_{23}, p_{31}, p_{12}, p_{14}, p_{24}, p_{34}$ (we use, as heretofore, small letters to denote points of ζ which correspond to points of ϑ denoted by the corresponding capital letters). If we take the three points p without the suffix i then the set of five points which consists of these three points and the two points o_i, o_i' is a set of five collinear points of ζ and is common to three of the six canonical sets. If we now refer to the seventh canonical set mentioned on ϑ we see that the eight points o_1, o_1', o_2, o_2', o_3, o_3', o_4, o_4' lie on a conic, and that the two remaining intersections of this conic with ζ coincide in a contact at o. Whence we have the following:

If the Jacobian curve ϑ of a net of quadrics in [4] *is such that there is a plane which meets it in the six vertices of a quadrilateral then, if ζ is any plane quintic which is in birational correspondence with ϑ, these six* *points of* ϑ *correspond to six points of* ζ *which are also the vertices of a quadrilateral. The remaining eight intersections of* ζ *with the sides of this quadrilateral are on a conic, and this conic touches* ζ.

It should be noticed that three collinear vertices of the quadrilateral on ϑ correspond to three non-collinear vertices of the quadrilateral on ζ, namely to the vertices of a triangle obtained by omitting one side of the quadrilateral. This configuration of four lines and a conic is clearly a special case of the configuration, obtained in § 3, of two lines and two conics; here one of the two conics touching the quintic curve has become a line-pair intersecting on the curve. We can also anticipate a further specialisation of the present configuration, namely when both conics in the configuration of § 3 have become line-pairs intersecting on the curve; the quintic is then circumscribed to a hexagram.

10. The existence of the quadrilateral inscribed in the plane quintic ζ can be established without any appeal to canonical sets. For it is known that a secant plane which passes through a trisecant of ϑ meets ϑ further in three points which are the vertices of three cones belonging to the same pencil, the vertices of the two remaining cones of this pencil being those two points of ϑ which are on the conjugate trisecant and which are not conjugate to the secant plane. Now the secant plane α contains the trisecant t_1 and meets ϑ further in the three points P_{34}, P_{42}, P_{23}; hence these three points are vertices of cones belonging to the same pencil. The vertices of the two remaining cones of this pencil are the two points, other than O, in which ϑ is met by the trisecant t_1'; i. e. they are the two points O_1, O_1'. Thus the five points O_1, O_1', P_{34}, P_{42}, P_{23} are the vertices of five cones of a pencil. Similar arguments give three other such sets of five points, and it follows that, if ζ is any plane quintic in birational correspondence with ϑ, the four sets of points

$$(o_1\, o_1'\, p_{34}\, p_{42}\, p_{23}),\quad (o_2\, o_2'\, p_{41}\, p_{13}\, p_{34}),\quad (o_3\, o_3'\, p_{12}\, p_{24}\, p_{41}),\quad (o_4\, o_4'\, p_{23}\, p_{31}\, p_{12})$$

are four sets of five collinear points. Thus the six points p_{ij} are the six vertices of a quadrilateral inscribed in ζ. In order to establish the fact that the eight remaining intersections of ζ with the sides of this quadrilateral are on a conic we show, as above, that they correspond to eight points of ϑ which belong to the same canonical set.

One or two further features of the correspondence between ϑ and ζ may be pointed out (cf. *G.N.Q.* §§ 21, 22). Since the six points P_{ij} lie in the secant plane of ϑ conjugate to O there is a cubic curve touching ζ at each of the six points p_{ij}, the three remaining intersections of this cubic with ζ lying on the tangent of ζ at o. Again, the secant plane conjugate to P_{12} meets ϑ in the six points O, O_1, O_1', O_2, O_2', P_{34}; hence there is a cubic curve touching ζ at the six points o, o_1, o_1', o_2, o_2', p_{34} and meeting ζ again in its three intersections with its tangent at p_{12}; there are five other cubic curves similarly associated with the five points p_{ij} other than p_{12}. Also, since the points O, O_i, O_i' on ϑ are collinear there is a conic touching ζ at each of the points o, o_i, o_i'; thus we have three tritangent conics of ζ with a common point of contact. There is also a conic touching ζ at the vertices of any one of the four triangles formed by three of the sides of the quadrilateral, since the vertices of such a triangle correspond to three collinear points of ϑ. Since the two tritangent conics associated with two conjugate trisecants of ϑ are such that their four intersections lie on ζ, the four intersections of the conic touching ζ at o, o_i, o_i' with the conic touching ζ at the three points p with the suffix i lie on ζ.

11. We can obtain a canonical form for the net of quadrics when its Jacobian curve has a secant plane meeting it in the vertices of a quadrilateral and also, by taking the discriminant of a quadric of the net, obtain an equation for the plane quintic which corresponds birationally to the Jacobian curve.

We take the four solids

$$X_1 = 0, \quad X_2 = 0, \quad X_3 = 0, \quad X_4 = 0,$$

to be those solids which pass through O and which contain the sets of three of the lines t_1', t_2', t_3', t_4', the solid $X_i = 0$ containing the three lines other than t_i'; also we take the plane α to be $X_5 = X_6 = 0$. The six forms X are homogeneous linear forms in five variables, so that they satisfy one linear identity; we may take this to be

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \equiv 0.$$

The plane α contains four trisecants t_1, t_2, t_3, t_4 of the Jacobian curve, and the equations of the trisecant t_i are $X_i = X_5 = X_6 = 0$. The four pairs of lines

$$\left.\begin{matrix} X_2 = X_3 = X_4 = 0 \\ X_1 = X_5 = X_6 = 0 \end{matrix}\right\}, \quad \left.\begin{matrix} X_3 = X_4 = X_1 = 0 \\ X_2 = X_5 = X_6 = 0 \end{matrix}\right\}, \quad \left.\begin{matrix} X_4 = X_1 = X_2 = 0 \\ X_3 = X_5 = X_6 = 0 \end{matrix}\right\}, \quad \left.\begin{matrix} X_1 = X_2 = X_3 = 0 \\ X_4 = X_5 = X_6 = 0 \end{matrix}\right\},$$

have the property that, if one point is taken on each line of any pair, such a pair of points is conjugate in regard to every quadric of the net. It then follows easily that the net of quadrics is given by a set of three equations of the form

$$\left.\begin{aligned} a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + a_4 X_4^2 + a_5 X_5^2 + a_6 X_6^2 + 2\alpha X_5 X_6 &= 0, \\ b_1 X_1^2 + b_2 X_2^2 + b_3 X_3^2 + b_4 X_4^2 + b_5 X_5^2 + b_6 X_6^2 + 2\beta X_5 X_6 &= 0, \\ c_1 X_1^2 + c_2 X_2^2 + c_3 X_3^2 + c_4 X_4^2 + c_5 X_5^2 + c_6 X_6^2 + 2\gamma X_5 X_6 &= 0. \end{aligned}\right\} \qquad \text{III}$$

This form III is not sufficiently general to represent a general net of quadrics in [4], although it apparently contains the same number of constants as the general canonical form I; in fact we may, by using a suitable binary transformation of X_5 and X_6, assume that two of α, β, γ are zero.

12. If we write

$$\xi_i \equiv x a_i + y b_i + z c_i, \qquad \eta \equiv x\alpha + y\beta + z\gamma,$$

the equation of an arbitrary quadric of the net III is

$$\xi_1 X_1^2 + \xi_2 X_2^2 + \xi_3 X_3^2 + \xi_4 X_4^2 + \xi_5 X_5^2 + \xi_6 X_6^2 + 2\eta X_5 X_6 = 0.$$

Writing this as

$$\xi_1 (X_2 + X_3 + X_4 + X_5 + X_6)^2 + \xi_2 X_2^2 + \xi_3 X_3^2 + \xi_4 X_4^2 + \xi_5 X_5^2 + \xi_6 X_6^2 + 2\eta X_5 X_6 = 0$$

we see that the discriminant of the quadric is

$$D \equiv \begin{vmatrix} \xi_1 + \xi_2 & \xi_1 & \xi_1 & \xi_1 & \xi_1 \\ \xi_1 & \xi_1 + \xi_3 & \xi_1 & \xi_1 & \xi_1 \\ \xi_1 & \xi_1 & \xi_1 + \xi_4 & \xi_1 & \xi_1 \\ \xi_1 & \xi_1 & \xi_1 & \xi_1 + \xi_5 & \xi_1 + \eta \\ \xi_1 & \xi_1 & \xi_1 & \xi_1 + \eta & \xi_1 + \xi_6 \end{vmatrix}$$

$$\begin{aligned} = \xi_2\xi_3\xi_4\xi_5\xi_6 + \xi_3\xi_4\xi_5\xi_6\xi_1 + \xi_4\xi_5\xi_6\xi_1\xi_2 + \xi_5\xi_6\xi_1\xi_2\xi_3 + \xi_6\xi_1\xi_2\xi_3\xi_4 + \xi_1\xi_2\xi_3\xi_4\xi_5 \\ - 2\eta\xi_1\xi_2\xi_3\xi_4 - \eta^2(\xi_2\xi_3\xi_4 + \xi_3\xi_4\xi_1 + \xi_4\xi_1\xi_2 + \xi_1\xi_2\xi_3). \end{aligned}$$

The equation of the plane quintic is $D = 0$. Writing this in the form

$$(\xi_2\xi_3\xi_4 + \xi_3\xi_4\xi_1 + \xi_4\xi_1\xi_2 + \xi_1\xi_2\xi_3)(\xi_5\xi_6 - \eta^2) + \xi_1\xi_2\xi_3\xi_4(\xi_5 + \xi_6 - 2\eta) = 0,$$

we see that the curve passes through the six vertices of the quadrilateral formed by the four lines

$$\xi_1 = 0, \quad \xi_2 = 0, \quad \xi_3 = 0, \quad \xi_4 = 0,$$

and also through all the intersections of these four lines with the conic $\xi_5 \xi_6 = \eta^2$. Furthermore the remaining two intersections of the conic with the quintic are its two intersections with the line $\xi_5 + \xi_6 = 2\eta$ and, as this is a tangent of the conic, these two intersections coincide in a single contact. The tangent of ζ at the point $\xi_i = \xi_j = 0$ $(i, j = 1, 2, 3, 4)$ is $\xi_i + \xi_j = 0$. The point p_{ij} on ζ is the intersection of the two sides of the quadrilateral other than $\xi_i = 0$ and $\xi_j = 0$.

This form of the equation for ζ enables us easily to verify the existence of the contact curves that have been mentioned. For example, the cubic

$$\xi_2\xi_3\xi_4 + \xi_3\xi_4\xi_1 + \xi_4\xi_1\xi_2 + \xi_1\xi_2\xi_3 = 0$$

touches ζ at each of the six vertices of the quadrilateral formed by the four lines $\xi_i = 0$; its remaining three intersections with ζ are its intersections with $\xi_5 + \xi_6 = 2\eta$, and this is the tangent of ζ at o. If we write the equation of ζ as

$$\xi_1\xi_2(\xi_5\xi_6 - \eta^2)(\xi_3 + \xi_4) + \xi_3\xi_4\{(\xi_1 + \xi_2)(\xi_5\xi_6 - \eta^2) + \xi_1\xi_2(\xi_5 + \xi_6 - 2\eta)\} = 0,$$

we see that the intersections of ζ with the cubic

$$(\xi_1 + \xi_2)(\xi_5\xi_6 - \eta^2) + \xi_1\xi_2(\xi_5 + \xi_6 - 2\eta) = 0$$

consist of the three intersections of ζ with its tangent $\xi_3 + \xi_4 = 0$, other than the point of contact p_{12} of this tangent, and of the twelve intersections of the cubic with the quartic curve $\xi_1\xi_2(\xi_5\xi_6 - \eta^2) = 0$. But it is clear, from the form of its equation, that the cubic passes through the intersection of the two lines $\xi_1 = 0$, $\xi_2 = 0$ and also through the intersections of both these lines with the conic $\xi_5\xi_6 = \eta^2$; also that the cubic touches the conic at its point of contact with $\xi_5 + \xi_6 = 2\eta$; hence the twelve intersections of the cubic and quartic coincide in pairs, and the cubic therefore touches ζ at six points; these points are p_{34}, o, o_1, o_1', o_2, o_2'.

When the equation of ζ is written

$$\xi_2\xi_3\xi_4\{\xi_5\xi_6 - \eta^2 + \xi_1(\xi_5 + \xi_6 - 2\eta)\} + \xi_1(\xi_5\xi_6 - \eta^2)(\xi_3\xi_4 + \xi_4\xi_2 + \xi_2\xi_3) = 0,$$

we see that the two conics

$$\xi_5\xi_6 - \eta^2 + \xi_1(\xi_5 + \xi_6 - 2\eta) = 0,$$

$$\xi_3\xi_4 + \xi_4\xi_2 + \xi_2\xi_3 = 0,$$

are both tritangent conics of ζ and that their four intersections lie on ζ; the first conic touches ζ at the three points o, o_1, o_1' and the second conic touches ζ at the three points p_{12}, p_{13}, p_{14}. And so on.

13. The locus of points in which any given secant plane α is met by the other secant planes of ϑ is, in general, a curve of order 13 with quintuple points at the intersections of α with ϑ (cf. *G.N.Q.* § 6). If however α meets ϑ in the six vertices of a quadrilateral the sides of this quadrilateral, each of which is a trisecant of ϑ and lies in two secant planes other than α, all count twice as parts of the curve of order 13; hence the residual part is a quintic curve passing through the vertices of the quadrilateral. Thus, for this particular curve ϑ, the double surface of the locus R_3^{15} generated by the secant planes of ϑ contains a plane quintic curve. This quintic curve circumscribes the quadrilateral formed by the four trisecants of ϑ which lie in α, and is in birational correspondence with ζ.

Apart from the four trisecants of ϑ which lie in α and the four trisecants of ϑ which pass through O ϑ has twelve other trisecants, these consisting of six conjugate pairs.

The Jacobian Curve whose Trisecants are the Edges of a Hexahedron.

14. We proceed now to consider the net of quadrics whose canonical form is

$$\left.\begin{aligned} a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_4X_4^2 + a_5X_5^2 + a_6X_6^2 = 0,\\ b_1X_1^2 + b_2X_2^2 + b_3X_3^2 + b_4X_4^2 + b_5X_5^2 + b_6X_6^2 = 0,\\ c_1X_1^2 + c_2X_2^2 + c_3X_3^2 + c_4X_4^2 + c_5X_5^2 + c_6X_6^2 = 0, \end{aligned}\right\} \qquad \text{II}$$

where

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \equiv 0.$$

The six solids $X = 0$ are the faces of a hexahedron; this hexahedron, which we may call $\mathfrak{H}$, has fifteen vertices, twenty edges, and fifteen planes. Since we can combine the three equations II linearly so that any two of the six squares disappear it follows that each of the fifteen vertices of $\mathfrak{H}$ is the vertex of a cone belonging to the net; hence *the Jacobian curve ϑ is circumscribed to $\mathfrak{H}$*. Moreover the polar prime of any vertex of $\mathfrak{H}$ in regard to every quadric II contains the opposite plane of $\mathfrak{H}$ (the plane of intersection of two faces of $\mathfrak{H}$ being 'opposite' to that vertex which is the intersection of the remaining four faces), so that the fifteen planes of $\mathfrak{H}$ are secant planes of ϑ, being conjugate, respectively, to the opposite vertices of $\mathfrak{H}$. Also, since each edge of $\mathfrak{H}$ contains three vertices, the twenty edges of $\mathfrak{H}$ are trisecants of ϑ and each plane of $\mathfrak{H}$, as it contains four edges, meets ϑ in the six vertices of a quadrilateral. The twenty trisecants of ϑ are thus all accounted for in this way; *the trisecants of the Jacobian curve of a net of quadrics given by the canonical form II are the edges of a hexahedron.* Two trisecants are conjugate when they are opposite edges of $\mathfrak{H}$; i.e. when the three faces of $\mathfrak{H}$ passing through one of the trisecants and the three faces of $\mathfrak{H}$ passing through the other together constitute all the six faces of $\mathfrak{H}$. The double surface of R_3^{15} now contains fifteen plane quintic curves; these are all in birational correspondence with ϑ and with each other.

Any solid which contains a secant plane of ϑ meets ϑ further in four points which are vertices of four cones belonging to a pencil; hence, if ζ is any plane quintic in birational correspondence with ϑ, the vertices of the tetrahedron in which any face of $\mathfrak{H}$ is met by four of the other faces correspond to four collinear points on ζ. It follows immediately that any five faces of $\mathfrak{H}$ form a simplex, inscribed in ϑ, whose five vertices correspond to five collinear points of ζ, and hence that the fifteen vertices of $\mathfrak{H}$ correspond to fifteen points of ζ which are the intersections of six lines. Hence

If the Jacobian curve of a net of quadrics in [4] *is circumscribed to a hexahedron, then any plane quintic which is in birational correspondence with it is circumscribed to a hexagram.*

The intersections of the quintic with a side of the hexagram correspond to the vertices of a simplex formed by five faces of the hexahedron; the ten vertices of a pentagram formed by five lines of the hexagram correspond to the ten intersections of the Jacobian curve with a face of the hexahedron. The ten vertices of the pentagram must therefore be the points of contact of the quintic with a contact quartic. Also the vertices of the quadrilateral formed by any four lines of the hexagram correspond to the six intersections of the Jacobian curve with a plane of the hexahedron; hence there is a cubic curve touching the quintic at each vertex of the quadrilateral, the remaining three intersections of the curves being the intersections of the quintic with its tangent at the point of intersection of the remaining two sides of the hexagram. The vertices of a triangle formed by three sides of the hexagram correspond to three collinear points of the Jacobian curve, so that there is a conic touching the quintic at the vertices of such a triangle. If we take two such triangles whose sides consist of all the six lines of the hexagram the associated trisecants of ϑ are conjugate, so that the four intersections of the two tritangent conics are on the quintic curve.

The equation of the quintic is obtained immediately by writing zero instead of η in the determinant D of § 12; it is therefore

$$\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1} + \xi_4^{-1} + \xi_5^{-1} + \xi_6^{-1} = 0,$$

and the curve passes through the intersection of any two of the six lines $\xi_i = 0$. The quartic curve

$$\xi_2^{-1} + \xi_3^{-1} + \xi_4^{-1} + \xi_5^{-1} + \xi_6^{-1} = 0$$

touches the quintic at the ten vertices of the pentagram whose sides are the sides of the hexagram with $\xi_1 = 0$ omitted; also the equations of the cubic curves and tritangent conics just referred to can at once be written down.

Although every plane quintic which is in birational correspondence with a Jacobian curve having an inscribed hexahedron has an inscribed hexagram it is not true, conversely, that every Jacobian curve which is in birational correspondence with a plane quintic having an inscribed hexagram has an inscribed hexahedron. Nevertheless, when a plane quintic has an inscribed hexagram, a Jacobian curve can always be found which is in birational correspondence with it and which has an inscribed hexahedron. For we can always suppose that the equation of such a plane quintic is

$$\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1} + \xi_4^{-1} + \xi_5^{-1} + \xi_6^{-1} = 0,$$

where

$$\xi_i \equiv x a_i + y b_i + z c_i = 0 \qquad (i = 1, 2, 3, 4, 5, 6)$$

are the equations to the sides of the hexagram. The equation to this quintic, when written in the determinantal form, is at once seen to be the condition that a quadric of the net II should be a cone, and the Jacobian curve of the net II fulfills the required conditions.

15. The locus of the poles of any solid S_3 in regard to the quadrics of the net is a sextic surface on which lie ten lines. The line conjugate to any point of this surface lies in S_3 and, conversely, if a point is such that the line conjugate to it lies in S_3 then the point must lie on the surface (*G.N.Q.* § 13). Consider now the particular case when S_3 is a face of the hexahedron. The remaining five faces of the hexahedron form a simplex whose ten edges are trisecants of ϑ; the line conjugate to any point of any of these trisecants is the conjugate trisecant of ϑ, and lies in S_3. Hence the sextic surface which is the locus of poles of S_3 passes through the ten edges of the simplex, and these are the ten lines which lie on the surface.

