# On the Notion of Essential Dimension for Algebraic Groups 

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#### Abstract

We introduce and study the notion of essential dimension for linear algebraic groups defined over an algebraically closed fields of characteristic zero. The essential dimension is a numerical invariant of the group; it is often equal to the minimal number of independent parameters required to describe all algebraic objects of a certain type. For example, if our group $G$ is $S_{n}$, these objects are field extensions, if $G=O_{n}$, they are quadratic forms, if $G=\mathrm{PGL}_{n}$, they are division algebras (all of degree $n$ ), if $G=G_{2}$, they are octonion algebras, if $G=F_{4}$, they are exceptional Jordan algebras. We develop a general theory, then compute or estimate the essential dimension for a number of specific groups, including all of the above-mentioned examples. In the last section we give an exposition of results, communicated to us by J.-P. Serre, relating essential dimension to Galois cohomology.


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## 1 Introduction

The purpose of this paper is to introduce and study the notion of essential dimension for linear algebraic groups. All algebraic groups we shall consider, along with all varieties, fields, etc., and all morphism between them will be defined over a fixed algebraically closed base field $k$ of characteristic zero. The essential dimension $\operatorname{ed}(G)$ of an algebraic
group $G$ is the smallest integer $d$ such that every principal $G$-bundle $X \longrightarrow B$ can be obtained (up to birational isomorphism) by pull-back from a diagram

where $Y \longrightarrow C$ is a principal $G$-bundle, $B \longrightarrow C$ is a dominant map, and $\operatorname{dim}(C) \leq d$; cf. Definition 3.5 and Remark 2.17. The essential dimension naturally comes up in many interesting situations; several examples are given below.
(a) The essential dimension of the symmetric group $S_{n}$ is the minimal number of parameters required to write down the general polynomial of degree $n$ in one variable. (Here we allow Tschirnhaus transformations which do not involve radicals.) Equivalently, ed $\left(S_{n}\right)$ is the smallest integer $d$ with the following property: every field extension $K \subset L$ of degree $n$ can be defined by (a degree $n$ ) polynomial with at most $d$ algebraically independent coefficients (over $k$ ). The question of computing ed $\left(S_{n}\right)$ is related to the algebraic form of Hilbert's 13 th problem. For more on this we refer the reader to $\left[\mathrm{BR}_{1}\right]$ and $\left[\mathrm{BR}_{2}\right]$; a discussion of essential dimensions of other finite groups can be found in $\left[\mathrm{BR}_{1}\right]$.
(b) Groups of essential dimension zero are precisely the "special groups" introduced by Serre $\left[\mathrm{Se}_{1}\right]$ and classified by Grothendieck [G] in the 1950s; see Section 5.
(c) Let $D$ be a division algebra with center $K$. We say that $D$ is defined over a subfield $F$ of $K$ if $D \simeq E \otimes_{F} K$ for some division algebra $E$ with center $F$. The essential dimension of the projective linear group $\mathrm{PGL}_{n}$ is the smallest integer $d$ such that every division algebra $D$ of degree $n$ is defined over some field $F$ with $\operatorname{trdeg}_{k}(F) \leq d$. This number is of interest in the theory of division algebras; see [ $\left.\mathrm{Pr}_{1}, \mathrm{Thm} .2 .1\right]$ and [Row]. We will discuss essential dimensions of projective linear groups in Section 9.
(d) The essential dimension of the orthogonal group $O_{n}$ (resp. the special orthogonal group $S O_{n}$ ) is the smallest integer $d$ such that every quadratic form (resp. every quadratic form of determinant 1) over every field $F$ is equivalent to one with $\leq d$ algebraically independent coefficients (over $k$ ). We will determine these integers in Section 10.
(e) The essential dimensions of the exceptional group $G_{2}$ (resp. $F_{4}$ ) is the smallest integer $d$ such that every octonion algebra (resp. every 27 -dimensional exceptional simple Jordan algebra) can be defined over a field of transcendence degree $\leq d$. These numbers are discussed in Section 11.

The methods we use for computing, or at least estimating, the essential dimension of a given algebraic group $G$ can be roughly divided into three categories: geometric, algebraic and cohomological. The geometric approach, based on (birational) invariant theory, is discussed in Sections 2-4. The algebraic approach, based on descent of "structured spaces" is developed in Sections 6-8; some applications are given in Sections 9-11. The cohomological approach is discussed in Sections 5 and 12; Section 5 is
mostly concerned with the vanishing of $H^{1}$ and Section 12 deals with the relationship between the essential dimension of $G$ and its cohomological invariants.

In order to illustrate the interplay among the geometric, algebraic, and cohomological methods, we will often prove the same result in several different ways, sometimes in different parts of the paper. For the convenience of the reader, we summarize most of what we now know about essential dimensions of specific groups in the table below and indicate where the proof of each result can be found.

| Group | EssentialDimension | Proof |
| :---: | :---: | :---: |
| $\overline{(\mathbb{Z} / n)^{r}}$ | $r$ | [ $\mathrm{BR}_{1}, 6.1$ ] |
| $S_{n}$ | $\left\{\begin{array}{l} \geq[n / 2] \\ \leq n-3 \end{array}\right.$ | $\begin{gathered} {\left[\mathrm{BR}_{1}, 6.5(\mathrm{~b})\right],\left[\mathrm{BR}_{2}, 1.1\right], \text { Ex. } 12.8} \\ {\left[\mathrm{BR}_{1}, 6.5(\mathrm{c})\right]} \end{gathered}$ |
| $\left(k^{*}\right)^{n}$ | $=0$ | Thm. 5.4, Ex. 3.9(a) |
| $\mathrm{GL}_{n}$ | $=0$ | Thm. 5.4, Ex. 3.9(b) |
| $\mathrm{SL}_{n}$ | $=0$ | Thm. 5.4, Ex. 3.9(c), 8.9(a) |
| $\mathrm{Sp}_{2 n}$ | $=0$ | Thm. 5.4, Ex. 8.9(b) |
|  |  | $\begin{aligned} & \text { Thm. } 4.5 \\ & \text { Thm. } 9.6 \end{aligned}$ |
| $\mathrm{PGL}_{n}$ | $\left\{\begin{array}{l}\geq 2 r \text { if } n=n_{0}^{r} \\ =2 \text { if } n=2,3,6 \\ \text { other results }\end{array}\right.$ | $\begin{gathered} \text { Thm. } 9.3 \\ \text { Lemma } 9.4(\mathrm{c}) \\ \text { Prop. 9.8, } \end{gathered}$ |
| $O_{n}$ | $\left\{\begin{array}{l}=n \\ \leq n \\ \geq n\end{array}\right.$ | $\begin{gathered} \text { Thm. } 10.3 \\ \text { Ex. 3.10(a), } 4.2 \\ \text { Ex. } 12.6 \end{gathered}$ |
| $S O_{n}, \quad n \geq 3$ | $\left\{\begin{array}{l}=n-1 \\ \leq n-1 \\ \geq n-1\end{array}\right.$ | $\begin{aligned} & \text { Thm. } 10.4 \\ & \text { Ex. } 3.10(\mathrm{~b}), 4.2 \\ & \text { Ex. } 12.7 \end{aligned}$ |
|  | $\{=3$ | Thm. 11.2 |
| $G_{2}$ | $\left\{\begin{array}{l}\leq 3 \\ \geq 3\end{array}\right.$ | Rem. 11.4 <br> Rem. 11.3, Ex. 12.9, Thm. 12.12 |
| $F_{4}$ | $\geq 5$ | Thm. 11.5, Ex. 12.10 |
| $E_{6}$ (simply conn.) | $\left\{\begin{array}{l}\geq 3 \\ \leq \operatorname{ed}\left(F_{4}\right)+1\end{array}\right.$ | Prop. 11.6, Thm. 12.12 Prop. 11.7 |
| $E_{7}$ (simply conn.) | $\geq 3$ | Prop. 12.11(a), Thm. 12.12 |
| $E_{8}$ | $\geq 3$ | Prop. 12.11(b), Thm. 12.12 |

For further results see the table in the appendix at the end of this paper.

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ular, Section 12 is based entirely on results he communicated to us. He also suggested the statements of Theorems $10.3,10.4,11.2$ and 11.5 ; his proofs of these results are outlined in Section 12. (We give somewhat different proofs in Sections 10 and 11.)

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## 2 Preliminaries

The purpose of this section is to introduce notation, terminology and a number of known results from birational invariant theory. Our emphasis will be on stating these results in a form suitable for subsequent use, not on developing the theory in a coherent and self-contained manner. For an excellent introduction to birational invariant theory we refer the reader to [PV, Sect 2]; see also [Po].

Our basic object of study will be primitive rather than irreducible $G$-varieties (see Definition 2.1); this (slightly) greater degree of generality will be needed the sequel.

### 2.1 Notation and terminology

The following notational conventions will be used throughout the paper.

| $k$ | algebraically closed base field of characteristic 0 |
| :--- | :--- |
| $F$ | usually a field extension of $k$ |
| $G$ | algebraic group defined over $k$ |
| $X$ | usually a $G$-variety |
| $G x$ | orbit of $x$ under the action of $G$ |
| $\operatorname{Stab}(x)$ | stabilizer of $x$ |
| $\operatorname{ed}(X)$ | essential dimension $X$; see Definition 3.1 |
| $\operatorname{ed}(G)$ | essential dimension of $G$; see Definition 3.5 |
| $(W, \beta)$ | structured space over $F ;$ see Definition 6.1 |
| $(V, \alpha)$ | structured space over $k$ |
| Aut $(V, \alpha)$ | automorphism group of $(V, \alpha)$, usually denoted by $G$ |
| $\tau(W, \beta)$ | see Definition 8.7 |
| $Z(A)$ | center of the central simple algebra $A$ |
| $\operatorname{UD}(m, n)$ | universal division algebra of $m n \times n$-matrices |
| $\ll a_{1}, \ldots, a_{n} \gg$ | $n$-fold Pfister form; see Section 10.1 |

Throughout this paper we shall work over a fixed base field $k$, which we assume to be algebraically closed and of characteristic 0 . All algebraic objects we will consider (e.g., rings, fields, algebraic groups, algebraic varieties) and all maps between them will be defined over $k$.

If $X$ is an algebraic variety, we shall denote the algebra of regular (resp. rational) functions on $X$ by $k[X]$ (resp. $k(X)$ ). If $X$ is irreducible then $k(X)$ is a field; in general, $k(X)=k\left(X_{1}\right) \oplus \ldots \oplus k\left(X_{n}\right)$ is a direct sum of fields; here $X_{1}, \ldots, X_{n}$ are the irreducible components of $X$. We shall say that a certain property holds for $x$ in general position in $X$ if it holds for all $x \in U$, where $U$ is a dense open subset of $X$ (i.e., an open subset which non-trivially intersects every irreducible component of $X$ ).

As usual, a rational map $X \longrightarrow Y$ is an equivalence class of regular maps $U \longrightarrow Y$, where $U$ is a dense open subset of $X ; f_{1}: U_{1} \longrightarrow Y$ and $f_{2}: U_{2} \longrightarrow Y$ are considered equivalent if they agree on $U_{1} \cap U_{2}$. The domain of a rational map $f: X \rightarrow Y$ is the union of all dense open subsets $U$ where $f$ is defined, and the range $f(X)$ is the union of $f(U)$. We will say that $f$ is dominant if $f(X)$ is dense in $Y$.

We shall call an algebraic variety $X$ a $G$-variety if $X$ is equipped with a regular action of $G$ (i.e., an action given by a regular morphism $G \times X \longrightarrow X$ ). If $X$ and $Y$ are $G$-varieties then by a regular map $X \longrightarrow Y$ of $G$-varieties we mean a regular $G$-equivariant map. The same applies to rational maps of $G$-varieties, biregular and birational isomorphisms of $G$-varieties, etc.

### 2.2 Primitive $G$-varieties

Definition 2.1 We shall call a $G$-variety $X$ primitive if $G$ transitively permutes the components of $X$.

Note that if $G$ is connected then $X$ is a primitive $G$-variety if and only if $X$ is irreducible.

Lemma 2.2 Let $X$ be a G-variety. Then
(a) $X$ is birationally isomorphic to a disjoint union of primitive $G$-varieties.
(b) $X$ is primitive if and only if $k(X)^{G}$ is a field.

Proof. (a) Write $X=X_{1} \cup \ldots \cup X_{n}$, where each $X_{i}$ is a union of components of $X$ transitively permuted by $G$. Then each $X_{i}$ is primitive and the natural projection from the disjoint union $\amalg_{i=1}^{m} X_{i}$ to $X$ is a $G$-equivariant birational isomorphism.
(b) Suppose $k(X)^{G}$ is not a field. Then $k(X)^{G}$ has a zero-divisor $f \neq 0$. Let $X_{0}$ be the union of those irreducible components of $X$ where $f=0$ and $X_{1}$ be the union of the remaining irreducible components. Then $X_{0}$ and $X_{1}$ are $G$-invariant and non-empty. Since $G$ cannot map a component in $X_{0}$ to a component in $X_{1}$, we conclude that $X$ is not primitive. Conversely, suppose $X$ is not primitive. By part (a) we may assume that $X$ is a disjoint union of $m \geq 2$ primitive $G$-varieties $X_{1}, \ldots, X_{m}$. Let $f \in k(X)^{G}$ be given by $f_{\mid X_{1}}=1$ and $f_{\mid X_{i}}=0$ for all $i=2, \ldots, m$. Then $f$ is a zero divisor in $k(X)^{G}$ and thus $k(X)^{G}$ is not a field.

### 2.3 Rational quotients

Theorem 2.3 (Rosenlicht $\left[\operatorname{Ros}_{1}\right]$, $\left[\operatorname{Ros}_{2}\right]$ ) Let $X$ be a $G$-variety. Suppose $k(W)=$ $k(X)^{G}$ and $\pi: X \rightarrow W$ is rational map induced by the inclusion $k(X)^{G} \hookrightarrow k(X)$. Then there exists an open dense subset $U$ of $X$ such that $\pi_{\mid U}: U \longrightarrow W$ is a regular map and for any $x, y \in U, \pi(x)=\pi(y)$ iff $G x=G y$.

We shall denote $W$ by $X / G$; this variety is uniquely defined up to birational equivalence. By Lemma 2.2 (b), $X$ is primitive if and only if $X / G$ is irreducible. We will refer to the map $\pi: X \rightarrow X / G$ as the rational quotient map or simply the quotient map, since no other kind of quotient map will be considered in this paper.

Remark 2.4 The identity $k(X / G)=k(X)^{G}$, which we used to define $X / G$, can be restated as a universal property of $X / G$ as follows. Any rational map $f: X \rightarrow Y$ of $G$-varieties gives rise to a rational map $\bar{f}: X / G \rightarrow Y / G$ such that the diagram

commutes. Here $\pi_{X}$ and $\pi_{Y}$ are the rational quotient maps.
Remark 2.5 The rational quotient map $\pi: X \rightarrow X / G$ is unique in the following sense.

Let $X$ be a primitive $G$-variety. Suppose $f: X \rightarrow X_{0}$ is a dominant rational map such that $f^{-1}(f(x))=G x$ for $x_{0} \in X$ in general position. Then there exists a birational isomorphism $\bar{f}: X / G \xrightarrow{\simeq} X_{0}$ such that $f=\bar{f} \circ \pi$.

The map $\bar{f}$ is given by diagram (1) with $Y=X_{0}$, where $X_{0}$ is viewed as a $G$ variety with trivial $G$-action. Here $\pi_{X}=\pi, Y / G=X_{0}$ and $\pi_{Y}: Y \rightarrow Y / G$ is the identity map. Since $X$ is primitive, both $X_{0}$ and $X / G$ are irreducible. Moreover, by our assumption on $f$, the map $\bar{f}: X / G \rightarrow X_{0}$ is dominant and $\bar{f}^{-1}\left(x_{0}\right)$ is a single point for $x_{0}$ in general position in $X_{0}$. Thus $\bar{f}$ is a birational isomorphism; see, e.g., [H, 4.6].

Remark 2.6 Suppose $X$ is a $G$-variety and $N$ is a closed normal subgroup of $G$. Then $G / N$ has a naturally defined rational action on $X / N$; see [PV, Proposition 2.6]. In general, this action will not be regular; however, by a theorem of Rosenlicht (see [Ros ${ }_{1}$, Theorem 1], [PV, Corollary 1.1]) there exists a model for $X / N$ such that the induced $G / N$-action on it is regular, i.e., $X / N$ is a $G / N$-variety. Moreover, this $G$-variety is uniquely defined up to birational isomorphism.

### 2.4 Generically free varieties

Definition 2.7 A $G$-variety $X$ is said to be generically free if $G$ acts freely (i.e., with trivial stabilizers) on a dense open subset of $X$.

Lemma 2.8 Suppose a rational map $f: X \rightarrow Y$ of generically free $G$-varieties induces a birational isomorphism $\bar{f}: X / G \rightarrow Y / G$. Then $f$ is a birational isomorphism.

Proof. We may assume without loss of generality that $X / G$ and $Y / G$ are irreducible, i.e., $X$ and $Y$ are primitive $G$-varieties. In particular, every irreducible component of $X$ has dimension $\operatorname{dim}(X)$ and every irreducible component of $Y$ has dimension $\operatorname{dim}(Y)$.
Moreover,

$$
\operatorname{dim}(X)=\operatorname{dim}(X / G)+\operatorname{dim}(G)=\operatorname{dim}(Y / G)+\operatorname{dim}(G)=\operatorname{dim}(Y)
$$

Replacing $X / G$ and $Y / G$ by sufficiently small open subsets we may assume that all four maps in diagram (1) are regular, $G$ acts freely on $X$ and $Y$, and $\pi_{X}$ separates the $G$-orbits in $X$. Then $f(X)$ is $G$-invariant subset of $Y$ which intersects every $G$-orbit in $Y$; hence, $f(X)=Y$. On the other hand, $f\left(x_{1}\right)=f\left(x_{2}\right)$ means that $\pi_{X}\left(x_{1}\right)=\pi_{X}\left(x_{2}\right)$, i.e., $x_{2}=g\left(x_{1}\right)$ for some $g \in G$. Consequently, $g f\left(x_{1}\right)=f\left(x_{2}\right)=f\left(x_{1}\right)$. Since $G$ acts freely on $Y$, we conclude that $x_{1}=x_{2}$.

We have therefore proved that $f$ is bijective. Let $X=X_{1} \cup \ldots \cup X_{n}$ be an irreducible decomposition of $X$. Let $Y_{i}$ be the closure of $f\left(X_{i}\right)$ in $Y$. Since $\operatorname{dim}\left(Y_{i}\right)=\operatorname{dim}\left(X_{i}\right)=$ $\operatorname{dim}(X)=\operatorname{dim}(Y), Y_{1}, \ldots, Y_{n}$ are the irreducible components of $Y$. Moreover the injectivity of $f$ implies that $Y_{i} \neq Y_{j}$ if $i \neq j$. For every $i=1, \ldots, n$, the map $f_{\mid X_{i}}$ : $X_{i} \longrightarrow Y_{i}$ is dominant and injective. Hence, each $f_{\mid X_{i}}$ is a birational isomorphism (see, e.g., $[H, 4.6])$ and therefore, so is $f$.

## $2.5(G, H)$-sections

Definition 2.9 (cf. [Ka $1_{1}$, Sect. 1], [Po, 1.7.6], [Do, Sect. 3].) Let $G$ be an algebraic group, $X$ be a $G$-variety, and $H$ be a closed subgroup of $G$. An irreducible subvariety $S \subset X$ is called a $(G, H)$-section if
(a) $G S$ is dense in $X$ and
(b) there is a dense open subvariety $S_{0} \subset S$ such that for any $s \in S_{0}$ we have $g s \in S$ if and only if $g \in H$.

Example 2.10 Let $X$ be a primitive $G$-variety and let $X_{1}$ be an irreducible component of $X$. Then $X_{1}$ is a $\left(G, G_{1}\right)$-section, where $G_{1}=\left\{g \in G \mid g\left(X_{1}\right)=X_{1}\right\}$.

Lemma 2.11 Let $X$ be a $G$-variety and $S$ be a $(G, H)$-section of $X$. Then $k(S)^{H}=$ $k(X)^{G}$.

Proof. Let $\pi: X \rightarrow X / G$ be the rational quotient map. Then the restriction $\pi_{\mid S}: S \rightarrow X / G$ is well-defined and separates $H$-orbits in $S_{0}$. By Remark $2.5 \pi_{\mid S}$ is the quotient map for the $H$-action on $S$. Equivalently, $k(S)^{H}=k(X / G)$, as claimed.

Definition 2.12 Let $G$ be an algebraic group, $H$ be a closed subgroup of $G$, and $W$ be an $H$-variety. Then $G \times W$ is a $G \times H$-variety via

$$
\begin{equation*}
(g, h):\left(g^{\prime}, w\right) \mapsto\left(g g^{\prime} h^{-1}, h w\right), \tag{2}
\end{equation*}
$$

for any $g, g^{\prime} \in G, h \in H$, and $w \in W$. By Remark 2.6 a model $Y$ for $(G \times W) / H$ can be chosen so that the $G$-action on $Y$ is regular and thus the quotient map $\pi: G \times W \rightarrow Y$ is a rational map of $G$-varieties. (Here we are identifying $H$ with a normal subgroup $N=\left\{1_{G}\right\} \times H$ of $G \times H$ and $G$ with $\left.(G \times H) / N\right)$.) We shall denote $Y$ by $G *_{H} W$; this $G$-variety is well-defined up to birational isomorphism.