The Jacobian Curve with a Scroll of Trisecants.

16. When a plane quintic has an inscribed hexagram the sides of the hexagram will not, in general, touch a conic. We now prove that if a plane quintic has an inscribed hexagram whose sides touch a conic then it has an infinity of inscribed hexagrams whose sides all touch this same conic.

Let us suppose that the hexagram consists of the tangents to the conic $xz = y^2$ at the six points $(\theta_i^2, \theta_i, 1)$, with $i = 1, 2, 3, 4, 5, 6$. Then the equation of the quintic is of the form

$$\sum_{i=1}^{6} \frac{\lambda_i}{x - 2y\theta_i + z\theta_i^2} = 0.$$

Now take any point on this curve; if φ and ψ are the parameters of the points of contact of the two tangents which can be drawn to the conic from this point,

$$\sum_{i=1}^{6} \frac{\lambda_i}{(\varphi - \theta_i)(\psi - \theta_i)} = 0.$$

If φ is given this is a quintic equation for ψ; it gives the parameters of the points of contact with the conic of those of its tangents which pass through the five intersections of the quintic with the tangent to the conic at the point whose parameter is φ. Suppose that ψ_1 and ψ_2 are two roots of this quintic. Then

$$\Sigma \lambda_i (\varphi - \theta_i)^{-1} (\psi_1 - \theta_i)^{-1} = 0,$$

$$\Sigma \lambda_i (\varphi - \theta_i)^{-1} (\psi_2 - \theta_i)^{-1} = 0.$$

Subtracting

$$(\psi_2 - \psi_1)\,\Sigma\,\lambda_i\,(\varphi - \theta_i)^{-1}(\psi_1 - \theta_i)^{-1}(\psi_2 - \theta_i)^{-1} = 0.$$

Adding

$$(\psi_2 + \psi_1)\,\Sigma\lambda_i\,(\varphi - \theta_i)^{-1}(\psi_1 - \theta_i)^{-1}(\psi_2 - \theta_i)^{-1} - 2\,\Sigma\,\lambda_i\,\theta_i\,(\varphi - \theta_i)^{-1}(\psi_1 - \theta_i)^{-1}(\psi_2 - \theta_i)^{-1} = 0.$$

The first of these last two equations gives

$$\Sigma\,\lambda_i\,(\varphi - \theta_i)^{-1}(\psi_1 - \theta_i)^{-1}(\psi_2 - \theta_i)^{-1} = 0,$$

and the second then gives

$$\Sigma\,\lambda_i\,\theta_i\,(\varphi - \theta_i)^{-1}(\psi_1 - \theta_i)^{-1}(\psi_2 - \theta_i)^{-1} = 0.$$

Combining these two results we finally obtain

$$\Sigma\,\lambda_i\,(\psi_1 - \theta_i)^{-1}(\psi_2 - \theta_i)^{-1} = 0,$$

and this is the condition that the tangents of the conic at the points ψ_1 and ψ_2 should intersect on the quintic. Hence it has been shown that, if any tangent of the conic is taken, the other five tangents which can be drawn to the conic from the five points in which this tangent meets the quintic are such that the intersection of any pair of them is on the quintic. Thus any tangent of the conic determines a hexagram whose sides are all tangents of the conic and which is inscribed in the quintic; since any tangent of the conic belongs to one and only one hexagram the sets of tangents of the conic which compose the hexagrams are the sets of an involution g_6^1. This same argument also proves that if a curve C^n of order n passes through the $\frac{1}{2}n(n+1)$ intersections of $n+1$ tangents of a conic then there is an infinity of sets of $n+1$ tangents of the same conic such that the intersection of any two tangents of the conic that belong to the same set is on C^n. In particular, since any five lines touch a conic, the case $n = 4$ gives Lüroth's porism: if a quartic curve is circumscribed to a pentagram then it is circumscribed to an infinity of pentagrams.

17. Conversely: if we taken an involution g_6^1 consisting of sets of tangents of a conic γ the locus of intersections of pairs of tangents of γ which belong to the same set of g_6^1 is a quintic curve. This can be shown if we appeal to the theory of correspondence; for the g_6^1 sets up a $(5, 5)$ correspondence between the tangents of the curve γ of class 2, and there are ten tangents of γ which

are self-corresponding; hence, since the correspondence is symmetrical, the order of the curve which is the locus of intersections of corresponding pairs of tangents is $\frac{1}{2}\{2.5 + 2.5 - 10\} = 5$. This result however can also be established easily by direct algebra, without any appeal to correspondence theory. For suppose that the six tangents, of the conic $xz = y^2$, which belong to a set of g_6^1 are those the parameters of whose points of contact are the roots of the sextic

$$a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5 + a_6\theta^6 + \\ + \lambda(b_0 + b_1\theta + b_2\theta^2 + b_3\theta^3 + b_4\theta^4 + b_5\theta^5 + b_6\theta^6) = 0,$$

different sets of g_6^1 being given by different values of λ. If (x, y, z) is the intersection of the tangents of the conic at two points whose parameters are the roots of this sextic it follows, since these two parameters are the roots of the quadratic $x - 2y\theta + z\theta^2 = 0$, that there must be an identity of the form

$$a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5 + a_6\theta^6 + \\ + \lambda(b_0 + b_1\theta + b_2\theta^2 + b_3\theta^3 + b_4\theta^4 + b_5\theta^5 + b_6\theta^6) \equiv \\ \equiv (x - 2y\theta + z\theta^2)(\alpha + \beta\theta + \gamma\theta^2 + \delta\theta^3 + \varepsilon\theta^4).$$

Equating the coefficients of the different powers of θ in this identity, and eliminating λ, α, β, γ, δ, ε from the resulting seven equations, we see that (x, y, z) must lie on the curve

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ x & -2y & z & 0 & 0 & 0 & 0 \\ 0 & x & -2y & z & 0 & 0 & 0 \\ 0 & 0 & x & -2y & z & 0 & 0 \\ 0 & 0 & 0 & x & -2y & z & 0 \\ 0 & 0 & 0 & 0 & x & -2y & z \end{vmatrix} = 0.$$

This is a quintic curve; it is circumscribed to all the hexagrams formed by sets of the g_6^1.

18. The method by which we have established that the locus of intersections of pairs of lines belonging to sets of a g_6^1 among the tangents of a

conic is a quintic curve will also establish similarly the fact that, if a g_5^1 is taken whose sets consist of tangents of a conic, the locus of intersections of pairs of lines belonging to sets of the g_5^1 is a quartic curve. Any two sets of five tangents of a conic determine a g_5^1; hence if any two pentagrams are taken which are circumscribed to the same conic, their twenty vertices lie on a quartic curve. Now consider the quintic curve ζ, obtained above as the locus of vertices of hexagrams given by the sets of a g_6^1 among the tangents of the conic γ. If we take any two of the hexagrams, and omit one side of each, we obtain two pentagrams whose vertices, as well as being on ζ, all lie on the same quartic curve; the twenty vertices of the two pentagrams therefore form the complete set of intersections of ζ and the quartic curve. If, for the moment, we keep one of these pentagrams fixed and allow the other to vary continuously (the second pentagram is determined when the omitted side of the hexagram, to which it belongs, is given) we see that if any pentagram is taken whose sides are all sides of the same hexagram inscribed in ζ its ten vertices are points of contact of ζ with a contact quartic. We have seen that any two such sets of contacts all lie on the same quartic, so that all the contact quartics obtained belong to the same system; moreover, since the set of vertices of a pentagram includes sets of four collinear points, the system of contact quartics cannot be one of those 2015 systems whose members are such that their sets of ten contacts with ζ lie on cubic curves; it must then be one of the 2080 systems of the first kind. Wherefore *ζ can be put into correspondence with the Jacobian curve ϑ of a net of quadrics in [4] in such a way that the ten vertices of a pentagram on ζ always correspond to ten cospatial points on ϑ.* The six vertices of a quadrilateral whose sides all belong to the same inscribed hexagram of ζ therefore correspond to six coplanar points of ϑ, and the three vertices of a triangle whose sides all belong to the same inscribed hexagram of ζ correspond to three collinear points of ϑ. *Thus ϑ has a scroll of trisecants and an infinity of inscribed hexahedra.* We have seen previously that, when ϑ has an inscribed hexahedron, ζ has a corresponding inscribed hexagram; hence, if ϑ has an infinity of inscribed hexahedra, any plane quintic in birational correspondence with it has an infinity of inscribed hexagrams. It can be shown directly (see § 20 below) that the sides of these hexagrams all touch the same conic; whence we have the following:

There exist in [4] *nets of quadrics whose Jacobian curves are circumscribed to an infinity of hexahedra; any plane quintic which is in birational correspondence with such a Jacobian curve is circumscribed to an infinity of hexagrams whose sides all touch the same conic. Conversely: if a plane quintic is circumscribed to an infinity of hexagrams whose sides all touch the same conic it can always be put in birational correspondence with a Jacobian curve in* [4] *that is circumscribed to an infinity of hexahedra.*

19. Through any point of [4] there pass a finite number of faces of the hexahedra inscribed in ϑ; if this number is n we say that the faces of the hexahedra generate a developable of class n. This developable is rational. For the ten intersections of ϑ with a face of a hexahedron correspond to ten points of ζ which are the vertices of a pentagram belonging to an inscribed hexagram; the sixth side of this hexagram may therefore be regarded as corresponding to the face of the hexahedron, so that the faces of the hexahedra are in (1, 1) correspondence with the tangents of a conic. The class n can be obtained immediately by means of the principle of correspondence; for the hexahedra are the sets of a linear series g_6^1 of sets of solids of the developable, and the plane of intersection of two faces of the same hexahedron is a secant plane of ϑ; hence the locus R_3^{15} of secant planes of ϑ is generated by the intersections of pairs of solids which correspond to one another in a symmetrical (5, 5) correspondence between the solids of the developable of class n. Since the developable is rational the number of self-corresponding solids is 10; hence

$$\frac{1}{2}(5\,n + 5\,n - 10) = 15,$$

$$n = 4.$$

The faces of the hexahedra which are inscribed in ϑ *generate a developable of class 4.* The hexahedra can therefore be obtained as the sets of a g_6^1 among the osculating solids of a rational normal quartic curve.

20. Having shown that the developable generated by the faces of the hexahedra is of class 4 it is interesting to prove, from the four-dimensional figure, that the sides of the hexagrams which are inscribed in any plane quintic which is in birational correspondence with ϑ all touch the same conic. These lines envelop a curve whose class is the number of sides of hexagrams passing

through an arbitrary point of the plane; this point represents a quadric Q belonging to the net in [4], and the five points of the quintic which lie on a side of a hexagram through the point correspond to five points of ϑ which are vertices of a simplex whose faces all belong to the same hexahedron and which is a self-conjugate simplex in regard to Q. The faces of such a simplex are common to two developables; namely the developable generated by the faces of the hexahedra and the developable generated by the polar solids of the points of ϑ in regard to Q. Conversely: suppose P_1 is a point of ϑ such that the polar solid of P_1 in regard to Q is a face of one of the hexahedra inscribed in ϑ; this solid meets ϑ in the six points of the secant plane conjugate to P_1 and in four further points P_2, P_3, P_4, P_5 which are the vertices of the remaining four cones that belong to the pencil determined by (P_1) and Q. Then the respective polar solids of P_2, P_3, P_4, P_5 in regard to Q are $P_3 P_4 P_5 P_1$, $P_4 P_5 P_1 P_2$, $P_5 P_1 P_2 P_3$, $P_1 P_2 P_3 P_4$; and these are four of the faces of the hexahedron to which $P_2 P_3 P_4 P_5$ belongs. Thus the five points p_1, p_2, p_3, p_4, p_5 on the plane quintic are its intersections with a side of a hexagram, and the line on which they lie passes through the point that represents Q. Wherefore the class of the envelope of the sides of the hexagrams is one-fifth of the number of solids common to the two developables. Now the solids of both these developables pass through the generating planes of the locus R_3^{15}, so that the number of solids common to them can be calculated at once by the formula dual to that which gives the number of intersections of two curves on a ruled surface.[1] The developable generated by the faces of the hexahedra is of class 4; each solid of this developable contains five planes of R_3^{15} and each plane of R_3^{15} lies in two solids of the developable. The developable generated by the polar solids of the points of ϑ in regard to Q is of class 10; each solid of this developable contains one plane of R_3^{15} while each plane of R_3^{15} lies in one solid of the developable. Hence, by the formula referred to, the number of solids common to the two developables is

$$4 \cdot 5 \cdot 1 + 10 \cdot 1 \cdot 2 - 15 \cdot 2 \cdot 1 = 10.$$

Hence the sides of the hexagrams envelop a curve of class 2.

[1] If we have, on a ruled surface of order n, a curve of order m meeting each generator in k points and such that s generators pass through each point of the curve, and also a curve of order m' meeting each generator in k' points and such that s' generators pass through each point of the curve, then the number of intersections of the two curves is $msk' + m's'k - nkk'$

21. When the quadrics of the net II are represented by the points of a plane the quintic curve whose points represent the cones of the net is circumscribed to the hexagram formed by the six lines $a_i x + b_i y + c_i z = 0$. The necessary and sufficient condition that these lines should touch a conic is

$$\Delta \equiv \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & a_6^2 \\ b_1^2 & b_2^2 & b_3^2 & b_4^2 & b_5^2 & b_6^2 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 & c_5^2 & c_6^2 \\ b_1 c_1 & b_2 c_2 & b_3 c_3 & b_4 c_4 & b_5 c_5 & b_6 c_6 \\ c_1 a_1 & c_2 a_2 & c_3 a_3 & c_4 a_4 & c_5 a_5 & c_6 a_6 \\ a_1 b_1 & a_2 b_2 & a_3 b_3 & a_4 b_4 & a_5 b_5 & a_6 b_6 \end{vmatrix} = 0.$$

Thus when the coefficients in the canonical form II satisfy the relation $\Delta = 0$ there is an infinity of canonical forms II for the same net of quadrics, and the coefficients in any one of these canonical forms satisfy the relation analogous to $\Delta = 0$.

22. Suppose now that a net of quadrics can be reduced to the canonical form II, and that the coefficients satisfy the relation $\Delta = 0$. Then the equations of the quadrics can be expressed in terms of either of two (among an infinite number of) sets of six squares; the first quadric, for example, in this way gives rise to an identity

$$a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + a_4 X_4^2 + a_5 X_5^2 + a_6 X_6^2 \equiv$$
$$\equiv a_1' Y_1^2 + a_2' Y_2^2 + a_3' Y_3^2 + a_4' Y_4^2 + a_5' Y_5^2 + a_6' Y_6^2.$$

This identity involves the squares of twelve linear forms. Suppose we take any quadric which touches k of the twelve solids obtained by equating these linear forms to zero, and then substitute differential operators for the coordinates in the prime equation of this quadric. If we then operate on the identity with the differential operator so constructed, the squares of the k linear forms corresponding to the solids touched by the quadric are annihilated. Suppose, in particular, that $k = 9$, and that the quadric touches all the twelve solids except the three $X_1 = 0$, $X_2 = 0$, $X_3 = 0$; the application of the operator corresponding to such a quadric gives

$$\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0,$$

where $\lambda_1, \lambda_2, \lambda_3$ are constants depending on the quadric. Also we obtain, by means of the same differential operator, relations

$$\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 = 0,$$

$$\lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3 = 0,$$

arising from identities associated with other quadrics of the net. But the coefficients in II do not satisfy any relation other than $\Delta = 0$, so that the three linear equations just obtained for $\lambda_1, \lambda_2, \lambda_3$ can only be satisfied by taking

$$\lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Thus, whatever quadric we choose to touch the nine solids, the differential operator constructed from its prime equation annihilates not only the nine squares corresponding to these solids, but also the other three squares as well; hence all quadrics which touch the nine solids touch all the twelve solids, and the twelve solids are touched by ∞^5 quadrics. The conclusion is, in general, that the twelve solids all osculate the same rational normal quartic curve.[1]

Conversely: if twelve solids osculate a rational normal quartic curve there are three linearly independent identities connecting their squares; for the equation of any one of the solids may be taken as

$$P_i \equiv x_0 + x_1 \theta_i + x_2 \theta_i^2 + x_3 \theta_i^3 + x_4 \theta_i^4 = 0,$$

and so the twelve squares P_i^2 are linearly dependent from nine quadratic functions of the coordinates x_0, x_1, x_2, x_3, x_4. If then the twelve solids are divided into two sets of six there are three linearly independent quadrics whose equations are expressible in terms of either set of six squares. We can indeed give the explicit form of the equations. For suppose we take the six primes

$$x_0 + x_1 \theta_i + x_2 \theta_i^2 + x_3 \theta_i^3 + x_4 \theta_i^4 = 0, \qquad (i = 1, 2, 3, 4, 5, 6)$$

which osculate the quartic curve at the points whose parameters are $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$ and also the six primes

$$x_0 + x_1 \varphi_i + x_2 \varphi_i^2 + x_3 \varphi_i^3 + x_4 \varphi_i^4 = 0, \qquad (i = 1, 2, 3, 4, 5, 6)$$

which osculate the quartic curve at the points whose parameters are $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6$. If we write

[1] See, for a corresponding argument in [3], A. C. Dixon, *Proc. London Math. Soc.* (2), 7 (1909), 153.

$$f(t) \equiv (t-\theta_1)(t-\theta_2)(t-\theta_3)(t-\theta_4)(t-\theta_5)(t-\theta_6),$$
$$g(t) \equiv (t-\varphi_1)(t-\varphi_2)(t-\varphi_3)(t-\varphi_4)(t-\varphi_5)(t-\varphi_6),$$

it can be verified that, for all values of the constants α, β, γ,

$$\sum_{i=1}^{6} \frac{(\alpha+\beta\theta_i+\gamma\theta_i^2)(x_0+x_1\theta_i+x_2\theta_i^2+x_3\theta_i^3+x_4\theta_i^4)^2}{f'(\theta_i)\,g(\theta_i)} + \\ + \sum_{i=1}^{6} \frac{(\alpha+\beta\varphi_i+\gamma\varphi_i^2)(x_0+x_1\varphi_i+x_2\varphi_i^2+x_3\varphi_i^3+x_4\varphi_i^4)^2}{f(\varphi_i)\,g'(\varphi_i)} \equiv 0.$$

We thus obtain the three quadrics

$$Q_0 \equiv \sum_{i=1}^{6} \frac{(x_0+x_1\theta_i+x_2\theta_i^2+x_3\theta_i^3+x_4\theta_i^4)^2}{f'(\theta_i)\,g(\theta_i)} \equiv \\ \equiv -\sum_{i=1}^{6} \frac{(x_0+x_1\varphi_i+x_2\varphi_i^2+x_3\varphi_i^3+x_4\varphi_i^4)^2}{f(\varphi_i)\,g'(\varphi_i)} = 0,$$

$$Q_1 \equiv \sum_{i=1}^{6} \frac{\theta_i(x_0+x_1\theta_i+x_2\theta_i^2+x_3\theta_i^3+x_4\theta_i^4)^2}{f'(\theta_i)\,g(\theta_i)} \equiv \\ \equiv -\sum_{i=1}^{6} \frac{\varphi_i(x_0+x_1\varphi_i+x_2\varphi_i^2+x_3\varphi_i^3+x_4\varphi_i^4)^2}{f(\varphi_i)\,g'(\varphi_i)} = 0,$$

$$Q_2 \equiv \sum_{i=1}^{6} \frac{\theta_i^2(x_0+x_1\theta_i+x_2\theta_i^2+x_3\theta_i^3+x_4\theta_i^4)^2}{f'(\theta_i)\,g(\theta_i)} \equiv \\ \equiv -\sum_{i=1}^{6} \frac{\varphi_i^2(x_0+x_1\varphi_i+x_2\varphi_i^2+x_3\varphi_i^3+x_4\varphi_i^4)^2}{f(\varphi_i)\,g'(\varphi_i)} = 0.$$

23. Let us now consider some of the loci associated with the Jacobian curve ϑ that is circumscribed to an infinity of hexahedra. The ∞^1 trisecants of ϑ are the edges of the hexahedra, and are associated in conjugate pairs, two trisecants being conjugate when they are opposite edges of the same hexahedron; the polar solids of any point on a trisecant of ϑ in regard to the quadrics of the net all pass through the conjugate trisecant. Each point P of ϑ is the vertex of one and only one hexahedron; the four faces of the hexahedron which intersect in this vertex intersect by threes in the four edges of the hexahedron passing through that vertex; *ϑ is a quadruple curve on the scroll of its trisecants.* We can determine the order ν of this scroll, for since its generators are all

trisecants of ϑ any cubic primal containing ϑ meets the scroll in ϑ, counted four times, and $3\nu - 40$ generators. Suppose, in particular, that the cubic primal containing ϑ is the primal Π which is the locus of lines conjugate, in regard to the net of quadrics, to the points of a plane π; there are ν trisecants of ϑ which meet π, and the ν trisecants which are conjugate to these lie on Π. Conversely: if a trisecant of ϑ lies on Π, and is conjugate to a point of π, its conjugate trisecant must be one of the ν trisecants which meet π. Moreover no trisecant of ϑ can lie on Π unless it is conjugate to some point of π. For suppose t is any trisecant of ϑ which lies on Π; through an arbitrary point O' on t, as through any point on Π, there passes a line j conjugate to some point O of π; the line which is conjugate to O' therefore passes through O. But since O' is on t its conjugate line must be the trisecant conjugate to t; hence the line conjugate to O is t, and j coincides with t. It follows that Π contains ν trisecants of ϑ, and no more; hence

$$3\nu - 40 = \nu,$$

$$\nu = 20.$$

The trisecants of ϑ generate a scroll R_2^{20} of order 20, on which ϑ is a quadruple curve.