Remark 2.13 (a) If $W$ is a generically free $H$-variety then it is easy to see that $G *_{H} W$ is a generically free $G$-variety.
(b) $W \hookrightarrow G *_{H} W$ given by $w \mapsto(1, w)$ is an $H$-equivariant embedding of $H$ varieties, whose image is a $(G, H)$-section of $G *_{H} W$. The following lemma shows that every $(G, H)$-section is of this form.

Lemma 2.14 (cf. [Po, 1.7.5]) Let $X$ be a $G$-variety and $S$ be a $(G, H)$-section of $X$. Then $G *_{H} S$ and $X$ are birationally isomorphic $G$-varieties.

Proof. Consider the map $\phi: G \times S \longrightarrow X$ given by $(g, s) \mapsto g s$. Then $\phi$ is a morphism of $G \times H$ varieties, where $G \times H$ acts via (2) on $G \times S$ and $H$ acts trivially on $X$. Moreover, this map is dominant and $\phi^{-1}(x)$ is a single $H$-orbit for $x$ in general position in $X$. Thus by Remark 2.5, $\phi$ is the rational quotient map for the $H$-action on $G \times S$. In other words, we have the following commutative diagram:


To complete the proof, note that since the $G$-action on $G \times S$ commutes with the $H$ action, every map in this diagram is $G$-equivariant.

### 2.6 Compressions and fiber products

Definition 2.15 Let $X$ be a generically free $G$-variety. A $G$-compression of $X$ is a dominant $G$-equivariant rational map $X \rightarrow Y$, where $Y$ is another generically free $G$-variety.

Let $Y$ be a primitive $G$-variety, $\pi: Y \rightarrow Y / G$ be the rational quotient map, and $\phi: X_{0} \rightarrow Y / G$ be a dominant rational map. Recall that the fiber product $Y \times_{Y / G} X_{0}$ is defined as the closure of

$$
\left\{\left(y, x_{0}\right) \mid \pi(y)=\phi\left(x_{0}\right)\right\}
$$

in $Y \times X_{0}$. This variety inherits a $G$-action from $Y$. Clearly $Y \times_{Y / G} X_{0}$ is a generically free $G$-variety if an only if so is $Y$.

Lemma 2.16 (a) Suppose $Y$ and $\phi: X_{0} \rightarrow Y / G$ are as above. Denote the fiber product $Y \times_{Y / G} X_{0}$ by $X$. Then the natural projection pr : $X=Y \times_{Y / G} X_{0} \longrightarrow X_{0}$ is the rational quotient map for the $G$-action on $X$.
(b) Conversely, suppose $X$ is a generically free $G$-variety, $X_{0}=X / G, \pi: X \rightarrow X_{0}$ is the rational quotient map and $\bar{f}: X_{0} \rightarrow Y / G$ is the dominant rational map induced by a $G$-compression $f: X \rightarrow Y$. Then

$$
\psi \stackrel{\text { def }}{=} f \times(\bar{f} \circ \pi): X \xrightarrow{\simeq} Y \times_{Y / G} X_{0}
$$

is a birational isomorphism of $G$-varieties.
Proof. (a) It is easy to see that $p r^{-1}\left(x_{0}\right)=G x_{0}$ for $x_{0}$ in general position in $X_{0}$. Part (a) now follows from Remark 2.5.
(b) By part (a), the map $\bar{\psi}: X / G \stackrel{\sim}{\simeq} X_{0}=\left(Y \times_{Y / G} X_{0}\right) / G$ is a birational isomorphism. The desired conclusion now follows from from Lemma 2.8.

Remark 2.17 Suppose $X$ is a generically free $G$-variety. One can show that the rational quotient map $\pi: X \rightarrow X / G$ is a principal $G$-bundle over a dense open subset $U \subset X / G$. This means that there exists an etale cover $U^{\prime} \longrightarrow U$ such that $U^{\prime} \times_{U} \pi^{-1}(U)$ is isomorphic to $U^{\prime} \times G$, as a $G$-variety. If $X$ is primitive, this is equivalent to the following: there exists a finite rational cover $X_{0} \rightarrow X / G$ of irreducible varieties such that $X_{0} \times_{X / G} X$ is birationally isomorphic to $X_{0} \times G$ (as a $G$-variety). The latter assertion follows from the existence of a rational quasisection for $X$; see [Po, 1.1.2]. Since we shall not use this result, except to motivate Definition 3.1, we omit the details of the argument.

### 2.7 The "no-name" lemma

Let $Y$ be a $G$-variety and let $\phi: E \longrightarrow Y$ be a $G$-vector bundle of rank $d$. This means that $\phi$ is an algebraic vector bundle of rank $d, G$ acts on both $E$ and $Y$ so that $\phi$ is a surjective morphism of $G$-varieties, and $g: \phi^{-1}(y) \mapsto \phi^{-1}(g(y))$ is a linear map for each $y \in Y$ and $g \in G$; see, e.g., [BK, Section 1].

Lemma 2.18 Let $Y$ be a generically free $G$-variety and let $E$ be a $G$-vector bundle on $Y$. Then there exists a birational isomorphism $\alpha$ so that the diagram

commutes. Here $k^{d}$ is the d-dimensional affine space and $Y / G \times k^{d} \longrightarrow Y / G$ is projection to the first factor.

Proof. We may assume without loss of generality that $Y$ is a primitive $G$-variety. Moreover, in view of Example 2.10 and Lemma 2.11, we may assume $Y$ is irreducible (otherwise we can replace $Y$ by an irreducible component $Y_{1}, G$ by the stabilizer $G_{1}$ of $Y_{1}$, and $E$ by $\left.\phi^{-1}\left(Y_{1}\right)\right)$.

If $Y$ is irreducible then Lemma 2.18 is the usual form of the so-called "no-name lemma"; see [BK, Lemma 1.2], [Ka 2 , p. 104], or [Do, p. 6].

In the sequel we will need the following variant of the no-name lemma.
Lemma 2.19 Let $Y$ be a generically free $G$-variety and let $\phi: E \longrightarrow Y$ be a $G$-vector bundle of dimension d. Then $E$ is birationally isomorphic to $Y \times k^{d}$ (as a $G$-variety), where $k^{d}$ is the affine $d$-space with trivial $G$-action.

Proof. By Lemma 2.16, $E$ is birationally isomorphic to $Y \times_{Y / G}(E / G)$. Let $\alpha$ be as in (3). Then we have the following birational isomorphisms of $G$-varieties:

$$
E \simeq Y \times_{Y / G} E / G \stackrel{1 \times \alpha}{\simeq} Y \times_{Y / G}\left(Y / G \times k^{d}\right) \simeq Y \times k^{d} .
$$

Corollary 2.20 Let $Y$ be a primitive generically free $G$-variety and let $V$ be a ddimensional linear representation of $G$. Then there exists a $G$-equivariant dominant rational map $f: Y \times k^{d} \rightarrow V$. Here $G$ acts on $Y \times k^{d}$ via $g(y, a)=(g y, a)$, as in the previous lemma.

Proof. Let $E=Y \times V$ and let $\phi: Y \times k^{d} \xrightarrow{\simeq} \rightarrow Y \times V$ be the birational isomorphism of Lemma 2.19. Now set $f=p r \circ \phi$, where $p r: Y \times V \longrightarrow V$ is projection to the second component.

## 3 Definition of essential dimension

### 3.1 The essential dimension of a $G$-variety

Recall that that, up to birational isomorphism, a generically free $G$-variety $X$ can be viewed as a principal $G$-bundle over the base $X / G$; see Remark 2.17. Moreover,

Lemma 2.16(b) says that if $X \rightarrow Y$ is a compression then $X$ is obtained from $Y$ by a (dominant) base extension $X / G \rightarrow Y / G$. Informally speaking, the $G$-bundle structure of $X \rightarrow X / G$ is completely determined by the $G$-bundle structure of $Y \rightarrow Y / G$; the base extension $X / G \rightarrow Y / G$ simply "spreads" this structure over a larger base. Thus given a generically free $G$-variety $X$ one can ask for a "minimal possible" base space $B=Y / G$ over which it is defined. Of course, such a base space cannot be expected to be unique, so we will limit ourselves to studying its dimension.

Definition 3.1 The essential dimension of a primitive generically free $G$-variety $X$ is the smallest possible value of $\operatorname{dim}(Y / G)$, where $X \rightarrow Y$ is a compression; see Definitions 2.1 and 2.15. We shall denote this number by ed $(X, G)$ or simply ed $(X)$ if the reference to $G$ is clear from the context.

Remark 3.2 If $X$ is primitive and $X \rightarrow Y$ is a $G$-compression then $Y$ is primitive as well. Indeed, $k(Y)^{G}$ is contained in $k(X)^{G}$, which is a field; see Lemma 2.2(b).

Note also that $\operatorname{dim}(Y / G)=\operatorname{dim}(Y)-\operatorname{dim}(G)$, where $\operatorname{dim}(Y)$ is the dimension of any irreducible component of $Y$. Thus we could have defined ed $(X, G)$ as the minimal value of $\operatorname{dim}(Y)$ rather than of $\operatorname{dim}(Y / G)=\operatorname{dim}(Y)-\operatorname{dim}(G)$. Subtracting $\operatorname{dim}(G)$ is simply a matter of convention; we will later find it useful.

We now record several simple observations for future reference.
Lemma 3.3 Let $X$ and $Y$ be primitive generically free $G$-varieties. Then
(a) $\operatorname{ed}(X, G) \leq \operatorname{dim}(X)-\operatorname{dim}(G)$.
(b) Suppose there exists a $G$-compression $X \rightarrow Y$. Then $\operatorname{ed}(X, G) \leq \operatorname{ed}(Y, G)$.
(c) Let $Y_{0}$ be the union of $\phi(X)$, as $\phi$ ranges over all $G$-equivariant rational maps $\phi: X \rightarrow Y .(A s$ usual, $\phi(X)$ is defined as $\phi(U)$, where $U$ is the domain of $\phi$.) Suppose $Y_{0}$ is dense in $Y$. Then $\operatorname{ed}(X, G) \leq \operatorname{ed}(Y, G)$.
(d) Let $S$ be a $G$-variety with trivial $G$-action. Then $\operatorname{ed}(X \times S, G)=\operatorname{ed}(X, G)$.
(e) If $G=\{1\}$ then $\operatorname{ed}(X, G)=0$ for any $X$.

Proof. Parts (a) and (b) are immediate consequences of the definition.
(c) Let $f: Y \rightarrow Z$ be a $G$-compression with $\operatorname{dim}(Z / G)=\operatorname{ed}(Y, G)$. Suppose $G$ acts freely on a dense $G$-invariant open subset $V$ of $Z$. Let $U=\phi^{-1}(V) \subset Y$. By our assumption $Y_{0} \cap U \neq \emptyset$. In other words, there exists a rational $G$-equivariant map $\phi$ : $X \rightarrow Y$ such that $\phi(X) \cap U \neq \emptyset$. Then $f \circ \phi: X \rightarrow Z$ is a well-defined $G$-equivariant rational map. Denote the closure of the image of this map in $Z$ by $Z_{0}$. By our choice of $U, Z_{0}$ is a generically free $G$-variety. Thus $f \circ \phi$ can be viewed as a compression $X \rightarrow Z_{0}$. Consequently, $\operatorname{ed}(X, G) \leq \operatorname{dim}\left(Z_{0} / G\right) \leq \operatorname{dim}(Z / G)=\operatorname{ed}(Y, G)$, as claimed.
(d) Since the natural projection $X \times S \longrightarrow X$ is a $G$-compression, part (b) says that ed $(X \times S, G) \leq \operatorname{ed}(X, G)$. On the other hand, given $s \in S$, let $\phi_{s}: X \longrightarrow X \times S$ be the (regular) $G$-equivariant map given by $\phi_{s}(x)=(x, s)$. Since the images of $\phi_{s}$ cover $X \times S$, as $s$ ranges over $S$, part (c) implies ed $(X, G) \leq \operatorname{ed}(X \times S, G)$.
(e) If $G=\{1\}$ then we can ( $G$-equivariantly) compress $X$ to a point.

### 3.2 The essential dimension of a group

Theorem 3.4 Let $G$ be an algebraic group, $X$ be a generically free primitive $G$-variety, $E \rightarrow X$ be a $G$-vector bundle over $X$ and $V, V^{\prime}$ be two generically free linear representations of $G$. Then
(a) $\operatorname{ed}(E, G)=\operatorname{ed}(X, G)$,
(b) $\operatorname{ed}(V, G)=\operatorname{ed}\left(V^{\prime}, G\right)$ and
(c) $\operatorname{ed}(X, G) \leq \operatorname{ed}(V, G)$.

Proof. (a) By Lemma $2.19, E$ is birationally equivalent to $X \times k^{d}$, as a $G$-variety. Thus ed $(E, G)=\operatorname{ed}\left(X \times k^{d}, G\right)=\operatorname{ed}(X, G)$; see Lemma 3.3(d).
(b) Let $E=V \times V^{\prime}$. Applying part (a) to the vector bundle $E \rightarrow V$ given by projection to the first factor, we obtain the equality $\operatorname{ed}(E, G)=\operatorname{ed}(V, G)$. Similarly, $\operatorname{ed}(E, G)=\operatorname{ed}\left(V^{\prime}, G\right)$, and thus ed $(V, G)=\operatorname{ed}\left(V^{\prime}, G\right)$.
(c) By Corollary 2.20 there exists a $G$-compression $X \times k^{d} \rightarrow V$, where $d=$ $\operatorname{dim}(V)$. Now combining parts (b) and (d) of Lemma 3.3, we obtain

$$
\operatorname{ed}(X, G)=\operatorname{ed}\left(X \times k^{d}, G\right) \leq \operatorname{ed}(V, G),
$$

as claimed. For an alternative proof, see Remark 7.2.
Note that every linear algebraic group can be embedded in some $\mathrm{GL}_{n}$ and thus has a generically free linear representation. Thus we can now give the following definition.

Definition 3.5 The essential dimension of an algebraic group $G$ is defined to be the essential dimension of a generically free linear $G$-representation. By Theorem 3.4(b) this number is is independent of the choice of the representation; we shall denote it by $\operatorname{ed}(G)$. Note that $\operatorname{ed}(G)=\max \{\operatorname{ed}(X, G)\}$, as $X$ ranges over all primitive generically free $G$-varieties; see Theorem 3.4(c).

Remark 3.6 In an earlier version of this paper the inequality $\operatorname{ed}(G, X) \leq \operatorname{ed}(G)$ was only established for reductive groups $G$. Theorem $3.4(c)$ in its current form was first pointed out to us by V. E. Kordonsky, who proved it by reducing the general case to one where $G$ is reductive and then appealing to our earlier result. His argument also yields the following theorem [ $\mathrm{Ko}_{1}$, Theorem 1]:

If $G$ is an algebraic group and $L$ is the Levi subgroup of $G$ then $\operatorname{ed}(G)=\operatorname{ed}(L)$.

### 3.3 First properties

Lemma 3.7 Suppose an algebraic subgroup $G$ is an (algebraic) semidirect product of its subgroups $N$ and $H$, with $N \triangleleft G$. Then $\operatorname{ed}(G) \geq \operatorname{ed}(H)$.

Proof. Let $V$ be a generically free linear representation of $G$ and let $\phi: V \rightarrow Y$ be a $G$-compression with $\operatorname{dim}(Y / G)=\operatorname{ed}(G)$. Then we can choose a regular model for $Y / N$ which is a generically free $H$-variety; see Remark 2.6 We can thus view $V$ as a generically free linear representation of $H$ and the composite map $V \rightarrow Y / N$ below as an $H$-compression:


Thus ed $(G)=\operatorname{dim}(Y)-\operatorname{dim}(G)=\operatorname{dim}(Y / N)-\operatorname{dim}(H) \geq \operatorname{ed}(H)$.
Lemma $3.8 \operatorname{ed}\left(G_{1} \times G_{2}\right) \leq \operatorname{ed}\left(G_{1}\right)+\operatorname{ed}\left(G_{2}\right)$ for any two algebraic groups $G_{1}$ and $G_{2}$.
Proof. For $i=1,2$ let $V_{i}$ be a generically free linear representation of $G_{i}$ and let $\phi_{i}: V_{i} \rightarrow Y_{i}$ be a $G_{i}$-compression such that $\operatorname{ed}\left(G_{i}\right)=\operatorname{dim}\left(Y_{i}\right)-\operatorname{dim}\left(G_{i}\right)$. Then $V_{1} \times V_{2}$ is a generically free linear representation of $G_{1} \times G_{2}$ and

$$
\phi_{1} \times \phi_{2}: V_{1} \times V_{2} \rightarrow Y_{1} \times Y_{2}
$$

is a $G_{1} \times G_{2}$-compression. Thus ed $(G) \leq \operatorname{dim}\left(Y_{1} \times Y_{2}\right)-\operatorname{dim}(G)=\operatorname{dim}\left(Y_{1}\right)+\operatorname{dim}\left(Y_{2}\right)-$ $\operatorname{dim}\left(G_{1}\right)-\operatorname{dim}\left(G_{2}\right)=\operatorname{ed}\left(G_{1}\right)+\operatorname{ed}\left(G_{2}\right)$, as claimed.

### 3.4 Examples

Example 3.9 (a) Let $T_{d}=\left(k^{*}\right)^{d}$ be ad-dimensional torus. Then $\operatorname{ed}\left(T_{d}\right)=0$ for any $d \geq 1$.
(b) $\operatorname{ed}\left(\mathrm{GL}_{n}\right)=0$,
(c) $\operatorname{ed}\left(\mathrm{SL}_{n}\right)=0$.

Proof. (a) The natural representation of $T_{d}$ on $V=k^{d}$ is generically free. Hence by Lemma 3.3(a), ed $\left(T_{d}\right)=\operatorname{ed}\left(V, T_{d}\right) \leq \operatorname{dim}(V)-\operatorname{dim}\left(T_{d}\right)=0$.
(b) Consider the linear representation of $\mathrm{GL}_{n}$ on $V=\mathrm{M}_{n}(k)$ given by left multiplication. By Lemma 3.3(a), $\operatorname{ed}\left(\mathrm{GL}_{n}\right) \leq \operatorname{dim}(V)-\operatorname{dim}\left(\mathrm{GL}_{n}\right)=0$.
(c) Let $\mathrm{SL}_{n}$ act on $V=\mathrm{M}_{n}(k)$ by left multiplication, as above. Let $Y \subset V$ be the set of matrices of determinant 1 and let $f: V \rightarrow Y$ be given by

$$
f(A)=\left(\frac{1}{\operatorname{det}(A)} v_{1}, v_{2}, \ldots, v_{n}\right)
$$

where $A=\left(v_{1}, \ldots, v_{n}\right)$ is the matrix whose columns are $v_{1}, \ldots, v_{n}$. It is easy to see that $f$ is a $\mathrm{SL}_{n}$-compression; thus $\operatorname{ed}\left(\mathrm{SL}_{n}\right) \leq \operatorname{dim}(Y)-\operatorname{dim}\left(S L_{n}\right)=0$.

Example 3.10 (a) $\operatorname{ed}\left(O_{n}\right) \leq n$, (b) $\operatorname{ed}\left(S O_{n}\right) \leq n-1$.
Proof. Let $W=k^{n}, q: W \times W \longrightarrow k$ be a non-degenerate symmetric bilinear form, and $O_{n}$ be the group of $q$-preserving linear transformations $W \longrightarrow W$. If $v \in W$ and $0 \neq v_{1}, \ldots, v_{r} \in W$, we define $P\left(v ; v_{1}, \ldots, v_{r}\right)$ by

$$
\begin{equation*}
P\left(v ; v_{1}, \ldots, v_{r}\right)=v-\sum_{i=1}^{r} \frac{q\left(v_{i}, v\right)}{q\left(v_{i}, v_{i}\right)} v_{i} \tag{4}
\end{equation*}
$$

Note that if $v_{1}, \ldots v_{i}$ are mutually orthogonal then $P\left(v, v_{1}, \ldots, v_{i}\right)$ is just the component of $v$ orthogonal to $v_{1}, \ldots, v_{r}$.