Any trisecant of ϑ is joined to its conjugate trisecant by a solid Σ, so that the solids Σ are determined as the joins of pairs of generators of R_2^{20} which correspond to one another in a symmetrical (1, 1) correspondence. Since no generator of R_2^{20} can intersect its corresponding generator *the solids Σ generate a developable D of class 20.*

24. It is known (*G. N. Q.* § 28) that any two canonical sets on ϑ form the complete intersection of ϑ with a quadric. Now a particular canonical set, as was seen in § 8 above, consists of any point P of ϑ, counted twice, and the eight other points in which the four trisecants through P meet ϑ; any quadric which contains this canonical set contains the four trisecants of ϑ through P and therefore, since these four trisecants do not lie in a solid, this quadric is a cone with vertex P. In particular: the quadric which contains the two canonical sets which arise in this way from two different points P_1 and P_2 of ϑ is a line-cone whose vertex is $P_1 P_2$; thus the projection of ϑ from any one of its chords is a plane octavic eight of whose fifteen nodes are on a conic. When the two points P_1 and P_2 coincide with the same point P of ϑ we have the re-

sult that the tangent of ϑ at any point P is the vertex of a line-cone $\{P\}$ which touches ϑ at each of the eight points in which ϑ is met by the four trisecants through P.

25. The tangent planes of a ruled surface at the different points of a generator all lie in the same solid; thus if any curve, in space of dimension greater than or equal to 4, has an infinity of trisecants, the three tangents of the curve at the points where it is met by any one of its trisecants lie in a solid, this solid being the tangent solid, along this particular trisecant, of the scroll generated by the trisecants. In particular the three tangents of ϑ at its intersections with any trisecant are cospatial[1]; if ϑ is projected on to a plane from one of its tangents, the tangent being supposed not to meet ϑ except at its point of contact, it becomes a plane octavic with four tacnodes; since the octavic is of genus 6 it will have seven ordinary nodes in addition to its tacnodes.

Any solid which contains a trisecant of ϑ touches R_2^{20} at some point of this trisecant; consider then the section of R_2^{20} by a solid which is a face of a hexahedron inscribed in ϑ. The other five faces of the hexahedron meet this face in five planes, and the ten lines of intersection of pairs of these planes are all trisecants of ϑ and so form part of the curve of section; there are three of these ten lines passing through each of the ten intersections of ϑ with the solid. Thus the curve ψ, of order ten, which is the remaining part of the curve of section, must pass through the ten intersections of ϑ with the solid, as these are to be *quadruple* points of the curve of section. Each of the ten lines is therefore met by ψ in three points; it is also met by ψ in that point of the line at which the solid touches R_2^{20}. The complete curve of section thus consists of the ten edges of a pentahedron and a curve ψ, of order ten, passing through the ten vertices of the pentahedron and meeting each edge in one point other than the three vertices which lie on that edge. The edges of the pentahedron are quadrisecants of ψ. It follows that a secant plane of ϑ does not meet R_2^{20} except in the four trisecants of ϑ which lie in the plane; for the curve ψ which lies in either of the two faces of the hexahedron which contain the secant plane

[1] This follows immediately also from the existence of the line-cones $\{P\}$. The tangents of ϑ at its three intersections with a trisecant are cospatial because, if P is any one of the intersections of the trisecant with ϑ, they lie in the solid which touches $\{P\}$ along the plane joining the trisecant to the tangent of ϑ at P. The four tacnodes of the plane projection of ϑ are on a conic which touches the four tacnodal tangents.

is such that all its ten intersections with the plane are on these four trisecants. There are ∞^1 curves ψ on R_2^{20}, and they are all in birational correspondence with each other.

26. Since there are three secant planes of ϑ passing through each of its trisecants the scroll R_2^{20} is a triple surface on the locus R_3^{15} generated by the secant planes; thus the double surface of R_3^{15}, which is, in general, of order 85, now consists of R_2^{20} counted three times and of a surface F_2^{25} of order 25. This surface meets each secant plane of ϑ in a quintic curve passing through the six points of ϑ which lie in that plane (cf. § 13); there are thus ∞^1 plane quintics on F_2^{25}; they are all in birational correspondence with each other and with ϑ, and any two of them intersect in the point which is common to their planes. Through an arbitrary point of F_2^{25} there pass two of the plane quintics, but if the point lies on ϑ there are six quintic curves passing through it. To find the multiplicity of ϑ on the surface F_2^{25} we consider the section of R_3^{15} by a plane meeting ϑ in a point P; this is a curve of order 15 and genus 6 and having therefore the equivalent of 85 double points. It has a sextuple point at P; also, since P is a quadruple point of R_2^{20}, the plane meets R_2^{20} in sixteen further points, and these are triple points of the curve of section; it follows that the curve must have 22 further double points, and these are the intersections, apart from P, of the plane with F_2^{25}. Hence *ϑ is a triple curve on F_2^{25}.* The scroll which is the section of R_3^{15} by an arbitrary prime has a triple curve of order 20 meeting each generator in four points and a double curve of order 25 meeting each generator in five points; both these multiple curves pass through the ten intersections of the prime with ϑ, these being sextuple points of the scroll, quadruple points of the triple curve and triple points of the double curve.

The section of F_2^{25} by a face of a hexahedron inscribed in ϑ consists of five plane quintics. The section of R_3^{15} by a face of a hexahedron inscribed in ϑ consists of five planes and a scroll Ψ of order ten; the five plane quintics in which the solid meets F_2^{25} are all on Ψ but are not multiple curves. The curve ψ in which the solid meets R_2^{20} is a triple curve on Ψ, and the generators of Ψ are all quadrisecants of ψ.

27. Suppose now that the rational quartic curve γ^4 which is osculated by the faces of the hexahedra inscribed in ϑ is given by the the equations[1]

[1] We shall assume as known the properties of the rational quartic curve and of loci associated with it; for example the loci generated by its tangents, by its chords and by its osculating

$$x_0 : x_1 : x_2 : x_3 : x_4 = \theta^4 : -4\theta^3 : 6\theta^2 : -4\theta : 1,$$

so that the equation of the osculating solid at the point whose parameter is θ_i is

$$x_0 + x_1\theta_i + x_2\theta_i^2 + x_3\theta_i^3 + x_4\theta_i^4 = 0.$$

Suppose also that the g_6^1 on γ^4 which determines the hexahedra is given by the pencil of sextics which includes

$$a_0\theta^6 + a_1\theta^5 + a_2\theta^4 + a_3\theta^3 + a_4\theta^2 + a_5\theta + a_6 = 0$$

and

$$b_0\theta^6 + b_1\theta^5 + b_2\theta^4 + b_3\theta^3 + b_4\theta^2 + b_5\theta + b_6 = 0.$$

Then, if any point on ϑ is taken, the four points of γ^4 at which the osculating solids pass through this point must all belong to the same member of the pencil of sextics; hence, given any point $(x_0, x_1, x_2, x_3, x_4)$ on ϑ it must be possible to find constants $\lambda, \mu, \alpha, \beta, \gamma$ so that

$$\begin{aligned}\lambda(a_0\theta^6 + a_1\theta^5 + a_2\theta^4 + a_3\theta^3 + a_4\theta^2 + a_5\theta + a_6) &+ \\ + \mu(b_0\theta^6 + b_1\theta^5 + b_2\theta^4 + b_3\theta^3 + b_4\theta^2 + b_5\theta + b_6) &\equiv \\ \equiv (x_0 + x_1\theta + x_2\theta^2 + x_3\theta^3 + x_4\theta^4)(\alpha + \beta\theta + \gamma\theta^2).&\end{aligned}$$

Equating the different powers of θ on the two sides of this identity, and eliminating $\lambda, \mu, \alpha, \beta, \gamma$ from the resulting seven equations, we see that the coordinates of any point on ϑ satisfy the equations

$$\left\| \begin{matrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ x_4 & x_3 & x_2 & x_1 & x_0 & 0 & 0 \\ 0 & x_4 & x_3 & x_2 & x_1 & x_0 & 0 \\ 0 & 0 & x_4 & x_3 & x_2 & x_1 & x_0 \end{matrix} \right\| = 0.$$

These equations are those of a curve of the tenth order.[1]

28. If we take a point on R_2^{20} then there are four osculating solids of γ^4 passing through it, and three of the four points of contact must belong to the

planes. Many of these properties can be obtained very simply either from the projective method of generating the curve or from its parametric representation (equivalent to the above) first given by Clifford.

[1] For the order of a locus given by the vanishing of the determinants of a matrix see Salmon: *Higher Algebra* (Dublin, 1885), Lesson 19.

same member of the pencil of sextics. Hence, given any point $(x_0, x_1, x_2, x_3, x_4)$ on R_2^{20} there must exist constants $\lambda, \mu, \varrho, \sigma, \alpha, \beta, \gamma, \delta$, such that

$$\{\lambda(a_0\theta^6 + a_1\theta^5 + a_2\theta^4 + a_3\theta^3 + a_4\theta^2 + a_5\theta + a_6) + \\ + \mu(b_0\theta^6 + b_1\theta^5 + b_2\theta^4 + b_3\theta^3 + b_4\theta^2 + b_5\theta + b_6)\}(\varrho\theta + \sigma) \equiv \\ \equiv (x_0 + x_1\theta + x_2\theta^2 + x_3\theta^3 + x_4\theta^4)(\alpha + \beta\theta + \gamma\theta^2 + \delta\theta^3);$$

and, conversely, if the constants can be determined so that this identity holds, the point $(x_0, x_1, x_2, x_3, x_4)$ lies on R_2^{20}. We can eliminate at once the eight quantities $\lambda\varrho, \mu\varrho, \lambda\sigma, \mu\sigma, \alpha, \beta, \gamma, \delta$ from the eight equations obtained by equating the coefficients of the different powers of θ on the two sides of the identity; we thus obtain the equation

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ x_4 & x_3 & x_2 & x_1 & x_0 & 0 & 0 & 0 \\ 0 & x_4 & x_3 & x_2 & x_1 & x_0 & 0 & 0 \\ 0 & 0 & x_4 & x_3 & x_2 & x_1 & x_0 & 0 \\ 0 & 0 & 0 & x_4 & x_3 & x_2 & x_1 & x_0 \end{vmatrix} = 0.$$

This is the equation of a primal of the fourth order; hence R_2^{20} *lies on a quartic primal* Φ. Also if $(x_0, x_1, x_2, x_3, x_4)$ satisfies the equations of ϑ, every first minor of this eight-rowed determinant vanishes; hence ϑ *is a double curve on* Φ. Since the surface of intersection of two quartic primals is only of order sixteen, and since any quartic primal on which ϑ is a double curve must contain all the trisecants of ϑ, which generate a surface of order twenty, there is no quartic primal other than Φ which has ϑ as a double curve. Also, since R_2^{20} is a triple surface on R_3^{15}, the complete intersection of R_3^{15} with Φ consists of R_2^{20} counted three times.

The polars of any three points of [4] in regard to Φ are cubic primals all containing the curve ϑ; their residual curve of intersection is of order 17 and has 30 intersections[1] with ϑ. But there are exactly 30 tangents of ϑ which

[1] If three primals in [4] of orders n_1, n_2, n_3 pass through a curve of order ε_0 and rank ε_1 their residual curve of intersection meets this curve in $\varepsilon_0(n_1+n_2+n_3-3)-\varepsilon_1$ points. See Veronese: *Math. Annalen* 19 (1882), 205.

meet the plane of the three points, so that these account for all the points common to ϑ and the residual curve of intersection of the three cubic primals; thus Φ cannot have any bispatial points on ϑ. It follows that the class of Φ is $4 \cdot 17 - 2 \cdot 30 = 8$.

The lines which pass through any point P of the double curve ϑ of the primal Φ all have two intersections with Φ at P, and those which have three intersections with Φ at P generate a quadric line-cone whose vertex is the tangent of ϑ at P. Now these lines include the four trisecants of ϑ which pass through P, so that the four planes which join the tangent of ϑ at P to these trisecants are generating planes of this line-cone, as they are of the cone $\{P\}$.

It can in fact be shown that $\{P\}$ *is actually the tangent cone of* Φ *at* P. The section of Φ by any solid which contains a trisecant of ϑ is a quartic surface with three collinear nodes P, Q, R; and it can easily be shown that any such quartic surface has the same tangent plane at every point of the line PQR and, furthermore, that this plane is also a tangent plane of each of the three nodal cones,[1] touching them all along PQR. It follows that Φ must have the same tangent solid at all points of a trisecant PQR of ϑ, and that this solid touches the tangent line-cones of Φ at each of the points P, Q, R. This solid must therefore contain the tangents of ϑ at the points P, Q, R; and it has been seen that this solid is a tangent solid of the line-cone $\{P\}$. Thus the tangent line-cone of Φ at P and the line-cone $\{P\}$ not only have four common generating planes, but they touch each other at any point of any one of these four planes; the two cones must therefore be the same.

If a chord PQ of ϑ lies on Φ then it must lie on the tangent cones of Φ at P and Q. Now the cone $\{P\}$ has no intersections with ϑ other than P itself and those points of ϑ which lie on the trisecants through P; it follows that there can be no chords of ϑ, other than the four trisecants, which pass through P and lie on Φ; *every chord of* ϑ *which lies on* Φ *is a trisecant of* ϑ.

[1] The equation of a quartic surface with three collinear nodes can be written

$$(x, y)_4 + z(x, y)_3 + w(x, y)'_3 + z^2(x, y)_2 + zw(x, y)'_2 + w^2(x, y)''_2 + zw(z-w)(ax+by) = 0.$$

Then $ax+by=0$ is the tangent plane of this surface at any point of the line $x=y=0$, and also touches along this line the three nodal cones

$$(x, y)_2 + w(ax+by) = 0, \quad (x, y)''_2 - z(ax+by) = 0,$$
$$(x, y)_2 + (x, y)'_2 + (x, y)''_2 + (z-w)(ax+by) = 0.$$

A similar statement holds for a surface of order n with $n-1$ collinear nodes.

The chords of ϑ generate a locus M_3^{30} on which R_2^{20} is a triple surface; the complete intersection of Φ with M_3^{30} must consist of the scroll R_2^{20} counted six times.

29. Write now

$$a_0\theta^6 + a_1\theta^5 + a_2\theta^4 + a_3\theta^3 + a_4\theta^2 + a_5\theta + a_6 = \\ = a_0(\theta-\theta_1)(\theta-\theta_2)(\theta-\theta_3)(\theta-\theta_4)(\theta-\theta_5)(\theta-\theta_6) = a_0 f(\theta);$$

$$b_0\theta^6 + b_1\theta^5 + b_2\theta^4 + b_3\theta^3 + b_4\theta^2 + b_5\theta + b_6 = \\ = b_0(\theta-\varphi_1)(\theta-\varphi_2)(\theta-\varphi_3)(\theta-\varphi_4)(\theta-\varphi_5)(\theta-\varphi_6) = b_0 g(\theta);$$

and multiply the eight-rowed determinant occurring in the equation of Φ by the eight-rowed determinant whose i^{th} row consists of the six elements

$$\theta_1^{8-i},\ \theta_2^{8-i},\ \theta_3^{8-i},\ \theta_4^{8-i},\ \theta_5^{8-i},\ \theta_6^{8-i};$$

the remaining elements, in the seventh and eighth columns, being arbitrary. Then the matrix of the first six rows of the product determinant is

$$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_1 g(\theta_1) & \theta_2 g(\theta_2) & \theta_3 g(\theta_3) & \theta_4 g(\theta_4) & \theta_5 g(\theta_5) & \theta_6 g(\theta_6) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ g(\theta_1) & g(\theta_2) & g(\theta_3) & g(\theta_4) & g(\theta_5) & g(\theta_6) \\ \theta_1^3 P_1 & \theta_2^3 P_2 & \theta_3^3 P_3 & \theta_4^3 P_4 & \theta_5^3 P_5 & \theta_6^3 P_6 \\ \theta_1^2 P_1 & \theta_2^2 P_2 & \theta_3^2 P_3 & \theta_4^2 P_4 & \theta_5^2 P_5 & \theta_6^2 P_6 \\ \theta_1 P_1 & \theta_2 P_2 & \theta_3 P_3 & \theta_4 P_4 & \theta_5 P_5 & \theta_6 P_6 \\ P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{matrix}\right\}$$

where, as before, $P_i \equiv x_0 + x_1\theta_i + x_2\theta_i^2 + x_3\theta_i^3 + x_4\theta_i^4$. Hence the equation of Φ may be written

$$\begin{vmatrix} \theta_1 g(\theta_1) & \theta_2 g(\theta_2) & \theta_3 g(\theta_3) & \theta_4 g(\theta_4) & \theta_5 g(\theta_5) & \theta_6 g(\theta_6) \\ g(\theta_1) & g(\theta_2) & g(\theta_3) & g(\theta_4) & g(\theta_5) & g(\theta_6) \\ \theta_1^3 P_1 & \theta_2^3 P_2 & \theta_3^3 P_3 & \theta_4^3 P_4 & \theta_5^3 P_5 & \theta_6^3 P_6 \\ \theta_1^2 P_1 & \theta_2^2 P_2 & \theta_3^2 P_3 & \theta_4^2 P_4 & \theta_5^2 P_5 & \theta_6^2 P_6 \\ \theta_1 P_1 & \theta_2 P_2 & \theta_3 P_3 & \theta_4 P_4 & \theta_5 P_5 & \theta_6 P_6 \\ P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{vmatrix} = 0.$$

This form of the equation shows immediately that Φ passes through the twenty edges of the hexahedron whose faces are the six primes $P_i = 0$, and that the fifteen vertices of this hexahedron are double points of Φ. The expansion of the determinant, by Laplace's rule, by means of the two top rows, gives the equation

$$\sum \frac{(\theta_i - \theta_j)^2 \, g(\theta_i) \, g(\theta_j)}{f'(\theta_i) \, f'(\theta_j)} P_k P_l P_m P_n = 0,$$

where the summation extends to the fifteen pairs of indices (ij), and k, l, m, n are the four numbers, other than i and j, of the set 123456. If we write $P_i \equiv X_i f'(\theta_i)$, so that the linear identity between the six forms X_i is

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \equiv 0,$$

the equation of Φ is

$$\sum \left\{ \frac{\theta_i - \theta_j}{f'(\theta_i) \, f'(\theta_j)} \right\}^2 g(\theta_i) \, g(\theta_j) \, X_k X_l X_m X_n = 0.$$

Similarly, if $Y_i g'(\varphi_i) \equiv x_0 + x_1 \varphi_i + x_2 \varphi_i^2 + x_3 \varphi_i^3 + x_4 \varphi_i^4$, we obtain an equation for Φ in the form

$$\sum \left\{ \frac{\varphi_i - \varphi_j}{g'(\varphi_i) \, g'(\varphi_j)} \right\}^2 f(\varphi_i) f(\varphi_j) \, Y_k Y_l Y_m Y_n = 0;$$

and we can similarly obtain an equation for Φ involving the six primes which are the faces of any one of the hexahedra inscribed in ϑ.

30. We obtained the equation of the primal Φ originally by equating to zero a determinant of eight rows and columns; we have now obtained another form of the equation in which a determinant of six rows and columns is equated to zero. We can proceed a step further in this direction and, by multiplying the determinant of six rows and columns by another determinant suitably chosen, reduce the number of rows and columns to four.

Take then the six-rowed determinant which occurs in the equation for Φ and multiply it by a six-rowed determinant the last four constituents of whose i^{th} row are

$$\frac{\theta_i^3}{g(\theta_i) \, f'(\theta_i)}, \quad \frac{\theta_i^2}{g(\theta_i) \, f'(\theta_i)}, \quad \frac{\theta_i}{g(\theta_i) \, f'(\theta_i)}, \quad \frac{1}{g(\theta_i) \, f'(\theta_i)};$$

the composition of the first two columns of this new determinant is immaterial save for restrictions that prevent the determinant from vanishing. Then, in virtue of the identity

$$\sum_{i=1}^{6} \frac{\theta_i^n}{f'(\theta_i)} \equiv 0,$$

which holds when $n = 0, 1, 2, 3, 4$, the last four constituents in each of the two top rows of the product determinant are zero. We thus obtain the equation of Φ in the form

$$\Theta \equiv \begin{vmatrix} \Theta_6 & \Theta_5 & \Theta_4 & \Theta_3 \\ \Theta_5 & \Theta_4 & \Theta_3 & \Theta_2 \\ \Theta_4 & \Theta_3 & \Theta_2 & \Theta_1 \\ \Theta_3 & \Theta_2 & \Theta_1 & \Theta_0 \end{vmatrix} = 0,$$

where

$$\Theta_n \equiv \sum_{i=1}^{6} \frac{\theta_i^n P_i}{g(\theta_i) f'(\theta_i)}.$$

The determinant Θ is symmetrical, so that *the quartic primal* Φ *is a symmetroid.* The section of Φ by an arbitrary solid is thus the well-known quartic symmetroid of Cayley, and incidentally we have the result that *if the Jacobian curve of a net of quadrics in* [4] *has a scroll of trisecants the ten points in which it is met by any solid are the nodes of a quartic symmetroid.*

It can be verified that the ten first minors of the symmetrical determinant Θ all vanish at any point of the curve ϑ, thus showing again that ϑ is a double curve on Φ. If we refer to the equations of the quadrics of the net as given in § 22 we see at once that the equations of the Jacobian curve ϑ are

$$\left\| \begin{matrix} \Theta_0 & \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 \\ \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 & \Theta_5 \\ \Theta_2 & \Theta_3 & \Theta_4 & \Theta_5 & \Theta_6 \end{matrix} \right\| = 0,$$

every three-rowed determinant of this matrix of five rows and three columns vanishing at any point on ϑ. There are ten three-rowed determinants belonging to this matrix; when equated to zero they represent ten cubic primals passing through ϑ. Also there are ten distinct first minors of the determinant Θ, and it is at once seen that seven of these ten occur also among the ten three-rowed determinants of the matrix; these seven minors therefore vanish on the curve ϑ.

The remaining three minors of the determinant, although not occurring explicitly as determinants in the matrix, are easily expressed linearly in terms of the actual determinants that do occur in the matrix; for example

$$\begin{vmatrix} \Theta_6 & \Theta_5 & \Theta_3 \\ \Theta_5 & \Theta_4 & \Theta_2 \\ \Theta_3 & \Theta_2 & \Theta_0 \end{vmatrix} = \begin{vmatrix} \Theta_6 & \Theta_5 & \Theta_2 \\ \Theta_5 & \Theta_4 & \Theta_1 \\ \Theta_4 & \Theta_3 & \Theta_0 \end{vmatrix} + \begin{vmatrix} \Theta_6 & \Theta_5 & \Theta_4 \\ \Theta_4 & \Theta_3 & \Theta_2 \\ \Theta_3 & \Theta_2 & \Theta_1 \end{vmatrix}.$$

Here the determinant on the left is a first minor of Θ which does not occur as a determinant of the matrix; the two determinants on the right do both occur in the matrix. Similarly each of the other two first minors of Θ which do not occur in the matrix can be expressed as the sum of two determinants which do. It then follows that all the first minors of Θ vanish on the curve ϑ.

31. When the equation of a primal is given by equating to zero a symmetrical determinant we can immediately write down, by bordering the determinant, the equation of a family of *contact primals*, i. e. of primals which have two intersections with the given primal wherever they meet it. A point common to the two primals is in general an ordinary point on both of them, the two primals having the same tangent prime there; but the point may also count for two among the points common to the two primals by being a node on one of them.

The identity

$$\begin{vmatrix} \Theta_6 & \Theta_5 & \Theta_4 & \Theta_3 & \alpha_1 \\ \Theta_5 & \Theta_4 & \Theta_3 & \Theta_2 & \alpha_2 \\ \Theta_4 & \Theta_3 & \Theta_2 & \Theta_1 & \alpha_3 \\ \Theta_3 & \Theta_2 & \Theta_1 & \Theta_0 & \alpha_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & 0 \end{vmatrix} \begin{vmatrix} \Theta_6 & \Theta_5 & \Theta_4 & \Theta_3 & \beta_1 \\ \Theta_5 & \Theta_4 & \Theta_3 & \Theta_2 & \beta_2 \\ \Theta_4 & \Theta_3 & \Theta_2 & \Theta_1 & \beta_3 \\ \Theta_3 & \Theta_2 & \Theta_1 & \Theta_0 & \beta_4 \\ \beta_1 & \beta_2 & \beta_2 & \beta_4 & 0 \end{vmatrix} - \begin{vmatrix} \Theta_6 & \Theta_5 & \Theta_4 & \Theta_3 & \alpha_1 \\ \Theta_5 & \Theta_4 & \Theta_3 & \Theta_2 & \alpha_2 \\ \Theta_4 & \Theta_3 & \Theta_2 & \Theta_1 & \alpha_3 \\ \Theta_3 & \Theta_2 & \Theta_1 & \Theta_0 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & 0 \end{vmatrix}^2$$

$$\equiv \begin{vmatrix} \Theta_6 & \Theta_5 & \Theta_4 & \Theta_3 & \alpha_1 & \beta_1 \\ \Theta_5 & \Theta_4 & \Theta_3 & \Theta_2 & \alpha_2 & \beta_2 \\ \Theta_4 & \Theta_3 & \Theta_2 & \Theta_1 & \alpha_3 & \beta_3 \\ \Theta_3 & \Theta_2 & \Theta_1 & \Theta_0 & \alpha_4 & \beta_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & 0 & 0 \end{vmatrix} \Theta \qquad \text{(A)}$$

shows at once that, whatever the values of the constants α_1, α_2, α_3, α_4, the cubic primal

$$\begin{vmatrix} \Theta_6 & \Theta_5 & \Theta_4 & \Theta_3 & \alpha_1 \\ \Theta_5 & \Theta_4 & \Theta_3 & \Theta_2 & \alpha_2 \\ \Theta_4 & \Theta_3 & \Theta_2 & \Theta_1 & \alpha_3 \\ \Theta_3 & \Theta_2 & \Theta_1 & \Theta_0 & \alpha_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & 0 \end{vmatrix} = 0$$

is a contact primal of Φ; we thus obtain a triply-infinite family of contact cubic primals. Each of these primals touches Φ along a sextic surface containing ϑ and, as follows at once from (A), the two sextic surfaces in which Φ is touched by any two contact primals of the family together form the complete intersection of Φ with a cubic primal.

32. The ∞^3 contact cubic primals of Φ all contain the curve ϑ, and we know that any one of the ∞^6 planes of [4] gives rise to a cubic primal Π also passing through ϑ; Π can be defined either as the locus of the polar lines of the plane in regard to the quadrics of the net or as the locus of the lines which are conjugate to the points of the plane (cf. *G. N. Q.* § 4). The question may then be asked whether any of the primals Π can be contact primals of Φ.

If the equations of a plane π are

$$a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0,$$
$$b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4 = 0,$$

then the equation of the associated primal Π is

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \\ \Theta_0 & \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 \\ \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 & \Theta_5 \\ \Theta_2 & \Theta_3 & \Theta_4 & \Theta_5 & \Theta_6 \end{vmatrix} = 0,$$

and we therefore have to enquire whether it is possible to choose constants a, b, α so that there is an identity

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \\ \Theta_0 & \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 \\ \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 & \Theta_5 \\ \Theta_2 & \Theta_3 & \Theta_4 & \Theta_5 & \Theta_6 \end{vmatrix} \equiv \begin{vmatrix} \Theta_6 & \Theta_5 & \Theta_4 & \Theta_3 & \alpha_1 \\ \Theta_5 & \Theta_4 & \Theta_3 & \Theta_2 & \alpha_2 \\ \Theta_4 & \Theta_3 & \Theta_2 & \Theta_1 & \alpha_3 \\ \Theta_3 & \Theta_2 & \Theta_1 & \Theta_0 & \alpha_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & 0 \end{vmatrix}.$$

When we attempt to determine the constants so that this identity is satisfied (it is not necessary to give the details of the algebra) we find that the identity must be of the form

$$\begin{vmatrix} \alpha^4 & \alpha^3 & \alpha^2 & \alpha & 1 \\ 4\alpha^3 & 3\alpha^2 & 2\alpha & 1 & 0 \\ \Theta_0 & \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 \\ \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 & \Theta_5 \\ \Theta_2 & \Theta_3 & \Theta_4 & \Theta_5 & \Theta_6 \end{vmatrix} \equiv \begin{vmatrix} \Theta_6 & \Theta_5 & \Theta_4 & \Theta_3 & 1 \\ \Theta_5 & \Theta_4 & \Theta_3 & \Theta_2 & \alpha \\ \Theta_4 & \Theta_3 & \Theta_2 & \Theta_1 & \alpha^2 \\ \Theta_3 & \Theta_2 & \Theta_1 & \Theta_0 & \alpha^3 \\ 1 & \alpha & \alpha^2 & \alpha^3 & 0 \end{vmatrix}, \quad \ldots\ldots \quad \text{(B)}$$

where α is arbitrary. The cubic primal Π must therefore arise from a plane π whose equations are of the form

$$\alpha^4 x_0 + \alpha^3 x_1 + \alpha^2 x_2 + \alpha x_3 + x_4 = 0,$$
$$4\alpha^3 x_0 + 3\alpha^2 x_1 + 2\alpha x_2 + x_3 \quad = 0.$$

This is an osculating plane of the quartic curve γ^4, and hence we have the result *that the cubic primal Π associated with a plane π is a contact primal of Φ when and only when π is an osculating plane of γ^4.* Such a cubic primal will be denoted by the symbol Π_ω, the suffix signalising the fact that the primal is associated not with a general plane of [4] but with an osculating plane of γ^4; there is then a singly-infinite set of cubic primals Π_ω, and they are contact primals of Φ.

It may be remarked in passing that the two determinants appearing in the identity (B) are also identically equal to the determinant

$$\begin{vmatrix} \alpha^2\Theta_4 - 2\alpha\Theta_3 + \Theta_2 & \alpha^2\Theta_5 - 2\alpha\Theta_4 + \Theta_3 & \alpha^2\Theta_6 - 2\alpha\Theta_5 + \Theta_4 \\ \alpha^2\Theta_3 - 2\alpha\Theta_2 + \Theta_1 & \alpha^2\Theta_4 - 2\alpha\Theta_3 + \Theta_2 & \alpha^2\Theta_5 - 2\alpha\Theta_4 + \Theta_3 \\ \alpha^2\Theta_2 - 2\alpha\Theta_1 + \Theta_0 & \alpha^2\Theta_3 - 2\alpha\Theta_2 + \Theta_1 & \alpha^2\Theta_4 - 2\alpha\Theta_3 + \Theta_2 \end{vmatrix}.$$

It follows that the quartic curve q whose equations are

$$\frac{\Theta_0 - 2\alpha\Theta_1 + \alpha^2\Theta_2}{\Theta_1 - 2\alpha\Theta_2 + \alpha^2\Theta_3} = \frac{\Theta_1 - 2\alpha\Theta_2 + \alpha^2\Theta_3}{\Theta_2 - 2\alpha\Theta_3 + \alpha^2\Theta_4} =$$

$$= \frac{\Theta_2 - 2\alpha\Theta_3 + \alpha^2\Theta_4}{\Theta_3 - 2\alpha\Theta_4 + \alpha^2\Theta_5} = \frac{\Theta_3 - 2\alpha\Theta_4 + \alpha^2\Theta_5}{\Theta_4 - 2\alpha\Theta_5 + \alpha^2\Theta_6}$$

is a double curve on Π_ω. Any point whose coordinates satisfy these equations is on the primal Φ, so that q lies on the surface along which Π_ω touches Φ.

The sextic surface φ_2^6 along which Π_ω touches Φ can be identified. The identity (B) is true for all values of α; we deduce from it, by differentiation, the identity

$$\begin{vmatrix} \alpha^4 & \alpha^3 & \alpha^2 & \alpha & 1 \\ 6\alpha^2 & 3\alpha & 1 & 0 & 0 \\ \Theta_0 & \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 \\ \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 & \Theta_5 \\ \Theta_2 & \Theta_3 & \Theta_4 & \Theta_5 & \Theta_6 \end{vmatrix} \equiv \begin{vmatrix} \Theta_6 & \Theta_5 & \Theta_4 & \Theta_3 & 0 \\ \Theta_5 & \Theta_4 & \Theta_3 & \Theta_2 & 1 \\ \Theta_4 & \Theta_3 & \Theta_2 & \Theta_1 & 2\alpha \\ \Theta_3 & \Theta_2 & \Theta_1 & \Theta_0 & 3\alpha^2 \\ 1 & \alpha & \alpha^2 & \alpha^3 & 0 \end{vmatrix}.$$

When either of these two determinants is equated to zero we obtain the equation of a cubic primal Π'. This is not a contact primal of Φ, but the form of the determinant on the right shows, when we refer to the identity (A), that Π' meets Φ in two sextic surfaces one of which is φ_2^6, the surface of contact of Π_ω and Φ. On the other hand the form of the determinant on the left shows that Π' is the locus of the poles of a plane π' in regard to the quadrics of the net, and that π' lies in the solid

$$\alpha^4 x_0 + \alpha^3 x_1 + \alpha^2 x_2 + \alpha x_3 + x_4 = 0.$$

But this is the osculating solid of γ^4 at the point where π is the osculating plane; hence π and π' both lie in this solid. Now when two planes π and π' lie in the same solid the cubic primals associated with them have in common a cubic scroll and a sextic surface, the sextic surface being the locus of the poles of the solid in regard to the quadrics of the net (*G. N. Q.* § 15). Hence *the cubic primal Π_ω which is associated with the osculating plane at a point P of γ^4 touches Φ along the sextic surface φ_2^6 which is the locus of the poles of the osculating solid of γ^4 at P in regard to the quadrics of the net.* Since the osculating solid of γ^4 at P is one of the six faces of a hexahedron inscribed in ϑ the surface φ_2^6 contains the ten edges of the simplex $\mathfrak{S}$ formed by the remaining five faces of this hexahedron (cf. § 15 above).

33. The equation of the contact primal Π_ω contains the parameter α to the sixth degree, so that through any point O of [4] there pass *six* of the primals Π_ω. This can also be seen otherwise; for in order that the cubic primal Π_ω associated with an osculating plane π of γ^4 should pass through O it is necessary and sufficient that the line conjugate to O should meet π. It is known[1] however that the osculating planes of γ^4 generate a sextic locus F_3^6, so that any line meets *six* osculating planes of γ^4. Suppose now that the point O lies on Φ. Then when the coordinates of O are substituted in the left-hand side of the equation of a contact primal Π_ω the resulting sextic polynomial in α is the square of a cubic polynomial; this is clear on referring to the identity (A), remembering that the coordinates of O now satisfy $\Theta = 0$. Hence through an arbitrary point of Φ there pass *three* of the primals Π_ω. Incidentally the surfaces φ_2^6 along which the primals Π_ω touch Φ form a singly-infinite family of surfaces of index three. It follows too that if O is any point of Φ the line which is conjugate to O must be such that its six intersections with F_3^6 coincide in pairs, or that the line must be a tritangent of F_3^6. This is so: for the line which is conjugate to any point of the sextic surface which is the locus of the poles of a solid S_3 must lie in S_3; hence, since there are three of the surfaces φ_2^6 passing through O, the line conjugate to O is common to three osculating solids of γ^4. But if π is the osculating plane of γ^4 at a point P the osculating solid of γ^4 at P is the tangent solid of F_3^6 at *every* point of π, so that any line which lies in the osculating solid touches F_3^6 at the point where it meets π; the line of intersection of three osculating solids of γ^4 is therefore a tritangent of F_3^6.

34. It has been remarked in § 28 that if three cubic primals pass through ϑ their residual curve of intersection meets Φ in eight points not on ϑ. Suppose now that the three primals are all contact primals of Φ; then any point which is common to Φ and the three primals and which does not lie on ϑ is such that Φ and the three primals all have the same tangent solid there; it therefore counts for eight among the points common to the four primals, and is therefore their *only* common intersection apart from the curve ϑ. It is thus also the only point, apart from the curve ϑ, which is common to the three sextic surfaces along which the primals touch Φ. In particular, when the three contact primals are all members of the singly-infinite family of primals Π_ω, we

[1] See, for example, Veronese, *loc. cit.*, 202.

see that any three of the surfaces φ_2^6 have in common, apart from the curve ϑ, one and only one point. Thus the line common to any three osculating solids of γ^4 is conjugate to a point of Φ. It has already been seen, conversely, that the line conjugate to any point of Φ is common to three osculating solids of γ^4; there is thus a (1, 1) correspondence between the points of Φ and the tritangents of F_3^6, and thus also between the points of Φ and the trisecant planes of γ^4.