For $d=1, \ldots, n$ let $Y_{d} \subset W^{d}$ be the set of elements ( $w_{1}^{\prime}, \ldots, w_{d}^{\prime}$ ) such that $q\left(w_{i}^{\prime}, w_{j}^{\prime}\right)=0$ and $q\left(w_{i}^{\prime}, w_{i}^{\prime}\right) \neq 0$ for any $1 \leq i<j \leq d$. Note that $Y_{d}$ is a locally closed $O_{n}$-invariant subvariety of $W^{d}$. An easy induction argument, using the projection $Y_{d+1} \longrightarrow Y_{d}$ to the first $d$ components, shows that $Y_{d}$ is an irreducible variety of dimension $n+(n-1)+\ldots+(n-d+1)=d(2 n-d+1) / 2$.
(a) Consider the linear representation of $O_{n}$ on $V=W^{n}$. Both $W^{n}$ and $Y_{n}$ are generically free $O_{n}$-varieties. We now observe that the usual diagonalization process gives rise to a compression $f: W^{n} \rightarrow Y_{n}$. That is, we define $f\left(w_{1}, \ldots, w_{n}\right)=$ $\left(w_{1}^{\prime}, \ldots, w_{n}^{\prime}\right)$, where $w_{1}^{\prime}=w_{1}$ and

$$
\begin{equation*}
w_{i}^{\prime}=P\left(w_{i} ; w_{1}^{\prime}, \ldots, w_{i-1}^{\prime}\right) \tag{5}
\end{equation*}
$$

for $i=2, \ldots, n$. Thus ed $\left(O_{n}\right) \leq \operatorname{dim}\left(Y_{n} / O_{n}\right)=\operatorname{dim}\left(Y_{n}\right)-\operatorname{dim}\left(O_{n}\right)=n(n+1) / 2-$ $n(n-2) / 2=n$, as claimed.
(b) Note that $W^{n-1}$ is a generically free linear representation of $S O_{n}$ (but not of $O_{n}$ ) and $Y_{n-1}$ is a generically free $S O_{n}$-variety of $W^{n-1}$. Moreover, the map $W^{n-1} \longrightarrow Y_{n-1}$ given by $\left(w_{1}, \ldots, w_{n-1}\right) \mapsto\left(w_{1}^{\prime}, \ldots, w_{n-1}^{\prime}\right)$, with $w_{1}^{\prime}, \ldots, w_{n-1}^{\prime}$ as in (5), is an $S O_{n^{-}}$ compression. Thus
$\operatorname{ed}\left(S O_{n}\right) \leq \operatorname{dim}\left(Y_{n-1}\right)-\operatorname{dim}\left(S O_{n}\right)=(n-1)(n+2) / 2-n(n-1) / 2=n-1$.

## $4(G-H)$-sections and essential dimension

### 4.1 The essential dimension of a section

Lemma 4.1 Let $X$ be a generically free $G$-variety and let $S \subset X$ be a $(G, H)$-section; see Definition 2.9. Then $\operatorname{ed}(X, G) \leq \operatorname{ed}(S, H)$.

Proof. Recall that $X \simeq G *_{H} S$; see Definition 2.12 and Lemma 2.14.
Choose a generically free $H$-variety $Y$ and an $H$-compression $\alpha: S \rightarrow Y$ such that $\operatorname{dim}(Y / H)=\operatorname{ed}(S, H)$. Then $G *_{H} Y$ is a generically free $G$-variety; see Remark 2.13(a). It is easy to see that $\alpha$ lifts to a $G$-compression $\beta: X \simeq G *_{H} S \rightarrow G *_{H} Y$ given by $\beta:(g, s) \mapsto(g, \alpha(s))$. Thus ed $(X, G)=\operatorname{ed}\left(G *_{H} S, G\right) \leq \operatorname{dim}\left(G *_{H} Y\right)-\operatorname{dim}(G)=$ $\operatorname{dim}(Y)-\operatorname{dim}(H)=\operatorname{ed}(S, H)$.

Example 4.2 To illustrate Lemma 4.1 we will give an alternative proof of the inequalities ed $\left(O_{n}\right) \leq n$ and $\operatorname{ed}\left(S O_{n}\right) \leq n-1$ of Example 3.10.

Let $W=k^{n}, q$ be a non-degenerate symmetric bilinear form of $W, e_{1}, \ldots, e_{n}$ be an orthonormal basis with respect to $q$, and $V=W^{n}$ be a generically free linear $O_{n-}$ variety as in Example 3.10. Set $V_{i}=\operatorname{Span}\left(e_{1}, \ldots, e_{i}\right)$. Then $S=V_{1} \times \ldots \times V_{n}$ is a $\left(O_{n},(\mathbb{Z} / 2)^{n}\right)$-section of $V=W^{n}$. Thus by Lemma 4.1

$$
\operatorname{ed}\left(O_{n}\right)=\operatorname{ed}\left(V, O_{n}\right) \leq \operatorname{ed}\left(S,(\mathbb{Z} / 2)^{n}\right)=\operatorname{ed}\left((\mathbb{Z} / 2)^{n}\right) \leq n .
$$

Note that $\operatorname{ed}\left(S,(\mathbb{Z} / 2)^{n}\right)=\operatorname{ed}\left((\mathbb{Z} / 2)^{n}\right)$ because $S$ is a linear space with a generically free $(\mathbb{Z} / 2)^{n}$-action. The last inequality follows from the fact that the group $(\mathbb{Z} / 2)^{n}$ has a generically free $n$-dimensional representation. (We remark that, in fact, $\operatorname{ed}\left((\mathbb{Z} / 2)^{n}\right)=n$ by $\left[\mathrm{BR}_{1}\right.$, Theorem 6.1]; however, all we need here is the inequality $\operatorname{ed}\left((\mathbb{Z} / 2)^{n}\right) \leq n$, which is immediate from the definition of essential dimension.)

Note that $S$ can also be viewed as a $\left(S O_{n},(\mathbb{Z} / 2)^{n-1}\right)$-section for the $S O_{n}$-action on $V$. Applying Lemma 4.1 to this section yields the inequality $\operatorname{ed}\left(S O_{n}\right) \leq n-1$ of Example 3.10(b).

Another application of Lemma 4.1 can be found in the proof of Proposition 11.7.
Proposition 4.3 Let $G$ be a connected semisimple group whose center is trivial. Let $N$ be the normalizer of a maximal torus in $G$. Then $\operatorname{ed}(G) \leq \operatorname{ed}(N)$.

Proof. Let $T$ be a maximal torus of $G$ and let $\operatorname{Lie}(G)$ and $\operatorname{Lie}(T)$ be the Lie algebras of $G$ and $T$ respectively. Then the linear representation of $G$ on

$$
V=\operatorname{Lie}(G) \times \operatorname{Lie}(G)
$$

given by the adjoint action in each component is generically free and

$$
S=\operatorname{Lie}(T) \times \operatorname{Lie}(G)
$$

is a $(G, N)$-section for this action; see [Po, 1.7.17]. Thus ed $(N)=\operatorname{ed}(S, N)$; see Definition 3.5. By Lemma 4.1 we have $\operatorname{ed}(G)=\operatorname{ed}(V, G) \leq \operatorname{ed}(S, N)=\operatorname{ed}(N)$, as claimed.

Remark 4.4 Note that ed $(G)$ can be strictly less than ed $(N)$. Indeed, let $G=\mathrm{SL}_{n}$. Then $\operatorname{ed}(G)=0$; see Example 3.9. On the other hand, $\operatorname{ed}(N)>0$, since $N$ is not connected; see Theorem 5.4.

We also remark that the group $N$ cannot be replaced by the Weyl group $W$ of $G$ in the statement of Proposition 4.3. In other words, the inequality $\operatorname{ed}(G) \leq \operatorname{ed}(W)$ is false in general. For example, if $G=\mathrm{PGL}_{4}$ then $W=S_{4}$ and thus ed $(W)=2$; see $\left[\mathrm{BR}_{1}\right.$, Thm. 6.5]. On the other hand, $\operatorname{ed}\left(\mathrm{PGL}_{4}\right) \geq 4$; see Theorem 9.3.

### 4.2 An application

Theorem $4.5 \operatorname{ed}\left(\mathrm{PGL}_{n}\right) \leq n^{2}-2 n$ for every $n \geq 4$.
Proof. Let $T$ be a maximal torus in $\mathrm{PGL}_{n}$ and let $N$ be the normalizer of $T$. By Proposition 4.3 it is sufficient to show that ed $(N) \leq n^{2}-2 n$. We therefore take a closer look at $N$. The maximal torus $T$ can be thought of as $\left(k^{*}\right)^{n} / \Delta$, where $\Delta \simeq k^{*}$ is the diagonal subgroup of $\left(k^{*}\right)^{n}$. Then $N$ can be written as a semidirect product $T \times_{\phi} S_{n}$, where $\phi: S_{n} \longrightarrow \operatorname{Aut}(T)$ is given by permuting the $n$ factors of $k^{*}$.

We will now construct a linear representation $V$ of $N$. Let $V$ be the $k$-vector space freely spanned by the $n(n-1)$ basis vectors $e_{i j}$, where $i, j=1, \ldots, n$ and $i \neq j$. We shall denote $\sum x_{i j} e_{i j}$ by $\left(x_{i j}\right)$. The action of $N$ on $V$ is defined as follows. Write an element of $N$ as $(t, \sigma)$, where $t=\left(t_{1}, \ldots, t_{n}\right)$ (modulo $\Delta$ ) and $\sigma \in S_{n}$. Then

$$
(t, \sigma): e_{i j} \mapsto t_{\sigma(i)} t_{\sigma(j)}^{-1} e_{\sigma(i) \sigma(j)} .
$$

One can check that this action is generically free and thus

$$
\operatorname{ed}(N)=\operatorname{ed}(V, N) \leq \operatorname{dim}(V / N)=\operatorname{dim}(V)-\operatorname{dim}(N)=n^{2}-2 n+1
$$

We shall, instead, derive the slightly better bound of Theorem 4.5 by considering the $N$-action on the projective space $P(V)$.

Lemma 4.6 The action of $N$ on $P(V)$ is generically free.
Proof. Let $a, a^{\prime}, b, b$ are four distinct integers between 1 and $n$. Define a rational function $z_{\left(a, a^{\prime}\right)\left(b, b^{\prime}\right)}$ on $P(V)$ by

$$
z_{\left(a, a^{\prime}\right)\left(b, b^{\prime}\right)}=\frac{x_{a b} x_{a^{\prime} b^{\prime}}}{x_{a b^{\prime}} x_{a^{\prime} b}} .
$$

Note that $z_{\left(a, a^{\prime}\right)\left(b, b^{\prime}\right)}=z_{\left(c, c^{\prime}\right)\left(d, d^{\prime}\right)}$ implies $\left\{a, a^{\prime}\right\}=\left\{c, c^{\prime}\right\}$ and $\left\{b, b^{\prime}\right\}=\left\{d, d^{\prime}\right\}$. Let $U$ be the dense $N$-invariant open subset of $V$ consisting of those $p \in P(V)$ satisfying
(i) $x_{i j}(p) \neq 0$ for every distinct $i, j=1, \ldots, n$, and
(ii) $z_{\left(a, a^{\prime}\right)\left(b, b^{\prime}\right)}(p) \neq z_{\left(c, c^{\prime}\right)\left(d, d^{\prime}\right)}(p)$ whenever $\left\{a, a^{\prime}\right\} \neq\left\{c, c^{\prime}\right\}$ or $\left\{b, b^{\prime}\right\} \neq\left\{d, d^{\prime}\right\}$.