Take now two contact primals of Φ; their surface of intersection meets Φ in ϑ and a residual curve. Now any point on this residual curve is such that Φ and the two contact primals all have the same tangent solid there, so that this residual curve must be counted four times as part of the curve common to Φ and the two primals. Since the complete curve of intersection is of order 36, and since ϑ is a double curve on Φ and a simple curve on the two contact primals and therefore counts twice as part of the curve of intersection, the residual curve must be a quartic. This quartic curve has a certain number of intersections with ϑ, and this number can be found at once from the fact that a third contact primal through ϑ must have a single contact with the quartic; of the twelve intersections of this third contact primal with the quartic ten must therefore lie on ϑ, so that the quartic meets ϑ in ten points. It follows also that the sextic surfaces along which any two of the ∞^3 contact primals touch Φ have in common the curve ϑ and a quartic curve meeting ϑ in ten points. Suppose, in particular, that the two contact primals are both primals Π_ω; the surfaces along which they touch Φ are then two of the surfaces φ_2^6; one of these two surfaces is the locus of the poles of an osculating solid Ω_1 of γ^4 in regard to the quadrics of the net, the other arising similarly from a second osculating solid Ω_2 of γ^4. Take now any point O on the quartic curve q_{12} which is common to these two surfaces φ_2^6; the line which is conjugate to O lies in Ω_1 and also in Ω_2, and hence in the plane of intersection ω_{12} of Ω_1 and Ω_2. But since O is a point of Φ the line conjugate to O must also lie in a third osculating solid of γ^4; it is therefore one of the lines in which ω_{12} is met by the osculating solids of γ^4. These lines, as is well-known, are the tangents of a conic c_{12}; this is in accordance too with the fact (cf. *G. N. Q.* § 16) that the order of the scroll generated by the lines conjugate to the points of a quartic curve having ten intersections with ϑ is $3 \cdot 4 - 10 = 2$. Now the solid Ω_1 is a face of a hexahedron inscribed in ϑ; the remaining five faces of this hexahedron form a simplex $\mathfrak{S}_1$ whose ten edges lie on the surface φ_2^6 associated with

Ω_1 and whose five vertices lie on ϑ. The secant plane of ϑ which is conjugate to any one of these five vertices lies in Ω_1 and meets ω_{12} in a tangent of c_{12}; hence the five vertices of $\mathfrak{S}_1$ are all on the quartic curve q_{12}. Similarly Ω_2 belongs to a hexahedron whose five remaining faces form a simplex $\mathfrak{S}_2$ whose five vertices are on q_{12}. Hence the quartic curve q_{12} is circumscribed to both the simplexes $\mathfrak{S}_1$ and $\mathfrak{S}_2$. The vertices of these two simplexes account for the ten intersections of q_{12} and ϑ. We have assumed that Ω_1 and Ω_2 are not two faces of the same hexahedron; in that case the curve q_{12} would degenerate into four concurrent trisecants of ϑ.

Consider now the limiting case in which Ω_1 and Ω_2 both coincide with the osculating solid Ω of γ^4 at a point P; the plane ω_{12} then becomes the osculating plane of γ^4 at P, and the conic c_{12} becomes the conic c, the envelope of the lines in which π is met by the osculating solids of γ^4. This conic c is also the locus of points in which π is met by the osculating planes of γ^4; it passes through P, and its tangent at P is also the tangent of γ^4 at P. The primal Π_ω which is generated by the lines conjugate to the points of π touches Φ along the surface φ_2^6 which is the locus of the poles of Ω in regard to the quadrics of the net; φ_2^6 contains ϑ and also ten trisecants of ϑ which are the edges of a simplex $\mathfrak{S}$. This surface φ_2^6 is met by the surface which is the locus of the poles of any other osculating solid Ω' of γ^4 in a quartic curve through the vertices of $\mathfrak{S}$, this quartic curve also meeting ϑ in the vertices of a second simplex $\mathfrak{S}'$ associated with Ω' in the same way that $\mathfrak{S}$ is with Ω. There is thus on φ_2^6 a singly-infinite family of quartic curves through the vertices of $\mathfrak{S}$; these include a *characteristic curve* q which touches ϑ at the five vertices of $\mathfrak{S}$, the lines conjugate to the points of q being the tangents of c. Since there are two tangents of c passing through any point of π the line conjugate to this point is a chord of q; conversely any chord of q is conjugate to a point of π, for the lines which are conjugate to its two intersections with q are tangents of c, so that the chord of q is the line conjugate to the point of intersection of these two tangents. Thus the primal Π_ω is the cubic primal generated by the chords of q, and q is the double curve on Π_ω that was noticed previously. If we take a point on c the line which is conjugate to it is a tangent of q; not only are the tangents of c conjugate to the points of q but the tangents of q are conjugate to the points of c.

35. Through any point O of [4] there pass four osculating solids of γ^4; any three of these meet in a tritangent of F_3^6, so that there are four tritangents of F_3^6 passing through O. These are the four tritangents of F_3^6 that are conjugate to the four points in which the line conjugate to O meets Φ. Suppose now that O lies on F_3^6 and so in an osculating plane π of γ^4. Then of the four osculating solids of γ^4 which pass through O two coincide with that osculating solid Ω of γ^4 which contains π; the plane of intersection of the remaining two meets Ω in a line which is a tritangent of F_3^6, this being the only tritangent whose point of contact is O. Correspondingly, of the four intersections of the line conjugate to O with Φ two coincide with the point to which this tritangent is conjugate; the two remaining intersections of the line with Φ lie, as has been seen, on the double curve q of the cubic primal Π_ω associated with π, being those points of q to which the two tangents of c which pass through O are conjugate.

The position of O may be particularised further. Suppose O lies on the tangent t of γ^4 at a point P, the osculating plane of γ^4 at P being π and the osculating solid Ω; then of the four osculating solids of γ^4 that pass through O three coincide with Ω, and the remaining one meets π in that tangent τ, other than t, of c which passes through O. Of the four intersections of Φ with the line conjugate to O three now coincide with the point to which τ is conjugate, the remaining one being that point, G say, to which t is conjugate; both these points are on the quartic curve q. Each generator of the cubic cone which projects q from G is conjugate to a point of t and has three-point contact with Φ at its other intersection with q. In particular the tangent of q at G has four-point contact with Φ at G; this tangent is the line conjugate to P, so that the lines conjugate to the points of γ^4 are all flecnodal tangents of Φ.

Consider lastly the case when O is on the conic c in the plane π; O is then the intersection of two osculating planes, π and π', of γ^4 and so lies not only on c but also on the conic c' in the plane π'. The line conjugate to O is therefore a tangent of two characteristic curves q and q'. The particular circumstance of Ω and Ω', the osculating solids of γ^4 which contain π and π', being two faces of the same hexahedron inscribed in ϑ may be remarked; the curves q and q' then touch ϑ at that vertex V of the hexahedron which is the point of intersection of its remaining four faces; the common tangent of q and q' is now the tangent of ϑ at V, meeting Φ in four points at V. The secant plane conjugate to V is the plane of the tangents of c and c' at their intersection, this being the plane of intersection of Ω and Ω'. Again: when we choose the point O on c to be at P, the point where π osculates γ^4, c' coincides with c and q' with q; we have again the tangent of q at G having four-point contact with Φ.

36. The characteristic curves q generate a surface χ whose order we can obtain; χ is the envelope of the singly-infinite set of surfaces φ_2^6. It has been seen that the lines conjugate to the points of a tangent t of γ^4 generate a cubic cone, and it is clear that the complete intersection of this cubic cone with Φ consists of the curve q counted three times. If then we take the primal which is generated by the lines conjugate to all the points of all the tangents of γ^4 its complete surface of intersection with Φ must be the surface χ counted three times. Now the tangents of γ^4 generate a sextic surface; hence (*G. N. Q.* § 16) the primal which is generated by the lines conjugate to the points of this surface is of order eighteen and has ϑ as a sextuple curve. The surface of intersection of such a primal with Φ is of order 72 and has ϑ as a curve of multiplicity 12; hence χ *is of order* 24 *and has* ϑ *as a quadruple curve.* Or we may argue as follows. The line which is conjugate to the point of intersection of any two osculating planes of γ^4 is, as has been seen, a bitangent of Φ, both its points of contact being on the surface χ. Now the points common to two osculating planes of γ^4 form, as is well-known, a quartic surface V_2^4 — the double surface of F_3^6. The primal generated by the lines which are conjugate to the points of V_2^4 therefore meets Φ in the surface χ counted twice. But the lines conjugate to the points of V_2^4 generate a primal of order twelve on which ϑ is a quadruple curve, and the surface of intersection of such a primal with Φ is of total order 48 and has ϑ as a curve of multiplicity eight. We have again therefore the result that χ is of order 24 and has ϑ as a quadruple curve. It may be noted, as a partial verification, that the primal which is generated by the lines conjugate to the points of a surface of order 24 on which ϑ is a quadruple curve is of order $3 \cdot 24 - 4 \cdot 15 = 12$. Now any point of χ lies on a characteristic curve q, and the line conjugate to the point is therefore a tangent of a conic c on V_2^4, and so lies in an osculating plane of γ^4. All the tangents of the ∞^1 conics c arise in this way and, since through any point of an osculating plane of γ^4 there pass two tangents of a conic c, the primal generated by the lines conjugate to the points of χ is F_3^6, counted twice. Thus it is of order twelve, as it ought to be.

37. We have obtained on each of the characteristic curves q a point G such that the tangent of q at G has four-point contact with Φ; the locus of these points G is a curve g lying on the surface χ. Now the line which is conjugate to any point of γ^4 has fourpoint contact with Φ, the point of contact being on g; hence the scroll of lines which are conjugate to the points of γ^4 meets Φ in the curve g counted four times. Since this scroll is of order 12, the curve g is of order 12.

38. It has been seen that through any point of Φ there pass three of the surfaces φ_2^6; the primal Φ may therefore be defined as *the locus of the poles of the osculating solids of γ^4 in regard to the quadrics of the net*, and properties of Φ can be deduced directly from this definition. Each point of Φ is the pole of three different osculating solids of γ^4 in regard to quadrics of the net; but for a point of χ two of the three solids coincide, while for a point of g all three solids coincide.

On the curve ϑ there is a linear series g_5^2, any set of this series being a set of vertices of five cones belonging to a pencil. Of these sets of five points ∞^1 are sets of five vertices of simplexes $\mathfrak{S}$ whose faces all osculate the quartic curve γ^4; and we have seen that the vertices of any two of the simplexes $\mathfrak{S}$ lie on a quartic curve q_{12}; there is thus a doubly-infinite system of curves q_{12}, including a singly-infinite system of characteristic curves q. This system of curves q_{12} was obtained as the system of curves in which pairs of surfaces φ_2^6 intersect, but it can also be obtained in other ways. In the first place there is a pencil of quadrics, belonging to the net, in regard to which any given simplex $\mathfrak{S}$ is self-conjugate; the locus of the poles of any solid in regard to the quadrics of this pencil is a quartic curve passing through the five vertices of $\mathfrak{S}$. If now we suppose that the solid is an osculating solid Ω of γ^4 we determine thereby a hexahedron of which Ω is one face; the five remaining faces of this hexahedron form a second simplex $\mathfrak{S}$. The quartic curve passes also through the vertices of this second simplex, for the secant plane conjugate to any one of these vertices lies in Ω, so that Ω is the polar of this vertex in regard to all the quadrics belonging to a pencil — which pencil has a quadric in common with the pencil we are considering. Thus the quartic curve, since it circumscribes two of the simplexes $\mathfrak{S}$, must be one of the quartics q_{12}, and all the quartics q_{12} are obtainable in this way. A characteristic curve q is obtained when we take the locus of the poles of an osculating solid Ω of γ^4 in regard to that pencil, of quadrics of the net, for which the simplex formed by the five remaining faces of the hexahedron determined by Ω is a self-conjugate simplex. In the second place the poles of the osculating solids of γ^4 in regard to any quadric Q of the net lie on a quartic curve. This quartic curve meets ϑ in the points which are such that their polar solids in regard to Q osculate γ^4; we have seen, however, in § 20, that there are ten such points on ϑ, and that they consist of the vertices of two of the simplexes $\mathfrak{S}$; the quartic curve we have obtained is again therefore one of the quartics q_{12}. Moreover Q is the quadric which is common to the two pencils of quadrics for which the two simplexes $\mathfrak{S}$ are respectively self-conjugate, so that every curve q_{12} can be obtained in this way. We may thus look upon the curves q_{12} as polar reciprocals of γ^4 in regard to the quadrics of the net.

The equation of the polar solid of the point $(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4)$ in regard to the quadric $\lambda Q_0 + \mu Q_1 + \nu Q_2 = 0$, the notation being as in § 22, is

$$\sum_{i=1}^{6} \frac{(x_0 + x_1\theta_i + x_2\theta_i^2 + x_3\theta_i^3 + x_4\theta_i^4)(\eta_0 + \eta_1\theta_i + \eta_2\theta_i^2 + \eta_3\theta_i^3 + \eta_4\theta_i^4)(\lambda + \mu\theta_i + \nu\theta_i^2)}{f'(\theta_i)\, g(\theta_i)} = 0.$$

It follows that the pole of the solid $x_0 + x_1\theta + x_2\theta^2 + x_3\theta^3 + x_4\theta^4 = 0$, which is an osculating solid of γ^4, lies, for all values of θ, on the quartic curve

$$\frac{\lambda\Theta_0 + \mu\Theta_1 + \nu\Theta_2}{\lambda\Theta_1 + \mu\Theta_2 + \nu\Theta_3} = \frac{\lambda\Theta_1 + \mu\Theta_2 + \nu\Theta_3}{\lambda\Theta_2 + \mu\Theta_3 + \nu\Theta_4} = \frac{\lambda\Theta_2 + \mu\Theta_3 + \nu\Theta_4}{\lambda\Theta_3 + \mu\Theta_4 + \nu\Theta_5} = \frac{\lambda\Theta_3 + \mu\Theta_4 + \nu\Theta_5}{\lambda\Theta_4 + \mu\Theta_5 + \nu\Theta_6}.$$

We have in this way, as the ratios $\lambda : \mu : \nu$ vary, the doubly-infinite system of curves q_{12}. A characteristic curve q is obtained when $\mu^2 = 4\nu\lambda$, as is seen by comparing these equations with those of a characteristic curve given in § 32.

39. The locus Φ is rational and can be mapped on a space Σ_3 by the usual method of mapping rational determinantal loci[1], so that its prime sections are mapped by quartic surfaces passing through a base curve ψ of order ten and genus eleven. The properties of Φ can be obtained *via* this representation, but we content ourselves by indicating in a table the main features of the mapping; the trisecants of ϑ are mapped by the *points* of ψ and the points of ϑ, each of which lies on four trisecants, are mapped by *quadrisecant lines* of ψ. These quadrisecants generate a scroll of order ten on which ψ is a triple curve.

[1] See, for example, Room: *Proc. London Math. Soc.* (2) 36, 1934, 12—15.

Φ	Σ_3
Trisecant of ϑ.	Point of ψ.
Point of ϑ.	Quadrisecant of ψ.
Prime section.	Quartic surface through ψ.
Surface along which Φ is touched by a contact cubic primal.	Plane.
Surface φ_2^6.	Plane of a certain cubic developable ∇.
Curve q_{12}.	Axis of ∇.
Surface χ	Quartic surface of generating lines of ∇.
Curve g.	Cuspidal edge of ∇.

The quartic symmetroid Φ is a particular case of one considered by L. Roth[1]; Roth's symmetroid has also a double curve of order ten with an infinity of trisecants generating a scroll of order twenty, but this curve is not necessarily the Jacobian curve of a net of quadrics.

Any secant plane of ϑ contains four trisecants of ϑ, any two of which intersect in a point of ϑ; correspondingly we have a set of four points of ψ the join of any two of which is a quadrisecant of ψ; in other words we have a tetrahedron whose vertices are on ψ and whose edges are quadrisecants of ψ. Any osculating solid Ω of γ^4 contains five secant planes of ϑ, any two of which have a trisecant of ϑ as their line of intersection; correspondingly we have five tetrahedra whose vertices are all on ψ, any two of which have three common faces; thus we have a pentahedron whose vertices are all on ψ and whose ten edges are quadrisecants of ψ. Proceeding a step further we find that, associated with any one of the hexahedra whose fifteen vertices lie on ϑ and whose twenty edges are trisecants of ϑ, we have in Σ_3 a hexahedron whose fifteen edges are quadrisecants of ψ and whose twenty vertices lie on ψ; thus *the curve ψ is circumscribed to a infinity of hexahedra.* A face of any one of these hexahedra is met by the other five faces in the five sides of a pentagram; these are quadrisecants of ψ and are associated with the vertices of a simplex $\mathfrak{S}$ on ϑ; it follows that the plane in Σ_3 is associated with the sextic surface φ_2^6 which contains the ten edges of $\mathfrak{S}$, and therefore is a plane of the developable ∇. The faces of the inscribed hexahedra of ψ therefore all belong to ∇, and we see that ψ may be generated very simply as the locus of the vertices of the hexahedra formed

[1] *Proc. London Math. Soc.* (2), 30, 1930, 305.

by the sets of an involution of sets of six planes of the cubic developable ∇. We can thus, if we wish, obtain the whole of the geometry by starting from a twisted cubic curve; we take a g_6^1 on the curve and consider the hexahedron formed by the planes which osculate the curve at the six points of a set of the g_6^1. As the set varies the vertices of the hexahedron trace out a curve ψ, and the system of ∞^4 quartic surfaces through ψ represents a symmetroid Φ in [4].

40. The lines of [4] which are cut in involution by the quadrics of the net are the lines of a cubic complex Λ; hence, since the trisecants of ϑ generate a scroll of order 20, there must be sixty trisecants of ϑ belonging to Λ.

Suppose that the trisecant PQR of ϑ belongs to Λ; since the involution which is cut out on PQR by the quadrics of the net has two, and only two, double points, the line must be a generator of one of the three cones (P), (Q), (R); suppose[1] that it is a generator of (P). Then Q and R are conjugate points in regard to every quadric of the net; hence, if UVW is that trisecant of ϑ which is conjugate to PQR, the secant plane conjugate to Q is $UVWR$ and the secant plane conjugate to R is $UVWQ$. Let X, Y be the remaining two intersections of the plane $UVWQ$ with ϑ, and Z, T the remaining two intersections of the plane $UVWR$ with ϑ.

There is, associated with any point K of ϑ, a solid ϖ; ϖ joins K to its conjugate secant plane and all the quadrics of the net which pass through K touch ϖ at K. Thus the solids ϖ_Q and ϖ_R associated with Q and R both coincide with the solid $UVWPQR$, which we will denote by Σ^*. The cone (P) touches ϖ_Q at Q and ϖ_R at R; hence Σ^* is the tangent solid of (P) along its generator PQR. Also the solid ϖ associated with the point K of ϑ meets ϑ in K, in the six points which lie in the secant plane conjugate to K, and in three other points; these three points are the vertices of those cones of the net, other than (K) itself, which pass through K. Hence the cones (X) and (Y) pass through Q while the cones (Z) and (T) pass through R.

Every secant plane of ϑ meets ϑ in the six vertices of a quadrilateral, and the line joining any two points of ϑ which lie in the same secant plane is a trisecant of ϑ. Hence QX, QY, RZ, RT, are trisecants of ϑ; moreover they

[1] We need not consider the possibility of the trisecant being a generator of more than one of the three cones. The only chords of ϑ which are generators of two cones are the 120 lines which are chords both of ϑ and of the base curve of the net, and none of these is a trisecant of ϑ.

are generators of (X), (Y), (Z), (T) respectively, so that we have four further trisecants of ϑ which belong to Δ. The third intersection of each of these four trisecants with ϑ must be one of U, V, W; since there are four trisecants and only three points U, V, W at least one of U, V, W must be common to two of the four trisecants; suppose that this point is V, and that QXV and RZV are trisecants of ϑ. Then $QXVZR$ is a secant plane of ϑ, and ZX passes through P; since this plane contains the trisecant PQR the point of ϑ to which it is conjugate must be either U or W, suppose it is W; then the trisecant conjugate to QXV must be RW, which we may suppose to pass through T, and the trisecant conjugate to RZV is QW, which now passes through Y. Then PYT, UXY, UZT are trisecants of ϑ; the figure now consists of a tetrahedron $QRVW$ and of a plane meeting its six edges QR, VW, QV, RW, QW, RV in the points P, U, X, T, Y, Z respectively. Also, since V and W are conjugate points in regard to every quadric of the net, the trisecant UVW belongs to Δ. We have thus shown that *those trisecants of ϑ which belong to Δ are distributed in cospatial sets of six*, each such set consisting of the six edges of a tetrahedron. Each pair of opposite edges of the tetrahedron is a pair of conjugate trisecants of ϑ, and each face of the tetrahedron is the secant plane conjugate to the opposite vertex. There are ten tetrahedra of this kind, lying in ten solids Σ^*. A solid Σ^* meets the quadrics of the net in a net of quadric surfaces all having $QRVW$ as a self-conjugate tetrahedron; moreover, since Σ^* is a tangent solid of each of the cones (P), (U), (X), (Y), (Z), (T) the net of quadric surfaces contain six plane-pairs. Each plane-pair has an edge of the common self-conjugate tetrahedron as its double line, and the Jacobian curve of the net of quadric surfaces consists of the six edges of the tetrahedron.

Since each solid Σ^* is the join of *three* pairs of conjugate trisecants, namely the three pairs of opposite edges of the tetrahedron, these ten solids are triple solids of the developable D generated by the solids Σ. Also each solid Σ^* joins *four* points of ϑ to their conjugate secant planes, so that the ten solids are quadruple solids of the developable generated by the solids ϖ.

41. Any solid which contains two secant planes passing through the same trisecant of ϑ is a face of a hexahedron inscribed in ϑ; in particular Σ^* is a face of such a hexahedron. Of the six faces of this hexahedron three pass through the trisecant PQR and the remaining three through the conjugate trisecant UVW; thus one of the first three faces must coincide with Σ^*, as also

must one of the remaining three; hence *two of the six faces of the hexahedron coincide with Σ^*.* That there are ten hexahedra, inscribed in ϑ, two of whose faces coincide is in accordance with the fact that, of the sets of a g_6^1 among the osculating primes of a rational curve, there are ten sets two of whose members coincide. The four faces, other than Σ^*, of this hexahedron intersect in that point O^* of ϑ to which the secant plane $PXYZTU$ is conjugate; the lines O^*Q, O^*R, O^*V, O^*W are the four trisecants of ϑ which pass through O^*, and these are the tangents of ϑ at Q, R, V, W respectively. *Of the trisecants of ϑ there are forty which touch the curve, and these consist of ten concurrent sets of four.* It has been shown elsewhere[1] that, for a Jacobian curve of a general net of quadrics in [4], there are 120 secant planes touching the curve; here, for this special Jacobian curve, these 120 secant planes consist of 60 bitangent planes, each counted twice; each of the ten points O^* is joined to the six edges of the corresponding tetrahedron $QRVW$ by six secant planes which are also bitangent planes of ϑ.

Since Σ^* passes through the secant plane $PXYZTU$ and meets ϑ further in the four points Q, R, V, W, the four cones (Q), (R), (V), (W) belong to a pencil, the fifth cone of this pencil being (O^*).

42. The locus of the poles of Σ^* in regard to the quadrics of the net is a sextic surface φ^* lying on Φ and containing the ten edges of the simplex O^*QRVW; on φ^* there is a quartic curve touching ϑ at the points O^*, Q, R, V, W; this curve consists however of the four lines O^*Q, O^*R, O^*V, O^*W. These four lines therefore constitute the characteristic curve on φ^*, and so lie on the surface χ; we have thus obtained forty lines lying on χ, namely those trisecants of ϑ two of whose intersections with ϑ coincide. These forty lines, together with ϑ, make up the complete intersection of the two surfaces χ and R_2^{20}. For we have seen that χ can be obtained as the surface of contact of Φ and a primal of order twelve on which ϑ is a quadruple curve; the complete intersection of such a primal with R_2^{20}, which is a curve of total order 240, consists of ϑ counted sixteen times (ϑ being quadruple on both surfaces) and a curve of order eighty which, since the primal is a contact primal of Φ and so touches R_2^{20} wherever it meets it, must consist of a curve of order forty counted twice.

A trisecant which touches ϑ corresponds in Σ_3 to a point of ψ such that two of the three quadrisecants which pass through it coincide; since there are

[1] *G.N.Q.* § 16.

four such trisecants passing through the point O^* of ϑ we have correspondingly four such points lying on a quadrisecant s^* of ψ; there are thus ten of these quadrisecants s^*; this result also follows from the fact that ψ can be generated by means of an involution of sets of six planes of the developable ∇. Now the line of Σ_3 which represents the characteristic curve consisting of the four lines O^*Q, O^*R, O^*V, O^*W must pass through the four points of ψ which correspond to these tangent trisecants of ϑ; it is therefore the quadrisecant s^*. But, since the line represents a characteristic curve, it must be a generating line of ∇; hence the ten quadrisecants s^* are generating lines of ∇.

43. We now obtain some relations between loci connected with the two curves ϑ and γ^4. On γ^4 there is an involution g_6^1, the osculating solids of γ^4 at the six points of any set of g_6^1 forming a hexahedron whose vertices are on ϑ. There are ten sets of g_6^1 which have double points; the ten points of γ^4 which are double points of sets of g_6^1 form the Jacobian set J of g_6^1. The solids which osculate γ^4 at the points of J are the ten solids Σ^*; the osculating plane of γ^4 at a point of J is to be regarded as the plane of intersection of the two coincident solids Σ^* belonging to a hexahedron inscribed in ϑ, and we thus have ten planes α^* which are secant planes of ϑ and osculating planes of γ^4; the plane $PXYZTU$ found above is one of these.

Let us consider the intersections of the curve γ^4 with the locus R_3^{15}. Each generating plane of R_3^{15} is the plane of intersection of two osculating solids of γ^4, the points of contact of these two solids with γ^4 belonging to the same set of g_6^1. But an osculating solid of γ^4 cannot meet γ^4 except at its point of osculation; hence a point j of R_3^{15} which lies on γ^4 must be such that the two osculating solids of γ^4 which contain the generating plane of R_3^{15} through j both coincide with the osculating solid of γ^4 at j. Thus j must be one of the ten points of J, and the generating plane of R_3^{15} which passes through j is the osculating plane of γ^4 at j, and one of the ten planes α^*. There can be no other points common to R_3^{15} and γ^4 apart from the ten points of J; hence, since γ^4 and R_3^{15} must have sixty intersections in all, *R_3^{15} has six-point contact with γ^4 at each point of J.*

Suppose we take any locus R_3 generated by a singly-infinite family of planes in [4], and consider the section of R_3 by an arbitrary solid S passing through one of its generating planes p: S meets the other generating planes in lines, and these lines generate a ruled surface of which one generator g lies in p. Then S touches R_3 at every point of g.

Apply this now to R_3^{15}, p being one of the planes α^* and S the corresponding solid Σ^*. In the first place let P be *any* point of γ^4, t the tangent there, π the osculating plane and S the osculating solid. The remaining osculating solids of γ^4 meet S in planes which osculate a twisted cubic γ^3; γ^3 passes through P, having t as its tangent and π as its osculating plane. A g_6^1 on γ^4 gives correspondingly a g_6^1 on γ^3. The planes of intersection of pairs of osculating solids of γ^4 whose points of osculation belong to the same set of g_6^1 generate a locus R_3^{15}, of which five generating planes lie in S; the lines of intersection of pairs of osculating planes of γ^3 whose points of osculation belong to the same set of the corresponding g_6^1 generate a ruled surface R_2^{10}. The five generating planes of R_3^{15} which lie in S, taken together with R_2^{10}, make up the complete intersection of S with R_3^{15}. There is a generator of R_2^{10} in each generating plane of R_3^{15}. Now let P be one of the ten points j belonging to J, so that π is a plane α^* and S a solid Σ^*. Then α^* is a generating plane of R_3^{15} and contains a generator of R_2^{10}. But, since j is a double point of the g_6^1 on γ^4 it must also be a double point of the g_6^1 on γ^3; thus α^* osculates γ^3 at a double point of the g_6^1, and therefore the tangent of γ^3 at j, which is also the tangent of γ^4 at j, is a generator of R_2^{10}. It follows that Σ^* touches R_3^{15} at every point of the tangent of γ^4 at j. Hence we have the result that *any one of the ten solids Σ^* is the tangent solid of R_3^{15} at all the points of the tangent to γ^4 at the point where it is osculated by Σ^*.*

Since the osculating plane α^* of γ^4 at any one of the ten points of J is a secant plane of ϑ, having six intersections with ϑ, all the sixty intersections of ϑ with the sextic locus F_3^6 generated by the osculating planes of γ^4 are accounted for in this way.

44. Consider now the curve of intersection of R_3^{15} and the sextic surface F_2^6 generated by the tangents of γ^4. This curve includes the tangents of γ^4 at the ten points of J. Now the generating plane of R_3^{15} at a point j of J is the osculating plane α^*, which touches F_2^6 at every point of the tangent to γ^4 at j, and the tangent solid of R_3^{15} at every point of this tangent is Σ^*, the osculating solid of γ^4 at j. Consider then the section of this figure by an arbitrary solid S. The section of F_2^6 is a curve f and that of R_3^{15} a ruled surface r; the point in which the tangent to γ^4 at j meets S is common to f and r, the tangent of f being a generator of r and the osculating plane of f, which is the

plane in which S meets Σ^*, being the tangent plane of r. It follows that this point counts for three among the intersections of f and r, and hence that the tangents of γ^4 at the ten points of J are to be reckoned three times as part of the curve of intersection of R_3^{15} and F_2^6.

The curve of intersection of R_3^{15} and F_2^6 will include, apart from these ten tangents of γ^4, a curve Γ. Through an arbitrary point there pass four osculating solids of γ^4; if however the point lies on a tangent of γ^4 the solid which osculates γ^4 at its point of contact with this tangent counts for three among these four, and so there is only one other osculating solid of γ^4 passing through the point. Now any point of R_3^{15} is common to the osculating solids of γ^4 at two points which belong to the same set of g_6^1; if then a tangent of γ^4 meets R_3^{15}, and its point of contact with γ^4 is not one of the ten points of J, its intersection with R_3^{15} can only be one of its five intersections with those solids which osculate γ^4 at the remaining five points of that set of g_6^1 which contains the point of contact of the tangent with γ^4. Hence a tangent of γ^4 meets R_3^{15} not in fifteen points but only in five distinct points; it will be an inflectional tangent of R_3^{15} at each of these five points. Again: an arbitrary plane meets six tangents of γ^4, but if the plane lies in an osculating solid of γ^4 the tangent of γ^4 at the point of osculation of this solid counts for three among the six tangents which meet the plane. But a plane of R_3^{15} is common to two osculating solids of γ^4, and therefore the only tangents of γ^4 which it can meet are the two which touch γ^4 at the points where these two solids osculate it. It follows that the curve Γ meets each generating plane of R_3^{15} in two points and each tangent of γ^4 in five points, and that Γ is to be included three times as part of the curve of intersection of R_3^{15} and F_2^6. Since the complete curve of intersection of these two loci is of order 90, and since it includes ten tangents of γ^4, each of which is to be counted three times, the curve Γ is of order 20.

45. Consider now the curve of intersection of the scroll R_2^{20}, generated by the trisecants of ϑ, and the locus F_3^6, generated by the osculating planes of γ^4. The generating planes of F_3^6 include the ten planes α^* and each of these, being a secant plane of ϑ, contains four generators of R_2^{20}; we have thus forty generators of R_2^{20} lying on F_3^6. The complete intersection of R_2^{20} and F_3^6 will also include, apart from these forty lines, a curve $\triangle$.

Any generator of R_2^{20} is the intersection of three osculating solids of γ^4, and therefore meets the three osculating planes of γ^4 which lie in these respective solids; moreover any osculating solid of γ^4 is the tangent solid of F_3^6 at *every* point of the corresponding osculating plane, so that *the generators of* R_2^{20} *are tritangents of* F_3^6. The intersections of a generator of R_2^{20} with the sextic locus F_3^6 are thus all accounted for. Again: take a generating plane π of F_3^6, other than the ten planes α^*. The point of contact of π and γ^4 determines a set of g_6^1; the osculating solids of γ^4 at the remaining five points of this set meet π in five lines, and the ten vertices of the pentagram formed by these lines are on R_2^{20}. Conversely: let P be an intersection of π and R_2^{20}. Then through P there pass three osculating solids of γ^4 whose points of osculation all belong to the same set of g_6^1. But there are only two osculating solids of γ^4 passing through P, apart from the osculating solid which contains π; hence, since π is not one of the planes α^*, these other two solids must osculate γ^4 in points which belong to that set of g_6^1 determined by the point of contact of π and γ^4. Thus P is one of the ten vertices of the pentagram just found, and the intersections of π and R_2^{20} consist of the ten vertices of this pentagram, counted twice. The curve $\triangle$ therefore meets each generator of R_2^{20} in three points and each generating plane of F_3^6 in ten points; R_2^{20} and F_3^6 intersect in the forty lines already noticed and in the curve $\triangle$ counted twice. Since the complete intersection is of order 120, $\triangle$ must be of order 40.

46. Let us now consider the configuration of points on the plane quintic ζ corresponding to the intersections of ϑ with Σ^*. Since Σ^* meets the six cones (P), (X), (Y), (Z), (T), (U) in plane-pairs the contact quartic of ζ whose points represent quadrics touching Σ^* has nodes at the six points p, x, y, z, t, u of ζ; hence this contact quartic, having six nodes, must consist of four lines whose intersections are these six points. The remaining intersections of the quartic with ζ consist of four contacts at q, r, v, w; since the cones (P), (X), (Y) all pass through Q the three points p, x, y of ζ lie on the tangent of ζ at q; similarly ptz touches ζ at r, uzx touches ζ at v and uyt touches ζ at w; *the contact quartic associated with* Σ^* *is an in-and-circumscribed quadrilateral*, and ζ has ten such in-and-circumscribed quadrilaterals. Since the cones (Q), (R), (V), (W), (O^*) all belong to a pencil the four points of contact of the sides of the in-and-circumscribed quadrilateral with ζ are collinear, and the line on which they lie meets ζ again in o^*.

The vertices of the quadrilateral formed by the four lines pxy, ptz, uzx, uyt correspond to the six intersections of ϑ with a secant plane; hence this quadrilateral is part of one of the inscribed hexagrams of ζ; the other two sides of this hexagram both coincide with $qrvwo^*$. Now through any point of ζ there pass two sides of an inscribed hexagram, these being the two tangents which can be drawn from the point to the conic γ which is the envelope of the sides of the hexagrams; for the point o^* however these two sides coincide, so that o^* must be on γ. This again shows that there are ten points o^*, since γ meets ζ in ten points.

Those quadrics of the net in [4] which touch any one of the solids Σ are represented in the plane of ζ by the points of a contact quartic which breaks up into two tritangent conics of ζ; of the system of ∞^4 contact quartics there are therefore ∞^1 which break up in this way, and these include ten special quartics which break up further into in-and-circumscribed quadrilaterals. Since the developable D is of class twenty there are forty solids Σ touching an arbitrary quadric; hence through an arbitrary point in the plane of ζ there pass forty of the composite contact quartics.

47. The quadrics of the net which are represented by the points of the conic γ have as their envelope a quartic primal Ξ^* on which the base curve C of the net of quadrics is a double curve ($G.N.Q.$ § 31). The forty intersections of Ξ^* with ϑ correspond to the forty points of ζ which are the points of contact with ζ of the forty common tangents of ζ and γ; in other words they correspond to the points where ζ is touched by the sides of its inscribed hexagrams. These forty points are the ten sets of four points such as q, r, v, w; hence *the forty points of contact of ϑ with those of its trisecants which touch it lie on a quartic primal Ξ^* having C as a double curve.* Now if K is any intersection of ϑ with a quartic primal on which C is a double curve the tangent solid of this primal at K is the solid ϖ associated with K; hence, since Σ^* is the solid ϖ associated with each vertex of the tetrahedron $QRVW$, each of the ten solids Σ^* is a quadritangent solid of Ξ^* and so meets Ξ^* in a quartic surface with twelve nodes. Through any point of ϑ there pass four chords of C, these all lying in the solid ϖ associated with the point; in particular the four chords of C which pass through any one of the points Q, R, V, W all lie in Σ^*. Wherefore the eight base points of the net of quadric surfaces in which Σ^* meets the quadrics of the net in [4] can be regarded as two tetrahedra in perspective from any one of the points Q, R, V, W; hence these eight base points form, when

taken with Q, R, V, W, a set of three desmic tetrahedra. The section of Ξ^* by Σ^* is thus *a desmic quartic surface.*

48. We can obtain a form for the equations of the quadrics if we take as simplex of reference the simplex formed by a solid Σ^* and the four other faces of the hexahedron of which Σ^* is two coincident faces. Take the equations of the four solids O^*RVW, O^*VWQ, O^*WQR, O^*QRV to be $X_1 = 0$, $X_2 = 0$, $X_3 = 0$, $X_4 = 0$ respectively, and let the solid $QRVW$ have the equation $X_5 = 0$. Since any quadric of the net is met by Σ^* in a surface with regard to which $QRVW$ is a self-polar tetrahedron its equation must be of the form

$$a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + a_4 X_4^2 + X_5 (\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 + \alpha_5 X_5) = 0.$$

The polar solid of O^* in regard to this quadric is

$$\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 + \alpha_5 X_5 = 0,$$

and this must meet $X_5 = 0$ in the secant plane conjugate to O^*; we can suppose that this secant plane is given by

$$X_1 + X_2 + X_3 + X_4 = X_5 = 0,$$

and therefore that the equation of the quadric is

$$a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + a_4 X_4^2 + a_5 X_5^2 = 2\alpha X_5 (X_1 + X_2 + X_3 + X_4 + X_5).$$

Hence, if a net of quadrics can be reduced to the canonical form II, with the coefficients satisfying $\Delta = 0$, it can be reduced, and that in ten ways, to the more special canonical form

$$\left.\begin{aligned} a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + a_4 X_4^2 + a_5 X_5^2 + 2\alpha X_5 X_6 = 0, \\ b_1 X_1^2 + b_2 X_2^2 + b_3 X_3^2 + b_4 X_4^2 + b_5 X_5^2 + 2\beta X_5 X_6 = 0, \\ c_1 X_1^2 + c_2 X_2^2 + c_3 X_3^2 + c_4 X_4^2 + c_5 X_5^2 + 2\gamma X_5 X_6 = 0, \end{aligned}\right\} \quad \ldots\ldots\ldots \mathrm{IV}$$

where

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \equiv 0.$$

If we write

$$\xi_i \equiv x a_i + y b_i + z c_i, \quad -\eta \equiv x\alpha + y\beta + z\gamma,$$

the discriminant of a general quadric of the net IV is

$$\begin{vmatrix} \xi_1 & 0 & 0 & 0 & \eta \\ 0 & \xi_2 & 0 & 0 & \eta \\ 0 & 0 & \xi_3 & 0 & \eta \\ 0 & 0 & 0 & \xi_4 & \eta \\ \eta & \eta & \eta & \eta & \xi_5 + 2\eta \end{vmatrix},$$

or

$$\xi_1 \xi_2 \xi_3 \xi_4 (\xi_5 + 2\eta) - \eta^2 (\xi_2 \xi_3 \xi_4 + \xi_3 \xi_4 \xi_1 + \xi_4 \xi_1 \xi_2 + \xi_1 \xi_2 \xi_3).$$

The Jacobian curve of the net is thus in birational correspondence with the plane quintic

$$\xi_1 \xi_2 \xi_3 \xi_4 (\xi_5 + 2\eta) = \eta^2 (\xi_2 \xi_3 \xi_4 + \xi_3 \xi_4 \xi_1 + \xi_4 \xi_1 \xi_2 + \xi_1 \xi_2 \xi_3).$$

This quintic curve is circumscribed to the quadrilateral formed by the four lines

$$\xi_1 = 0, \quad \xi_2 = 0, \quad \xi_3 = 0, \quad \xi_4 = 0.$$

Also each of these four lines touches the quintic, the four points of contact all lying on $\eta = 0$.

49. We can write down the equation of the primal Ξ^* when the quadrics of the net are given by IV. Calling the three equations of IV

$$Q_0 = 0, \quad Q_1 = 0, \quad Q_2 = 0$$

respectively, the equation of Ξ^* is

$$A Q_0^2 + B Q_1^2 + C Q_2^2 + 2F Q_1 Q_2 + 2G Q_2 Q_0 + 2H Q_0 Q_1 = 0,$$

where

$$A l^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$$

is the tangential equation of γ. But γ is determined from the fact that it touches the five lines $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$, $\xi_4 = 0$, $\eta = 0$, and we have the equation of Ξ^* in the form

$$\begin{vmatrix} Q_0^2 & Q_1^2 & Q_2^2 & Q_1 Q_2 & Q_2 Q_0 & Q_0 Q_1 \\ a_1^2 & b_1^2 & c_1^2 & b_1 c_1 & c_1 a_1 & a_1 b_1 \\ a_2^2 & b_2^2 & c_2^2 & b_2 c_2 & c_2 a_2 & a_2 b_2 \\ a_3^2 & b_3^2 & c_3^2 & b_3 c_3 & c_3 a_3 & a_3 b_3 \\ a_4^2 & b_4^2 & c_4^2 & b_4 c_4 & c_4 a_4 & a_4 b_4 \\ \alpha^2 & \beta^2 & \gamma^2 & \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} = 0.$$

When X_5 is put identically equal to zero in this equation there results the equation of the desmic surface which is the section of Ξ^* by Σ^*.