We claim that $N$ acts freely on $U$. Indeed, let $p \in U$ and suppose $(t, \sigma) p=p$ for some $(t, \sigma) \in N$. Note that the functions $z_{\left(a, a^{\prime}\right)\left(b, b^{\prime}\right)}$ are $T$-invariant and

$$
(t, \sigma) z_{\left(a, a^{\prime}\right)\left(b, b^{\prime}\right)}=z_{\left(\sigma(a), \sigma\left(a^{\prime}\right)\right)\left(\sigma(b) \sigma\left(b^{\prime}\right)\right)} .
$$

Thus if $p \in U$, then $\left\{\sigma(a), \sigma\left(a^{\prime}\right)\right\}=\left\{a, a^{\prime}\right\}$ and $\left\{\sigma(b), \sigma\left(b^{\prime}\right)\right\}=\left\{b, b^{\prime}\right\}$ for every 4-tuple $a, a^{\prime}, b, b^{\prime}$ of distinct integers between 1 and $n$. If $n \geq 4$ then the only $\sigma \in S_{n}$ with this property is the identity element.

We have thus proved that if $p \in U$ and $(t, \sigma) p=p$ then $\sigma=1$ in $S_{n}$. We now want to show that $t=1$ in $T$. Indeed, since every coordinate $x_{i j}$ of $p$ is assumed to be
non-zero, $(t, 1) p=p$ implies $t_{a} t_{b}^{-1}=t_{c} t_{d}^{-1}$ for every $1 \leq a, b, c, d \leq n$ with $a \neq b$ and $c \neq d$. Letting $b=d=1$, we see that $t_{2}=t_{3}=\ldots=t_{n}$. Similarly for $b=d=2$ we obtain $t_{1}=t_{3}=\ldots=t_{n}$. Thus $t_{1}=\ldots=t_{n}$, i.e., $t=1$ in $T$. This completes the proof of the lemma.

To finish the proof of the theorem, consider the natural ( $N$-equivariant) projection map $V \rightarrow P(V)$. By Lemma 4.6 this map is an $N$-compression. Thus by Lemma 3.3(a-b) ed $(N) \leq \operatorname{dim}(P(V))-\operatorname{dim}(N)=\left(n^{2}-n-1\right)-(n-1)=n^{2}-2 n$, as claimed.

Remark 4.7 In Section 9 we will strengthen the upper bound of Theorem 4.5 for every $n$ which is not a power of 2 ; see Theorem 9.6 and Remark 9.9.

## 5 Groups of low essential dimension

The results of this section have been suggested to us by V. L. Popov.

### 5.1 Rational sections

Let $X$ be a primitive $G$-variety (see Definition 2.1) and let $\pi: X \rightarrow X / G$ be the rational quotient map for the $G$-action on $X$. A subvariety $S \subset X$ is said to be a rational section if $\pi_{\mid S}: S \rightarrow X / G$ is a birational isomorphism.

Remark 5.1 $S$ is a rational section of $X$ if and only if $S$ is a $(G, H)$-section with $H=\left\{1_{G}\right\}$. Indeed, if $S$ is a $(G,\{1\})$-section then $\pi_{\mid S}: S \rightarrow X / G$ is a birational isomorphism by Lemma 2.11. The opposite implication follows from Theorem 2.3.

Lemma 5.2 Let $X$ be a primitive generically free $G$-variety. Then $\operatorname{ed}(X, G)=0$ if and only if $X$ has a rational section.

Proof. Suppose $\operatorname{ed}(X, G)=0$. Then there exists a $G$-compression $f: X \rightarrow Y$, where $Y / G$ is a point. In other words, there is a point $y \in Y$ such that $G y$ is dense in $Y$ and $\operatorname{Stab}(y)=\left\{1_{G}\right\}$. It is now easy to see that $S=f^{-1}(y)$ is a $(G,\{1\})$-section, and, hence, a rational section; see Remark 5.1.

Conversely, suppose $S \subset X$ is a rational section. Then $S$ is a ( $G,\{1\}$ )-section; hence, by Lemmas 4.1 and $3.3(\mathrm{e}), \mathrm{ed}(X, G) \leq \operatorname{ed}(S,\{1\})=0$.

### 5.2 Special groups

Let $G$ be an algebraic group defined over $k$. Recall that $G$ is called special if $H^{1}(K, G)=$ $\{1\}$ for every field $K$ containing $k$. Special groups were introduced by Serre $\left[\mathrm{Se}_{1}\right]$ and classified by Grothendieck [G].

Proposition 5.3 The following conditions are equivalent:
(a) $G$ is a special group,
(b) $\operatorname{ed}(X, G)=0$ for every primitive generically free $G$-variety $X$,
(c) $\operatorname{ed}(G)=0$.

Proof. By Lemma 5.2 condition (b) is equivalent to
$\left(b^{\prime}\right)$ every primitive generically free $G$-variety has a rational section and (c) is equivalent to
$\left(c^{\prime}\right)$ every generically free linear representation of $G$ has a rational section.
It is well known that conditions ( a ) , ( $\mathrm{b}^{\prime}$ ) and ( $\mathrm{c}^{\prime}$ ) are equivalent; see e.g., [Po, 1.4], [PV, Section 2.6], or [Do, Sect. 7]. Hence, so are (a), (b), and (c).

Combining Proposition 5.3 with Grothendieck's classification, of special groups (see [G, Section 5] or [PV, Theorem 2.8]), we obtain the following result.

Theorem 5.4 Let $G$ be an algebraic group defined over $k$. Then $\operatorname{ed}(G)=0$ if and only if $G$ is connected and its maximal connected semisimple subgroup is a direct product $G_{1} \times \ldots \times G_{r}$, where each $G_{i}$ is of type SL or Sp .

## $5.3 \quad n$-special groups

Definition 5.5 We will say that an algebraic group $G$ defined over $k$ is $n$-special if $H^{1}(K, G)=\{1\}$ for any extension $K / k$ of transcendence degree $\leq n$.

Note that a special group is $n$-special for every $n$.

Proposition 5.6 Let $G$ be an n-special algebraic group and let $X$ be a generically free primitive $G$-variety. Then
(a) either $\operatorname{ed}(X, G)=0$ or $\operatorname{ed}(X, G) \geq n+1$ and
(b) either $\operatorname{ed}(G)=0$ or $\operatorname{ed}(G) \geq n+1$.

Proof. (a) Assume the contrary: $\operatorname{ed}(X, G)=i$, where $1 \leq i \leq n$. Then there exists a compression $\alpha: X \rightarrow Y$ such that $\operatorname{trdeg}_{k} k(Y)^{G}=\operatorname{dim}(Y / G)=i$. Recall that the generically free action of $G$ on $Y$ determines an element of $H^{1}(k(Y / G), G)$; following Popov [Po, 1.3], we shall denote this class by $\operatorname{cl}(G: Y)$. By our assumption $H^{1}(k(Y / G), G)=\{1\}$ and thus $c l(G, Y)=1$. Consequently, $Y$ has a rational section; see $[\mathrm{Po}, 1.4 .1]$. By Lemma 5.2, this implies that $\operatorname{ed}(Y, G)=0$. Thus ed $(X, G) \leq$ $\operatorname{ed}(Y, G)=0$ (see Lemma $3.3(b)$ ), contradicting our assumption.

Part (b) is an immediate consequence of (a) and Definition 3.5.

Corollary 5.7 Let $G$ be a connected algebraic group. Then
(a) $\operatorname{ed}(X, G) \neq 1$ for any irreducible $G$-variety $X$.
(b) $\operatorname{ed}(G) \neq 1$.

Proof. By a theorem of Steinberg [St, Thm. 1.9] every connected group is 1-special.

Remark 5.8 Note that Corollary 5.7 fails if $G$ is not assumed to be connected. For example, $\operatorname{ed}(G)=1$ if $G$ is a (finite) cyclic or odd dihedral group; see $\left[\mathrm{BR}_{1}, 6.2\right]$.

Remark 5.9 A conjecture of Serre [Se 3 , III.3.1] (often referred to as "Conjecture II") says that every simply connected semisimple group $G$ is 2 -special. This conjecture has been verified for all simple groups other than $E_{8}$; see [Se 3 , Sect 5], [Se 4 , III.3], [BP], [ $\mathrm{Ko}_{2}$ ], and [Gi].

## 6 Structured spaces I

### 6.1 Definition and examples

Definition 6.1 Let $F$ be a field.
(a) A structured space over $F$ is a pair $(W, \beta)$, where $W$ is a finite-dimensional $F$ vector space and $\beta$ is a tensor on $W$, i.e., $\beta \in T(W) \otimes T\left(W^{*}\right)$. If $F^{\prime} / F$ is a field extension then $(W, \beta) \otimes_{F} F^{\prime}$ is defined to be the structured space ( $W^{\prime}, \beta^{\prime}$ ), where $W^{\prime}=W \otimes_{F} F^{\prime}$ and $\beta^{\prime}=\beta \otimes 1 \in T(W) \otimes T\left(W^{*}\right) \otimes F^{\prime}=T\left(W^{\prime}\right) \otimes T\left(W^{\prime *}\right)$.
(b) Two structured spaces $(W, \beta)$ and ( $W^{\prime}, \beta^{\prime}$ ) over $F$ are isomorphic if there exists an isomorphism $\phi: W \longrightarrow W^{\prime}$ of $F$-vector spaces such that $\phi^{*}(\beta)=\beta^{\prime}$, where $\phi^{*}$ is the isomorphism $T(W) \otimes T\left(W^{*}\right) \longrightarrow T\left(W^{\prime}\right) \otimes T\left(W^{\prime *}\right)$ induced by $\phi$.
(c) In the special case where $A$ is a finite-dimensional $F$-algebra, $W$ is the underlying $F$-vector space of $A$ and the multiplicative structure is given by a tensor $\beta \in W^{*} \otimes$ $W^{*} \otimes W$, we will write $A=(W, \beta)$. Note that here the algebra $A$ is not assumed to be commutative or associative, or to have an identity element.

Remark 6.2 The above definition is essentially the same as the one in [ $\mathrm{Se}_{2}, \mathrm{X} .2$ ] or [Se ${ }_{4}$, III.1.1]. The only difference is that we do not require $\alpha$ to be homogeneous, i.e., to lie in $T^{p}(W) \otimes T^{q}\left(W^{*}\right)$ for some $p, q \geq 0$; see Example 6.3(e) below.

We will always assume (as we do elsewhere in this paper) that $F$ is a field extension of $k$; if $F=k$ we will usually denote our structured spaces by ( $V, \alpha$ ). In this case we will be interested in the algebraic $\operatorname{group}_{\operatorname{Aut}}^{k}(V, \alpha) \stackrel{\text { def }}{=}\left\{g \in \operatorname{GL}(V) \mid g^{*}(\alpha)=\alpha\right\}$. The following examples will be of special interest to us in the sequel.