It has been seen that the quadrics of the net reciprocate the curve γ^4 into the curves q_{12} lying on the primal Φ: the primal Ξ^* is the envelope of those quadrics of the net which reciprocate γ^4 into the characteristic curves q.

The Net of Polar Quadrics of Points of a Plane in regard to a Segre Cubic Primal.

50. The equation to a Segre cubic primal is

$$X_1^3 + X_2^3 + X_3^3 + X_4^3 + X_5^3 + X_6^3 = 0,$$

where the six forms X, which are linear homogeneous functions of five coordinates, satisfy the identity

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \equiv 0.$$

The ten points for which

$$X_1^2 = X_2^2 = X_3^2 = X_4^2 = X_5^2 = X_6^2,$$

i. e. those points for which three of the six coordinates have the value $+1$ and the remaining three the value -1, are nodes of the primal; and the fifteen planes whose equations are

$$X_i + X_j = X_k + X_l = X_m + X_n = 0,$$

where $(ijklmn)$ is a permutation of (123456), all lie on the primal. Each plane contains four nodes and each node lies in six planes. Call the primal Ω.

The hexahedron $\mathfrak{h}$ bounded by the six solids $X = 0$ has important relations[1] with Ω, and Ω is uniquely determined when $\mathfrak{h}$ is given. The fifteen vertices of $\mathfrak{h}$ can be divided in fifteen ways into sets of three such that no two vertices of the same set lie on the same edge of $\mathfrak{h}$; the fifteen planes determined by these sets of three vertices are the fifteen planes on Ω. The three vertices of $\mathfrak{h}$ belonging to any one of the sets are the diagonal points of the quadrangle formed by the four nodes of Ω which lie in the plane of the set; e. g. the plane

[1] Concerning the Segre primal and its associated hexahedron see Castelnuovo: *Atti Ist. Veneto* (6), 6 (1888), 547—565.

$$X_1 + X_2 = X_3 + X_4 = X_5 + X_6 = 0$$

contains the four nodes

$$(1, -1, 1, -1, 1, -1); \quad (-1, 1, 1, -1, 1, -1); \quad (1, -1, -1, 1, 1, -1); \quad (1, -1, 1, -1, -1, 1)$$

and the three points

$$(1, -1, 0, 0, 0, 0); \quad (0, 0, 1, -1, 0, 0); \quad (0, 0, 0, 0, 1, -1).$$

The last three points are vertices of $\mathfrak{h}$ and are the diagonal points of the quadrangle formed by the preceding four; no two of them lie on the same edge of $\mathfrak{h}$.

The Hessian of Ω is the quintic primal Y whose equation is

$$X_1^{-1} + X_2^{-1} + X_3^{-1} + X_4^{-1} + X_5^{-1} + X_6^{-1} = 0;$$

this also contains the fifteen planes which lie on Ω, these planes forming the complete intersection of Ω and Y. The polar quadric of the point x, whose coordinates are $(x_1, x_2, x_3, x_4, x_5, x_6)$, in regard to Ω is

$$x_1 X_1^2 + x_2 X_2^2 + x_3 X_3^2 + x_4 X_4^2 + x_5 X_5^2 + x_6 X_6^2 = 0.$$

If this quadric is a cone then x lies on Y; also the vertex x' of the cone lies on Y, and the vertex of the polar cone of x' is x. The primal Y, as well as containing the fifteen planes of Ω, also contains the fifteen planes of intersection of the pairs of faces of $\mathfrak{h}$; the twenty edges of $\mathfrak{h}$ are double lines on Y and the fifteen vertices of $\mathfrak{h}$ are triple points of Y; this follows immediately from the form of the equation of Y. The polar cones of the points in a plane of $\mathfrak{h}$ all have the opposite vertex of $\mathfrak{h}$ for their common vertex. The polar cones of the points of an edge of $\mathfrak{h}$ are line-cones with the opposite edge of $\mathfrak{h}$ for vertex; the polar cone of a vertex of $\mathfrak{h}$ is a pair of solids passing through the opposite plane of $\mathfrak{h}$ and harmonically conjugate in regard to the two faces of $\mathfrak{h}$ which intersect in that plane. All this follows at once if the appropriate values for the coordinates x are substituted in the equation of the polar quadric. It also follows easily that the polar cone of any point in one of the fifteen planes of Ω has its vertex in that same plane; e. g. the polar cone of the point $(x, -x, x', -x', x'', -x'')$, which lies in the plane

$$X_1 + X_2 = X_3 + X_4 = X_5 + X_6 = 0,$$

is

$$x(X_1^2 - X_2^2) + x'(X_3^2 - X_4^2) + x''(X_5^2 - X_6^2) = 0,$$

and the vertex of this cone is $(x^{-1}, -x^{-1}, x'^{-1}, -x'^{-1}, x''^{-1}, -x''^{-1})$.

51. Consider now the net of polar quadrics of the points of a plane π in regard to Ω. The cones of the net are the polars of the points of the section ζ of Y by π; ζ is a quintic curve circumscribed to the hexagram formed by the lines in which the faces of $\mathfrak{h}$ meet π, and ζ is in birational correspondence with the curve ϑ, lying on Y, which is the locus of the vertices of the cones. Since each plane of $\mathfrak{h}$ meets π in a vertex of the hexagram ϑ passes through each vertex of $\mathfrak{h}$, and the vertices of $\mathfrak{h}$ correspond, in the birational correspondence between ϑ and ζ, to the vertices of the hexagram in π. Also each plane on Ω meets π, so that ϑ meets each plane of Ω in one point other than the three vertices of $\mathfrak{h}$ which lie on that plane. Since all the polar quadrics pass through all the nodes of Ω the fifteen planes on Ω are quadrisecant planes both of the Jacobian curve ϑ and of the base curve C of the net of quadrics. The Jacobian curve is circumscribed to $\mathfrak{h}$ and is such that *any plane which contains three vertices, no two of which lie on the same edge of* $\mathfrak{h}$, *meets the curve in a fourth point.* This property of the Jacobian curve of the net of polar quadrics of the points of a plane in regard to Segre cubic primal does not hold for the more general net of quadrics II whose Jacobian curve is circumscribed to a hexahedron. It can easily be shown that if the plane

$$X_1 + X_2 = X_3 + X_4 = X_5 + X_6 = 0,$$

which contains three vertices of $\mathfrak{h}$ no two of which lie on the same edge, meets the Jacobian curve of the net II in a fourth point, then the coefficients in II must satisfy the condition

$$|12, 34, 56| \equiv \begin{vmatrix} a_1 + a_2 & a_3 + a_4 & a_5 + a_6 \\ b_1 + b_2 & b_3 + b_4 & b_5 + b_6 \\ c_1 + c_2 & c_3 + c_4 & c_5 + c_6 \end{vmatrix} = 0.$$

Moreover, when this determinant vanishes the conics in which the plane is met by the quadrics of the net all belong to the same pencil, so that the plane is a quadrisecant plane not only of the Jacobian curve but also of the base-curve of the net.

This condition, together with the fourteen similar conditions, is certainly satisfied for the net of polar quadrics; for the quadric

$$a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + a_4 X_4^2 + a_5 X_5^2 + a_6 X_6^2 = 0$$

is the polar quadric of a point in regard to Ω if and only if

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 0.$$

Similar conditions for the coefficients b and c ensure that the fifteen determinants $|ij, kl, mn|$ all vanish.

52. It follows, exactly as in § 14, that ϑ is in birational correspondence with the plane quintic

$$\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1} + \xi_4^{-1} + \xi_5^{-1} + \xi_6^{-1} = 0;$$

but now we have the additional information that one of the linear identities satisfied by the six forms ξ is

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 \equiv 0.$$

Now this identity shows that three lines such as

$$\xi_i + \xi_j = 0, \quad \xi_k + \xi_l = 0, \quad \xi_m + \xi_n = 0,$$

are concurrent, and it will be remembered that these lines are tangents of the plane quintic ζ at vertices of its inscribed hexagram. Moreover the equation of ζ is satisfied when the equations of these three lines are simultaneously satisfied. Hence *the Jacobian curve of the net of polar quadrics is in birational correspondence with a plane quintic which is circumscribed to a hexagram, and which is such that if any three vertices of the hexagram are taken such that no two of them lie on the same side, the tangents of the quintic at these three vertices are concurrent in a point of the curve.* The quadrilateral formed by any four sides of the hexagram is such that the tangents of the quintic at any pair of opposite vertices intersect on the curve; the three points of the quintic so obtained from the three pairs of opposite vertices are collinear, and the line on which they lie is the tangent of the quintic at the intersection of the remaining two sides of the hexagram.

The plane which contains any set of three vertices of $\mathfrak{h}$ no two of which lie on the same edge of $\mathfrak{h}$ meets ϑ in a fourth point; since the plane is a quadrisecant plane of the base curve of the net of quadrics, and since this fourth point of ϑ is *not* a diagonal point of the quadrangle formed by the four points of C, the cone whose vertex is at this fourth point must contain the plane entirely. Thus the point of the plane quintic ζ which corresponds to this fourth point of ϑ must lie on the three tangents of ζ at the points which correspond to the three vertices of $\mathfrak{h}$. We thus obtain again the fifteen concurrent triads of tangents of ζ; the points of contact are vertices of the inscribed hexagram and the points of concurrence are on ζ.

The linear identity between the six forms ξ shows that the two lines

$$\xi_i + \xi_j + \xi_k = 0 \text{ and } \xi_l + \xi_m + \xi_n = 0$$

coincide: the tangents of ζ at the three vertices of a triangle whose sides belong to the inscribed hexagram meet the opposite sides of the triangle in three collinear points; the lines so obtained from two triangles whose sides together constitute the whole hexagram coincide for the special quintic curve we are now considering.

If PQR is a trisecant of ϑ the corresponding points p, q, r are the vertices of a triangle whose sides belong to the hexagram inscribed in ζ; the tangents of ζ at the vertices of this triangle meet the opposite sides in three collinear points, and it has been shown (*G.N.Q.* § 26) that the points of the line on which these three points lie represent the quadrics of a pencil in [4], the pencil being defined by the fact that the Hessian points of the triad PQR are a pair of conjugate points in regard to all the quadrics belonging to it. If then the net of quadrics in [4] consists of the polar quadrics of the points of a plane in regard to a Segre primal, and if PQR and UVW are a pair of opposite edges of $\mathfrak{h}$ and so of conjugate trisecants of ϑ, every quadric of the net which is such that the Hessian points of either triad, PQR or UVW, are conjugate in regard to it is also such that the Hessian points of the other triad are conjugate in regard to it.

53. It has been shown that any plane quintic which is in birational correspondence with the Jacobian curve of a net of polar quadrics has an inscribed hexagram with certain properties; it may now be shown, conversely, that such a plane quintic can always be put into birational correspondence with a Jacobian curve of a net of polar quadrics.

The plane quintic ζ is circumscribed to a hexagram; it may therefore be supposed that, absorbing certain constants into the linear forms ξ if necessary, its equation is

$$\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1} + \xi_4^{-1} + \xi_5^{-1} + \xi_6^{-1} = 0.$$

The tangent of ζ at the intersection of the two lines $\xi_i = 0$ and $\xi_j = 0$ is $\xi_i + \xi_j = 0$. Take now any quadrilateral formed by four of the six sides of the hexagram, say by $\xi_i = 0$, $\xi_j = 0$, $\xi_k = 0$, $\xi_l = 0$; then the two tangents of ζ at any pair of opposite vertices of this quadrilateral intersect on the line $\xi_i + \xi_j + \xi_k + \xi_l = 0$. But ζ is to be such that this line is the tangent at the intersection of the two remaining sides of the hexagram; in other words it must be the same line as $\xi_m + \xi_n = 0$. Hence there is an identity

$$\xi_i + \xi_j + \xi_k + \xi_l + \varrho(\xi_m + \xi_n) \equiv 0,$$

where ϱ is a constant. Similarly there is another identity

$$\xi_i + \xi_j + \xi_k + \xi_m + \varrho'(\xi_n + \xi_l) \equiv 0.$$

From these two identities it follows that

$$\xi_l + \varrho(\xi_m + \xi_n) \equiv \xi_m + \varrho'(\xi_n + \xi_l),$$

and therefore, since the three lines $\xi_l = 0$, $\xi_m = 0$, $\xi_n = 0$ are not concurrent, we must have $\varrho = \varrho' = 1$. Wherefore the six linear forms ξ satisfy the identity

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 \equiv 0.$$

Now the quintic curve ζ, having an inscribed hexagram, can, as has been seen, be put into birational correspondence with the Jacobian curve of the net of quadrics II, the six linear forms X being such that

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \equiv 0.$$

But now the identity which is satisfied by the six forms ξ shows that all the quadrics of the net II are, in the present instance, polar quadrics of points in regard to the primal

$$X_1^3 + X_2^3 + X_3^3 + X_4^3 + X_5^3 + X_6^3 = 0,$$

and this is, in virtue of the identity satisfied by the six forms X, a Segre primal.

54. In order that the net of quadrics II should be the net of polar quadrics of points of a plane in regard to a Segre primal it is necessary and sufficient

that the conditions

$$\Sigma a_i = \Sigma b_i = \Sigma c_i = 0$$

should be satisfied; when these relations are satisfied the fifteen determinants $|ij, kl, mn|$ all vanish. It is therefore natural to enquire whether the net II can be specialised in such a way that some, but not all, of these fifteen determinants vanish. Associated with each vanishing determinant there is a quadrisecant plane of ϑ. Also when $|ij, kl, mn| = 0$ we have a set of three concurrent tangents of ζ, namely

$$\xi_i + \xi_j = 0, \quad \xi_k + \xi_l = 0, \quad \xi_m + \xi_n = 0;$$

and conversely. In a general net II there is no such concurrent triad: for the net of polar quadrics there are fifteen such concurrent triads: we wish to consider whether there are any intermediate cases. It must always be borne in mind that none of the twenty determinants such as

$$|i, j, k| \equiv \begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{vmatrix}$$

vanishes; for the vanishing of such a determinant means that a quadric of the net II can be found whose equation is obtained by equating to zero the sum of only three squares X_l^2, X_m^2, X_n^2; such a quadric is a line-cone, and it is always supposed that no member of the net of quadrics can be a line-cone. The six linear forms ξ satisfy three linear identities; in the intermediate cases that we now wish to consider it must not be possible to deduce from these three the identity $\Sigma\xi \equiv 0$, nor must it be possible to obtain, by combining the three identities in any way, an identity involving only three of the six forms ξ.

If one of the identities satisfied by the six forms forms ξ is

$$\varrho(\xi_i + \xi_j) + \sigma(\xi_k + \xi_l) + \tau(\xi_m + \xi_n) \equiv 0,$$

where ϱ, σ, τ are unequal numerical constants, then a single determinant $|ij, kl, mn|$ vanishes, and we have one concurrent triad of tangents of ζ. Also there is one plane which contains three vertices of $\mathfrak{h}$, no two of which lie on the same edge of $\mathfrak{h}$, and which meets ϑ in a fourth point other than these three vertices; the equations of this plane are

$$X_i + X_j = X_k + X_l = X_m + X_n = 0.$$

Suppose now that the six forms ξ satisfy the identity

$$\varrho(\xi_i + \xi_j) + \xi_k + \xi_l + \xi_m + \xi_n \equiv 0,$$

where ϱ is a constant different from unity; then

$$|ij, kl, mn| = |ij, km, nl| = |ij, kn, lm| = 0,$$

three of the fifteen determinants vanishing. The plane quintic ζ now has an inscribed hexagram with three concurrent triads of tangents. The tangent $\xi_i + \xi_j = 0$ belongs to each of these three triads; it meets ζ in three points other than its point of contact, and through each of these three points there pass two other tangents of ζ whose points of contact are a pair of opposite vertices of the quadrilateral formed by the four sides of the hexagram other than $\xi_i = 0$ and $\xi_j = 0$.

The net of quadrics II with coefficients satisfying the three relations

$$|ij, kl, mn| = |ij, km, nl| = |ij, kn, lm| = 0$$

has a Jacobian curve ϑ of which the three planes

$$\begin{aligned} X_i + X_j &= X_k + X_l = X_m + X_n = 0, \\ X_i + X_j &= X_k + X_m = X_n + X_l = 0, \\ X_i + X_j &= X_k + X_n = X_l + X_m = 0 \end{aligned}$$

are all quadrisecant planes; these three planes all lie in the solid $X_i + X_j = 0$ and intersect where $X_k = X_l = X_m = X_n = 0$.

It may be pointed out that if two of the three determinants

$$|ij, kl, mn|, \quad |ij, km, nl|, \quad |ij, kn, lm|$$

vanish then the other one must vanish also. For suppose, for example, that

$$|ij, km, nl| = |ij, kn, lm| = 0.$$

Then there exist two identities

$$\begin{aligned} (\xi_i + \xi_j) + \sigma\,(\xi_k + \xi_m) + \tau\,(\xi_n + \xi_l) &\equiv 0, \\ (\xi_i + \xi_j) + \sigma'(\xi_k + \xi_n) + \tau'(\xi_l + \xi_m) &\equiv 0. \end{aligned}$$

If either $\sigma = \tau$ or $\sigma' = \tau'$ then the third determinant $|ij, kl, mn|$ will certainly vanish. If neither $\sigma = \tau$ nor $\sigma' = \tau'$ we deduce from these two identities the third identity.

$$(\sigma - \sigma' - \tau + \tau')(\xi_i + \xi_j) + (\sigma\tau' - \sigma'\tau)(\xi_k + \xi_l + \xi_m + \xi_n) \equiv 0.$$

If the coefficients here vanish we must have $\sigma = \sigma'$ and $\tau = \tau'$; the two previous identities then give $\sigma = \sigma' = \tau = \tau'$, and we are supposing this not to be so. If the coefficients in the last identity do not vanish, then we deduce $|ij, kl, mn| = 0$.

55. The Jacobian curve ϑ of the net of polar quadrics of the points of a plane π in regard to a Segre cubic primal is circumscribed to a hexahedron $\mathfrak{h}$ and, in general, $\mathfrak{h}$ is the only hexahedron inscribed in ϑ; but π can be chosen so that ϑ is circumscribed to an infinity of hexahedra. For suppose, the cubic primal Ω and the associated hexahedron $\mathfrak{h}$ being given, π is chosen so that the six lines in which it is met by the faces of $\mathfrak{h}$ are tangents to the same conic; of the ∞^6 planes of [4] there are ∞^5 satisfying this condition, those planes which fulfil the condition and lie in an arbitrary solid being the tangent planes of the surface reciprocal to a Weddle surface. The plane π meets the Hessian Y in a quintic curve ζ, and a hexagram is inscribed in ζ and circumscribed to a conic; hence ζ has an infinity of inscribed hexagrams whose sides all touch this same conic (cf. § 16). Now the polar cones, in regard to Ω, of the points of ζ lie on a curve ϑ, and those points of π whose polar quadrics, in regard to Ω, touch an arbitrary solid lie on a contact quartic of ζ. But if the arbitrary solid is chosen to be one of the six faces of $\mathfrak{h}$ the ten points of contact of the quartic curve with ζ are the ten vertices of the pentagram formed by the lines in which the five remaining faces of $\mathfrak{h}$ meet π. It follows that the system of contact quartics of ζ associated with the solids of [4] must be the same system as that which is determined, as in § 18, by the pentagrams which belong to the inscribed hexagrams of ζ. Hence, just as in § 18, ϑ has an infinity of inscribed hexahedra.