Example 6.3 (a) $V=k^{n}$ and $\alpha=0$. Then $\operatorname{Aut}_{k}(V, \alpha)=\mathrm{GL}_{n}$.
(b) $V=k^{n}$ and $0 \neq \alpha \in \Lambda^{n}\left(V^{*}\right)$ is a volume form on $V$. Then $\operatorname{Aut}_{k}(V, \alpha)=\mathrm{SL}_{n}$.
(c) $V=k^{2 n}$ and $\alpha \in \Lambda^{2}\left(V^{*}\right)$ is a symplectic (i.e., non-degenerate skew-symmetric bilinear) form. Then $\operatorname{Aut}_{k}(V, \alpha)=\mathrm{Sp}_{2 n}$.
(d) $V=k^{n}$ and $\alpha \in S^{2}\left(V^{*}\right)$ is a non-degenerate symmetric bilinear form. Then $\operatorname{Aut}_{k}(V, \alpha)=O_{n}$.
(e) $V=k^{n}$ and $\alpha=\alpha_{1}+\alpha_{2}$, where $\alpha_{1} \in \Lambda^{n}\left(W^{*}\right)$ is a volume form and $\alpha_{2} \in S^{2}\left(W^{*}\right)$ is a non-degenerate symmetric bilinear form. Then any element $g \in \mathrm{GL}(V)$ preserving $\alpha$ has to preserve both $\alpha_{1}$ and $\alpha_{2}$. Thus $\operatorname{Aut}_{k}(V, \alpha)=\mathrm{SL}_{n} \cap O_{n}=S O_{n}$.
(f) $(V, \alpha)$ is the algebra $k \oplus \ldots \oplus k$ ( n times) ; see Definition 6.1(c). Then $\operatorname{Aut}_{k}(V, \alpha)$ is the symmetric group $S_{n}$.
(g) $(V, \alpha)=\mathrm{M}_{n}(k)$ is the algebra of $n \times n$-matrices over $k$. Then $\operatorname{Aut}_{k}(V, \alpha)=$ $\mathrm{PGL}_{n}$.
(h) $(V, \alpha)=\mathbf{O}$, where $\mathbf{O}$ is the (split) octonion algebra over $k$; see Section 11.1. Then $\operatorname{Aut}_{k}(\mathbf{O})$ is the exceptional group $G_{2}$.
(i) $(V, \alpha)=\mathbf{A}$, where $\mathbf{A}$ is the (split) Albert algebra over $k$; see Section 11.3. Then $\operatorname{Aut}_{k}(V, \alpha)$ is the exceptional group $F_{4}$.
(j) $V=k^{27}$ is the underlying vector space of the Albert algebra $\mathbf{A}$ and $\alpha \in V^{*} \otimes$ $V^{*} \otimes V^{*}$ be the trilinear form associated to the cubic norm on $\mathbf{A}$; see Sections 11.3 and 11.5. Then $\operatorname{Aut}(V, \alpha)$ is the (simply connected) exceptional group $E_{6}$.

### 6.2 First properties

Let $(W, \beta)$ be a structured space defined over a field $F$ containing $k$. Suppose $B=$ $\left\{b_{1}, \ldots, b_{n}\right\}$ is an $F$-basis of $W$. Denote the dual basis of $W^{*}$ by $B^{*}=\left\{\left(b_{1}\right)^{*}, \ldots,\left(b_{n}\right)^{*}\right\}$. Then the tensors $b_{I}=b_{i_{1}} \otimes \ldots \otimes b_{i_{r}}$ form an $F$-basis of $T(W)$, as $r$ ranges over the positive integers, $I=\left(i_{1}, \ldots, i_{r}\right)$, and each $i_{j}$ ranges over $\{1, \ldots, n\}$. Similarly, the tensors $\left(b_{J}\right)^{*}=\left(b_{j_{1}}\right)^{*} \otimes \ldots \otimes\left(b_{j_{s}}\right)^{*}$ form an $F$-basis of $T^{*}(W)$. Thus we can write

$$
\begin{equation*}
\beta=\sum_{I, J} \beta_{I}^{J}(B) b_{I} \otimes b_{J}^{*}, \tag{6}
\end{equation*}
$$

where each coefficient $\beta_{I}^{J}(B)$ lies in $F$. Note that only finitely many of these coefficients are non-zero.

Let $\left(V=k^{n}, \alpha\right)$ be a structured space and $G=\operatorname{Aut}_{k}(V, \alpha)$. Note that the $G$-action on $V$ may not be generically free; however, the (diagonal) $G$-action on $V^{n}$ is always free on the open subset $\left(V^{n}\right)^{\prime}$ consisting of $n$-tuples of linearly independent elements. (We prefer the notation $\left(V^{n}\right)^{\prime}$ to $\mathrm{GL}(V)$, since we view this set as a Zariski open subset of $V^{n}$ and ignore the group structure.)

Let $B \in\left(V^{n}\right)^{\prime}$ be a basis of $V$. Write $\alpha=\sum_{I, J} \alpha_{I}^{J}(B) b_{I} \otimes b_{J}^{*}$, as in (6). Then each $\alpha_{I}^{J}$ can be viewed as a function $\left(V^{n}\right)^{\prime} \longrightarrow k$, which takes $B \in\left(V^{n}\right)^{\prime}$ to $\alpha_{I}^{J}(B)$. It is easy to see that the functions $\alpha_{I}^{J}$ are regular on $\left(V^{n}\right)^{\prime}$ and thus rational on $V^{n}$.

Lemma 6.4 Let $V=k^{n}, \alpha \in T(V) \otimes T\left(V^{*}\right), G=\operatorname{Aut}_{k}(\alpha)$ and $\left(V^{n}\right)^{\prime} \subset V^{n}$ be as above. Suppose $B, B^{\prime} \in\left(V^{n}\right)^{\prime}$. Then
(a) $B^{\prime}=g B$ for some $g \in G$ if and only if $\alpha_{I}^{J}(B)=\alpha_{I}^{J}\left(B^{\prime}\right)$ for every $I$ and $J$.
(b) If $Z$ is a $G$-invariant subvariety of $V^{n}$ such that $Z \cap\left(V^{n}\right)^{\prime} \neq \emptyset$ then $k(Z)^{G}=$ $k\left(\alpha_{I}^{J} \mid Z\right)$, where I ranges over $\{1, \ldots, n\}^{r}$ and $J$ ranges over $\{1, \ldots, n\}^{s}$.
Proof. (a) Write $B^{\prime}=g B$ for some $g \in G L(V)$ and observe that $g \in G=\operatorname{Aut}_{k}(V, \alpha)$ if and only if $\alpha_{I}^{J}(B)=\alpha_{I}^{J}\left(B^{\prime}\right)$ for every $I$ and $J$.
(b) Each $\alpha_{I}^{J}$ defines a regular map $\left(V^{n}\right)^{\prime} \longrightarrow k$; together they define a regular map $f:\left(V^{n}\right)^{\prime} \longrightarrow k^{N}$, where $N$ is the number of non-zero $\alpha_{I}^{J}$,s. Denote $f\left(Z \cap\left(V^{n}\right)^{\prime}\right)$ by $Z_{0}$. By part (a) $f$ separates the $G$-orbits in $\left(V^{n}\right)^{\prime}$ and, hence, in $Z \cap\left(V^{n}\right)^{\prime}$. Thus by Remark 2.5, $Z_{0}$ is birationally isomorphic to $Z / G$; hence, $k(Z)^{G}=k(Z / G)=k\left(Z_{0}\right)=$ $k\left(\alpha_{I}^{J} \mid Z\right)$, as claimed.

## 7 Spaces of rational maps

### 7.1 Equivariant maps into vector spaces

Proposition 7.1 Let $G$ be an algebraic group and let $V$ be a linear representation of $G$. Then for every generically free $G$-variety $X$ and every $v \in V$, there exists a $G$-equivariant rational map $f: X \rightarrow V$ whose image contains $v$.

Proof. Suppose the action of $G$ on $V$ is given by the group homomorphism $\phi: G \longrightarrow$ $\mathrm{GL}_{n}$, where $n=\operatorname{dim}(V)$. We may assume without loss of generality that $\phi$ is injective. Indeed, otherwise we can replace $V$ by $V^{\prime}=V \oplus W$, where $W$ is a generically free linear representation of $G$. If we can construct a $G$-equivariant rational map $X \rightarrow V^{\prime}$ whose image contains $(v, 0)$, then the composition of this map with the projection $V^{\prime}=V \oplus W \longrightarrow V$ will have the desired properties.

Note that we may also assume without loss of generality that $X$ is primitive. Indeed, by Lemma $2.2 X$ is isomorphic to a disjoint union of primitive varieties. It is enough to define $f$ on one of them; after that the others can be sent to 0 .

Thus we can think of $G$ as a subgroup of $\mathrm{GL}_{n}=\mathrm{GL}(V)$; the $G$-action on $V$ is then given by left multiplication. Let $X^{\prime}=\mathrm{GL}_{n} *_{G} X$; see Definition 2.12. Recall that $X^{\prime}$ is a generically free $\mathrm{GL}_{n}$-variety; see Remark 2.13 . Since $\mathrm{GL}_{n}$ is a special group, Proposition 5.3 tells us that $\operatorname{ed}\left(X^{\prime}, \mathrm{GL}_{n}\right)=0$. This means that there exists a $\mathrm{GL}_{n}$-compression $\alpha: X^{\prime} \rightarrow O$, where $O$ is a single $\mathrm{GL}_{n}$-orbit with trivial stabilizer. In other words, up to birational isomorphism of $\mathrm{GL}_{n}$-varieties, $O \simeq V^{n} \simeq \mathrm{M}_{n}$, where $\mathrm{GL}_{n}$ acts on $\mathrm{M}_{n}$ by left multiplication.

Note that the image of $\alpha$ contains every non-singular matrix in $\mathrm{M}_{n}$ (these matrices form a single dense $\mathrm{GL}_{n}$-orbit). In particular, there exists an $x^{\prime}=(g, x) \in X^{\prime}$ such that $\alpha\left(x^{\prime}\right)=I_{n}=n \times n$-identity matrix. Then $\alpha\left(1_{G}, x\right)=g^{-1}$. Let $v_{1}, \ldots, v_{n}$ be the columns of $g^{-1}$; since they are linearly independent, we can write

$$
v=c_{1} v_{1}+\ldots+c_{n} v_{n}
$$

for some $c_{1}, \ldots, c_{n} \in k$. Let $\beta: \mathrm{M}_{n} \longrightarrow V$ be the linear map given by

$$
\beta(A)=A\left(\begin{array}{c}
c_{1} \\
\vdots \\
c_{n}
\end{array}\right) .
$$

Note that $\beta$ is $\mathrm{GL}_{n}$-equivariant and $\beta\left(g^{-1}\right)=v$. Now the desired map $f: X \rightarrow V$ is obtained as a composition

$$
X \hookrightarrow X^{\prime}=\mathrm{GL}_{n} *_{G} X \quad \stackrel{\alpha}{\longrightarrow} \mathrm{M}_{n} \xrightarrow{\beta} V,
$$

where the inclusion $X \hookrightarrow X^{\prime}$ is given by $x \mapsto\left(1_{G}, x\right)$; see Remark 2.13. This inclusion is clearly $G$-equivariant; the maps $\alpha$ and $\beta$ are $\mathrm{GL}_{n}$-equivariant (and, hence, $G$-equivariant) by our construction. Hence, $f$ is $G$-equivariant and $f(x)=v$, as desired.