56. Of the inscribed hexagrams of ζ there is one such that the tangents at its fifteen vertices are a set of fifteen lines which are concurrent in threes in fifteen points. The equation of a plane quintic with these properties can easily be obtained. For suppose the sides of the hexagrams all touch the conic $xz = y^2$, and that ζ is circumscribed to the hexagram formed by the six lines

$$x - 2y\theta_i + z\theta_i^2 = 0. \qquad (i = 1, 2, 3, 4, 5, 6).$$

Let us write

$$f(t) \equiv (t-\theta_1)(t-\theta_2)(t-\theta_3)(t-\theta_4)(t-\theta_5)(t-\theta_6),$$

$$x - 2y\theta_i + z\theta_i^2 \equiv \xi_i f'(\theta_i);$$

then the three linear identities connecting the six forms ξ are

$$\Sigma\xi_i \equiv 0, \quad \Sigma\theta_i\xi_i \equiv 0, \quad \Sigma\theta_i^2\xi_i \equiv 0.$$

Since the quintic circumscribes the hexagram formed by the six lines $\xi_i = 0$ we may suppose that it has an equation of the form

$$A_1\xi_1^{-1} + A_2\xi_2^{-1} + A_3\xi_3^{-1} + A_4\xi_4^{-1} + A_5\xi_5^{-1} + A_6\xi_6^{-1} = 0,$$

the tangent at the point $\xi_i = \xi_j = 0$ then being $A_i^{-1}\xi_i + A_j^{-1}\xi_j = 0$. The fifteen tangents at the vertices of the hexagram will then certainly form fifteen concurrent triads if we choose

$$A_i^{-1} = \varrho + \sigma\theta_i + \tau\theta_i^2,$$

the constants ϱ, σ, τ being the same for all six values of i. Hence, given the conic and the hexagram, any quintic curve of the doubly-infinite family given by the equation

$$\sum_{i=1}^{6} f'(\theta_i)(\varrho + \sigma\theta_i + \tau\theta_i^2)^{-1}(x - 2y\theta_i + z\theta_i^2)^{-1} = 0$$

fulfils the required conditions.

The Freedoms of the Different Kinds of Jacobian Curves.

57. The word »freedom» is here used in the sense of the German *Konstantenzahl;* when we say that a curve, with certain specified properties and assumed to lie in a definite space $[n]$, is of freedom f we mean that the manifold of curves in $[n]$ with these specified properties is of dimension f; or, otherwise, that a curve with these specified properties can be regarded as belonging to a set of a finite number of such curves if we assign the values of f parameters on which the curve depends. For example: elliptic plane cubics are of freedom 9, rational plane cubics of freedom 8; plane cubics with a node at a fixed point of freedom 6, and so on.

The freedom of spaces $[k]$ in $[n]$ is $(k+1)(n-k)$; we shall appeal to this result occasionally in the particular case $k=2$, using the fact that the freedom of planes in $[n]$ is $3n-6$.

58. The Jacobian curve of a net of quadrics in [4] is of order 10 and genus 6, and is a particular case of the *determinantal curve* whose equations are obtained by equating to zero all the three-rowed determinants of a matrix of three rows and five columns whose elements are all linear functions of the coordinates. Such a determinantal curve is generated by the intersections of sets of corresponding solids of five projectively related doubly-infinite systems; it is not, however, even for the most general of these projective generations, the most general curve of order 10 and genus 6 in [4]. It follows from a result obtained by Brill and Nöther[1] that the freedom of the most general curve of order 10 and genus 6 in [4] is 45, while it is found by Room[2] that the freedom of the determinantal curve is 42. The freedom of the Jacobian curve can be calculated quite easily. The quadrics in [4] are of freedom 14, this being one less than the number of terms in a homogeneous quadratic polynominal in five variables; moreover they form a linear system, and so can be regarded as points in [14]. The freedom of nets of quadrics in [4] is therefore the same as that of planes in [14], which is 36. Since each net of quadrics determines a Jacobian curve, and conversely, we have our first result: *the Jacobian curve of a general net of quadrics in [4] is of freedom 36.* Incidentally it follows that in order to be a Jacobian curve a general curve of order 10 and genus 6 in [4] must be subjected to nine conditions while a determinantal curve must be subjected to six conditions.

59. The freedom of the general Jacobian curve can also be obtained by an appeal to the canonical form I. A given net of quadrics can be reduced to the canonical form I in a finite number of ways, and the net can be identified in two stages; first by identifying the seven linear forms whose squares occur in I and secondly by identifying the coefficients. Now of the seven linear forms Z is completely determined when $X_1, X_2, X_3, Y_1, Y_2, Y_3$ are given, and these last six forms can represent any six solids in the space. Since the identities

$$X_1 + X_2 + X_3 \equiv Z \equiv Y_1 + Y_2 + Y_3$$

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[1] *Math. Annalen* 7 (1874), 308.

[2] *Proc. London Math. Soc.* (2), 36 (1933), 25—26.

must be satisfied the seven linear forms are, except for a constant multiplier the same for all, completely identified when the six solids have once been chosen. Then the quadrics whose equations are obtained by equating to zero a sum of multiples of squares of these seven forms constitute a linear system of dimension 6 and so can be regarded as points in [6]; the freedom of the nets I when the six solids are chosen is therefore the same as the freedom of planes in [6], which is 12. But six arbitrary solids in [4] make up a configuration whose freedom is $6.4 = 24$; hence the freedom of the general net of quadrics I is $24 + 12 = 36$, as before.

If now we wish to calculate the freedom of the Jacobian curve which has two pairs of concurrent trisecants we may again appeal to the canonical form I. As before we have a configuration of six arbitrary solids with freedom 24, but we cannot now choose every net of quadrics I, only those for which a determinant

$$\begin{vmatrix} a_1 & c & b_1 \\ a_1' & c' & b_1' \\ a_1'' & c'' & b_1'' \end{vmatrix}$$

vanishes being eligible; in other words, when we regard the quadrics as points of [6], the six solids in [4] having been previously chosen, we cannot take every plane of [6] and so obtain a net of quadrics; if we wish to obtain a Jacobian curve with two pairs of concurrent trisecants we may only choose those planes one of whose coordinates vanishes. The freedom of the planes that may be chosen is therefore not 12 but 11, and the Jacobian curve now has freedom 35.

Similarly if two or three of the determinants

$$\begin{vmatrix} a_1 & c & b_1 \\ a_1' & c' & b_1' \\ a_1'' & c'' & b_1'' \end{vmatrix}, \quad \begin{vmatrix} a_2 & c & b_2 \\ a_2' & c' & b_2' \\ a_2'' & c'' & b_2'' \end{vmatrix}, \quad \begin{vmatrix} a_3 & c & b_3 \\ a_3' & c' & b_3' \\ a_3'' & c'' & b_3'' \end{vmatrix}$$

vanish the Jacobian curve has freedom 34 or 33 respectively.

60. We pass now to the calculation of the freedom of the Jacobian curve with four concurrent trisecants.

We can choose any point O of [4] to be the point through which the four trisecants pass, thus imposing four conditions on the complete configuration. We then choose any four solids through O; when O has been fixed such a set of four solids has freedom 12, the same as the freedom of tetrahedra in [3]; the lines of intersection of sets of three of these four solids can be taken to be the four trisecants through O. We then may choose the secant plane α conjugate to O to be any general plane of [4]; thus when O and the four trisecants through it are given the choice of α depends on 6 constants, this being the number of constants on which a plane in [4] depends. We now have to choose the net of quadrics so that each of its members has the following property: every point of the line of intersection of α with any one of the four solids through O is conjugate, in regard to the quadric, to every point of the line of intersection of the remaining three solids through O. We saw in § 11 that, when α and the four trisecants through O are given, such quadrics form a linear system of freedom 6: hence the nets consisting of such quadrics are of freedom 12, this being the freedom of planes in [6]. Wherefore the freedom of a Jacobian curve with four concurrent trisecants is

$$4 + 12 + 6 + 12 = 34.$$

A Jacobian curve must therefore be subjected to two conditions if it is to have four concurrent trisecants.

61. A net of quadrics whose Jacobian curve has an inscribed hexahedron can be reduced to the canonical form II, and in only one way. If then we choose six arbitrary solids to be the faces of the hexahedron the six linear forms X, which are to satisfy the identity

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \equiv 0,$$

are completely determined except for a constant multiplier the same for all. Every quadric of the net must then belong to the linear system of quadrics determined by the squares of these six forms X, and this linear system is of dimension 5; hence, when the six solids are given, the net of quadrics has the same freedom as a plane in [5], and this is 9. Hence, since a configuration of six arbitrary solids in [4] has freedom 24, the net of quadrics has total freedom 33. This then is the freedom of the Jacobian curve whose twenty trisecants are the edges of a hexahedron. If however, as in § 54, we restrict the coefficients in II so that they satisfy the relation $|ij, kl, mn| = 0$ the resulting Jacobian curve has only freedom 32, while if the coefficients are to be such that

$$|ij, kl, mn| = |ij, km, nl| = |ij, kn, lm| = 0,$$

a set of conditions which we saw to be the equivalent of two conditions only, the Jacobian curve has freedom 31.

62. When a Jacobian curve has an infinity of inscribed hexahedra its freedom can be determined quickly from the fact that it is the locus of intersections of sets of four osculating solids of a quartic curve when their four points of osculation are all members of the same set of a g_6^1. It is known, and can be established, for example, by appealing to the general result of Brill and Nöther already quoted, that the freedom of a rational quartic curve in [4] is 21, while the freedom of a g_6^1 on any rational curve is 10; hence the freedom of a Jacobian curve with a scroll of trisecants is 31.

63. Consider now the Jacobian curve of the net of polar quadrics of the points of a plane in regard to a Segre primal. The primal determines a hexahedron $\mathfrak{h}$ and, conversely, is determined when $\mathfrak{h}$ is given; hence a Segre primal in [4] has freedom 24. Moreover a plane in [4] has freedom 6; hence the freedom of the Jacobian curve in this case is 30. This statement assumes tacitly that the same Jacobian curve cannot arise for two Segre primals or two planes, but if the same curve arose for two different primals it would have to be circumscribed to both the corresponding hexahedra, whereas the Jacobian curve has, it is supposed, only one inscribed hexahedron. Thus there can only be one Segre primal associated with any Jacobian curve, and hence, clearly, only one plane also. In the particular case when the Jacobian curve of the net of polar quadrics has an infinity of inscribed hexahedra we can again, in order to obtain such a net of quadrics, choose an arbitrary Segre primal, but we are then restricted to ∞^5 of the ∞^6 planes of [4], and so this particular net of quadrics has only freedom 29.

64. The Jacobian curve of a net of quadrics in [4] can always be put into birational correspondence with a plane quintic and, conversely, a plane quintic can always be put into birational correspondence with the Jacobian curve of a net of quadrics in [4]. The Jacobian curve of a general net of quadrics in [4] is, as we have seen, of freedom 36, while the freedom of a general quintic curve in a plane is 20. If then we take a specialised net of quadrics whose Jacobian curve has freedom $36-x$ it is to be expected that the specialised plane quintic with which it is in birational correspondence will be of freedom $20-x$. We can verify this statement by calculating directly the freedoms of the various special types of plane quintic curves that we have obtained.

Suppose then that we calculate the freedom of a plane quintic which has, as in § 3, an associated configuration of a line-pair and two conics; the quintic passes through the intersections of the two conics and through all the intersections of the line-pair with the conics; moreover the two remaining intersections of the quintic with either conic are to coincide in a single contact, and the quintic is also to pass through the intersection of the two lines. Since the freedom of a conic in a plane is 5 and the freedom of a line is 2, the freedom of the configuration which consists of two conics and a line-pair is 14. Now, when such a configuration is given, we have to impose 15 conditions on the quintic curve, namely 4 to pass through the intersections of the conics, 8 to pass through the intersections of the line-pair and the conics, 2 to touch the conics and 1 to pass through the intersection of the line-pair. Hence, when the configuration of the two conics and the line-pair is given, the freedom of a quintic curve with which this particular configuration is associated is $20-15=5$. Hence the aggregate of plane quintics with which such configurations are associated is $14+5=19$. Like the net of quadrics in [4] with whose Jacobian curve it is in birational correspondence, the quintic is subjected to one condition.

The freedom of this curve can also be obtained from its equation; the equation of a general plane quintic being of the form

$$\xi_1^{-1}+\xi_2^{-1}+\xi_3^{-1}+\eta_1^{-1}+\eta_2^{-1}+\eta_3^{-1}+\zeta(\eta_1^{-1}+\eta_2^{-1}+\eta_3^{-1})(\xi_1^{-1}+\xi_2^{-1}+\xi_3^{-1})=0,$$

where the seven letters ξ_1, ξ_2, ξ_3, η_1, η_2, η_3, and ζ denote linear forms in three homogeneous coordinates; the quintic becomes one of the special kind that we are now considering if we assume that ζ is linearly dependent on ξ_1 and η_1. The equation of the general plane quintic contains 21 arbitrary coefficients (three for each of the seven linear forms), so that the general curve is of freedom 20; when, however, the line $\zeta=0$ is supposed to pass through the intersection of $\xi_1=0$ and $\eta_1=0$ the number of arbitrary coefficients which enter into the equation is one less than in the general case, so that the freedom is one less also; thus the special quintic curve that we are considering is of freedom 19.

Similarly, if we suppose that $\zeta=0$ joins the intersection of $\xi_1=0$ and $\eta_1=0$ to that of $\xi_2=0$ and $\eta_2=0$ the quintic curve so arising is of freedom 18; while the still more specialised curve for which $\zeta=0$ is the axis of perspective of two triangles, one triangle being formed by the three lines $\xi=0$ and the other by the three lines $\eta=0$, is of freedom 17.

65. Suppose now that a plane quintic passes through the six vertices of a quadrilateral, that its eight remaining intersections with the sides of this quadrilateral are on a conic and that its two remaining intersections with this

conic coincide in a single contact. The plane configuration of four lines and a conic is of freedom $4 \cdot 2 + 5 = 13$. When such a configuration is given a plane quintic associated with it is subjected to 15 conditions, namely six to pass through the vertices of the quadrilateral, eight to pass through the intersections of the sides of the quadrilateral with the conic and one to touch the conic. Hence the freedom of quintic curves associated with a particular configuration is $20 - 15 = 5$. Wherefore the freedom of a plane quintic associated with an arbitrary configuration is $5 + 13 = 18$. Like the Jacobian curve with four concurrent trisecants, with which it is in birational correspondence, such a plane quintic is subjected to two conditions.

66. We next suppose that the plane quintic has an inscribed hexagram; this is in birational correspondence with the Jacobian curve whose trisecants are the edges of a hexahedron, and we have seen that this Jacobian curve is of freedom 33. We therefore expect that the plane quintic has freedom 17. This is immediately verified; for the freedom of a hexagram in a plane is $6 \cdot 2 = 12$, and in order that a quintic curve should pass through the vertices of the hexagram it must be subjected to 15 conditions; hence, given a definite hexagram, the freedom of the quintic curves which pass through its vertices is $20 - 15 = 5$. Wherefore the freedom of quintic curves circumscribed to an arbitrary hexagram is $5 + 12 = 17$. This result also follows immediately from the equation to the quintic which, when it has an inscribed hexagram, must be of the form

$$\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1} + \xi_4^{-1} + \xi_5^{-1} + \xi_6^{-1} = 0,$$

and so contains eighteen arbitrary coefficients.

When a plane quintic has an inscribed hexagram we have seen that it can be specialised in certain ways so as to have triads of tangents that are concurrent in points of the curve; the point of contact of each of the tangents of the triad is an intersection of two sides of the hexagram, and the three tangents of the triad give all the six sides of the hexagram in this way. The existence of one concurrent triad of tangents is secured, as we have seen, by imposing one condition on the coefficients; the corresponding quintic curve is therefore of freedom 16. Also, if one of the identities between the six forms is

$$\varrho(\xi_i + \xi_j) + \xi_k + \xi_l + \xi_m + \xi_n \equiv 0$$

there are three concurrent triads of tangents; we have seen that this can be

Peculiarity of Jacobian Curve	Freedom of ϑ	Canonical Form
—	36	I
Two pairs of conjugate trisecants (t_1, t_1') and (t, t') such that t meets t_1 and t' meets t_1'.	35	I, with the condition $\lvert a_1 c' b_1'' \rvert = 0$.
Three pairs of conjugate trisecants (t_1, t_1'), (t_2, t_2') and (t, t') such that t meets t_1 and t_2 while t' meets t_1' and t_2'.	34	I, with $\lvert a_1 c' b_1'' \rvert = \lvert a_2 c' b_2'' \rvert = 0$.
Four pairs of conjugate trisecants (t_1, t_1'), (t_2, t_2'), (t_3, t_3') and (t, t') such that t meets t_1, t_2 and t_3 while t' meets t_1', t_2' and t_3'.	33	I, with $\lvert a_1 c' b_1'' \rvert = \lvert a_2 c' b_2'' \rvert = \lvert a_3 c' b_3'' \rvert = 0$.
Four concurrent trisecants.	34	III
Inscribed hexahedron.	33	II
Inscribed hexahedron of which one set of three opposite vertices determines a quadrisecant plane.	32	II, with the condition $\lvert ij, kl, mn \rvert = 0$,
Inscribed hexahedron of which three sets of three opposite vertices determine quadrisecant planes.	31	II, with $\lvert ij, kl, mn \rvert = \lvert ij, km, nl \rvert = \lvert ij, kn, lm \rvert = 0$.
Scroll of trisecants.	31	II, with $\varDelta = 0$. Or, alternatively, IV.
Jacobian curve of the net of polar quadrics of the points of a plane in regard to a Segre cubic primal.	30	II, with $\Sigma a_i = \Sigma b_i = \Sigma c_i = 0$.
As in the preceding case, but having also a scroll of trisecants.	29	II, with $\Sigma a_i = \Sigma b_i = \Sigma c_i = \varDelta = 0$.

secured by imposing two conditions on the coefficients, so that the corresponding quintic curve is of freedom 15. Further; if the six forms ξ are such that

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 \equiv 0$$

the plane quintic has fifteen concurrent triads of tangents. Here we may choose any five of the six forms ξ to be arbitrary, but the sixth is then completely determined; hence there are now only fifteen arbitrary coefficients entering into the equation of the quintic curve, which is therefore now of freedom 14. Like the Jacobian curve of the net of polar quadrics of the points of a plane in regard to a Segre primal, with which it is in birational correspondence, such a plane quintic is subjected to six conditions.

67. There remain those quintic curves which are circumscribed to an infinity of hexagrams, the sides of these hexagrams all touching the same conic. The most general quintic of this type, which is in birational correspondence with a Jacobian curve having a scroll of trisecants, is obtained by taking a conic and a g_6^1 thereon; the locus of the vertices of those hexagrams which consist of the tangents of the conic at the points forming the sets of the g_6^1 is the quintic curve that we are to consider. Since the freedom of a conic in a plane is 5, and the freedom of a g_6^1 on a conic is 10, the freedom of quintic curves of this kind is $5 + 10 = 15$. The quintic curve can, however, be specialised so that its tangents at the fifteen vertices of one of the hexagrams fall into fifteen concurrent triads; we have seen in § 56 that when a conic and a hexagram, consisting of tangents of this conic, are given, there is a doubly-infinite family of such quintic curves. Hence the freedom of quintic curves of this type is $5 + 6 + 2 = 13$.

Table Showing the Various Nets of Quadrics in [4].

68. In conclusion we exhibit in tabular form the different kinds of nets of quadrics that have been encountered; the table shows the peculiarity of the Jacobian curve that characterises the net, the freedom of the net and its canonical form. In each case any plane quintic in birational corrrespondence with the Jacobian curve has a corresponding peculiarity, and the freedom of the plane quintic is always less by 16 than that of the corresponding Jacobian curve.