Remark 7.2 Let $G$ be an algebraic group, $X$ be a generically free primitive $G$-variety and $V$ be a generically free linear representation of $G$. Then combining Proposition 7.1 with Lemma $3.3(c)$, we obtain the inequality $\operatorname{ed}(X, G) \leq \operatorname{ed}(V, G) \stackrel{\text { def }}{=} \operatorname{ed}(G)$. This gives another proof of Theorem 3.4(c).

Definition 7.3 Let $G$ be an algebraic group, $G \longrightarrow \mathrm{GL}(V)$ be a linear representation of $G$ and $X$ be a $G$-variety. We shall denote the $k(X)^{G}$-vector space of $G$-equivariant rational maps $X \rightarrow V$ by $R M a p s_{G}(X, V)$.

Equivalently, $R M a p s_{G}(X, V)=\left(V \otimes_{k} k(X)\right)^{G}$.
Lemma 7.4 Let $V=k^{n}$, let $G \longrightarrow \mathrm{GL}(V)$ be a linear representation of $G$, and let $X$ be a primitive $G$-variety. Then
(a) $b_{1}, \ldots, b_{m} \in R M a p s_{G}(X, V)$ are linearly independent over $k(X)^{G}$ if and only if $b_{1}(x), \ldots, b_{m}(x)$ are linearly independent over $k$ for $x$ in general position in $X$.
(b) If $X$ is generically free then $\operatorname{dim}_{k(X)^{G}}\left(R M a p s_{G}(X, V)\right)=n$.

Proof. (a) Suppose $b_{1}(x), \ldots, b_{m}(x)$ are linearly independent for $x \in X$ in general position and $\sum_{i=1}^{m} f_{i} b_{i}=0$ for some $f_{1}, \ldots, f_{m} \in k(X)^{G}$. Then evaluating both sides at $x \in X$ in general position, we conclude that $f_{1}=\ldots=f_{m}=0$ in $k(X)^{G}$ and thus $b_{1}, \ldots, b_{m}$ are $k(X)^{G}$-linearly independent.

Conversely, suppose $b_{1}(x), \ldots, b_{m}(x)$ are $k$-linearly dependent for $x \in X$ in general position. Then we may assume without loss of generality that $b_{1}(x), \ldots, b_{m-1}(x)$ are linearly independent for $x$ in general position; otherwise, we can simply replace $b_{1}, \ldots, b_{m}$ by $b_{1}, \ldots, b_{m-1}$. Thus for each $x$ in general position in $X$, there exist uniquely defined $f_{1}(x), \ldots, f_{m-1}(x) \in k$ such that

$$
\begin{equation*}
b_{m}=f_{1} b_{1}+\ldots+f_{m-1} b_{m-1} \tag{7}
\end{equation*}
$$

It remains to prove that $f_{1}, \ldots, f_{m-1} \in k(X)^{G}$. By Cramer's rule $f_{1}, \ldots, f_{m-1} \in k(X)$. To prove these functions are $G$-invariant, choose $x \in X$ and $g \in G$, then substitute $g x$ into (7) and apply $g^{-1}$ to both sides. Since each $b_{i}$ is a $G$-equivariant map $X \rightarrow V$, we have $g^{-1} b_{i}(g x)=b_{i}(x)$ and thus

$$
b_{m}(x)=f_{1}(g x) b_{1}(x)+\ldots+f_{m-1}(g x) b_{m-1}(x)
$$

By uniqueness of $f_{i}(x)$, we conclude that $f_{i}(g x)=f_{i}(x)$, i.e., $f_{i} \in k(X)^{G}$, as claimed.
(b) The inequality $\operatorname{dim}_{k(X)^{G}} R M a p s(X, V) \leq n$ is a direct consequence of part (a). To prove the opposite inequality, let $v_{1}, \ldots, v_{n}$ be a $k$-basis of $V, v=\left(v_{1}, \ldots, v_{n}\right) \in V^{n}$. By Proposition 7.1 there exists a rational $G$-equivariant map $b: X \rightarrow V^{n}$ whose image contains $v$. Write $b=\left(b_{1}, \ldots, b_{n}\right)$, where each $b_{i}$ is a rational $G$-equivariant map $X \rightarrow V$, i.e., and element of $R M a p s_{G}(X, V)$. In order to complete the proof of part (b), it is enough to show that $b_{1}, \ldots, b_{n}$ are $k(X)^{G}$-linearly independent. Let $S$ be the set of $x \in X$ such that $b_{1}(x), \ldots, b_{n}(x)$ are $k$-linearly independent. Clearly $S$ is an open $G$-invariant subset of $X$; moreover, since the image of $b$ contains $v, S \neq \emptyset$. Thus $S$ is open and dense in $X$; the desired conclusion now follows from part (a).

### 7.2 Structured spaces of rational maps

Let ( $V=k^{n}, \alpha$ ) be a structured space, let $G=\operatorname{Aut}_{k}(V, \alpha)$, and let $H$ be an algebraic subgroup of $G$, as above. In this setting $R M a p s_{H}(X, V)$ carries a naturally defined tensor, which we shall denote by $\alpha_{H}^{X}$. To define $\alpha_{H}^{X}$, choose a $k(X)^{H}$-basis $B=\left\{b_{1}, \ldots, b_{n}\right\}$ of $R M a p s_{H}(X, V)$. Then

$$
\alpha_{H}^{X} \stackrel{\text { def }}{=} \sum_{I, J}\left(\alpha_{H}^{X}\right)_{I}^{J} b_{I} \otimes\left(b^{*}\right)_{J},
$$

where we define $\left(\alpha_{H}^{X}\right)_{I}^{J} \in k(X)^{H}$ (as a rational function on $X$ ) by

$$
\begin{equation*}
\left(\alpha_{H}^{X}\right)_{I}^{J}(x)=\alpha_{I}^{J}(B(x)) . \tag{8}
\end{equation*}
$$

Here $B(x)=\left\{b_{1}(x), \ldots, b_{n}(x)\right\}$ is a basis of $V$ (for $x$ in general position in $X$ ). Note that since $H$ is a subgroup of $G=\operatorname{Aut}_{k}(V, \alpha)$, each function $x \mapsto \alpha_{I}^{J}(B(x))$ is, indeed, an element of $k(X)^{H}$. Moreover, it is easy to see that our definition of $\alpha_{H}^{X}$ does not depend on the choice of $B$.

Example 7.5 Suppose $\alpha \in V^{*} \otimes V^{*} \otimes V$ defines the structure of an algebra $A$ on $V$, i.e., $(V, \alpha)=A$; see Definition $6.1(c)$. Then the structured space $\left(\right.$ RMaps $\left._{H}(X, V), \alpha_{H}^{X}\right)$ is the algebra of $H$-equivariant rational maps $X \rightarrow V$, with pointwise multiplication.

Lemma 7.6 Let $\left(V=k^{n}, \alpha\right)$ be a structured space, $H$ be an algebraic subgroup of $G=\operatorname{Aut}_{k}(V, \alpha)$, and $X$ be a generically free primitive $G$-variety.
(a) Suppose $f: X \rightarrow Y$ is an $H$-compression. Then

$$
\left(\text { RMaps }_{H}(X, V), \alpha_{H}^{X}\right) \simeq\left(\operatorname{RMaps}_{H}(Y, V), \alpha_{H}^{Y}\right) \otimes_{k(Y)^{H}} k(X)^{H},
$$

where the field extension $k(X)^{H} / k(Y)^{H}$ is induced by $f$.
(b) Suppose $S$ is an $\left(H, H^{\prime}\right)$-section of $X$. Then

$$
\left(\text { RMaps }_{H}(X, V), \alpha_{H}^{X}\right) \simeq\left(R M a p s_{H^{\prime}}(S, V), \alpha_{H^{\prime}}^{S}\right) .
$$

(c) Suppose $X=H \times X_{0}$, where $X_{0}$ is an irreducible variety and $H$ acts on $X$ by $h\left(h^{\prime}, x_{0}\right)=\left(h h^{\prime}, x_{0}\right)$.Then $\left(R M a p s_{H}(X, V), \alpha_{H}^{X}\right) \simeq(V, \alpha) \otimes_{k} k\left(X_{0}\right)$.
Proof. (a) Let $\phi: \operatorname{RMaps}_{H}(Y, V) \otimes_{k(Y)^{H}} k(X)^{H} \longrightarrow$ RMaps $_{H}(X, V)$ be $k(X)^{H_{-} \text {-linear }}$ map given by $a \otimes 1 \longrightarrow a \circ f$. If $C=\left\{c_{1}, \ldots, c_{n}\right\}$ is a $k(Y)^{H}$-basis of $R M a p s_{H}(Y, V)$ then Lemma 7.4 shows that $\phi(C)$ is a $k(X)^{H}$-basis of $R M a p s_{H}(Y, V)$. This proves that $\phi$ is an isomorphism of $k(X)^{H}$-vector spaces. Comparing the coefficients of $\alpha_{H}^{X}$ in the basis $C$ to the coefficients of $\alpha_{H}^{Y}$ in the basis $\phi(C)$, we conclude that $\phi$ is an isomorphism of structured spaces.
(b) Let $B=\left\{b_{1}, \ldots, b_{n}\right\}$ be a $k(X)^{H}$-basis of $R M a p s_{H}(X, V)$. Recall that by Lemma 2.11, the restriction $a \mapsto a_{\mid S}$ induces an isomorphism between the fields $k(X)^{H}$ and $k(S)^{H^{\prime}}$. We will therefore identify these fields. Moreover, by Lemma 7.4, $B_{\mid S}=$ $\left\{\left(b_{1}\right)_{\mid S}, \ldots,\left(b_{n}\right)_{\mid S}\right\}$ is a basis of RMaps $_{H^{\prime}}(S, V)$. Hence, $\phi: a \mapsto a_{\mid S}$ gives rise to an isomomorphism $\phi:$ RMaps $_{H}(X, V) \longrightarrow$ RMaps $_{H^{\prime}}(S, V)$ of $k(X)^{H^{-}}$-vector spaces. To show that $\phi$ induces the desired isomorphism of structured spaces, observe that if we restrict the coefficients of $\alpha_{H}^{X}$ in the basis $B$ to S , we obtain the coefficients of $\alpha_{H^{\prime}}^{S}$; see (8).
(c) Note that $\{1\} \times X_{0}$ is an $(H,\{1\})$-section of $X$. Thus by part (b), we may assume $H=\{1\}$ and $X=X_{0}$. Now apply part (a) to the $\{1\}$-compression $X \longrightarrow\{p t\}$, where $\{p t\}$ is a single point. Since $\left(R \operatorname{Maps}_{\{1\}}(\{p t\}, V), \alpha_{\{1\}}^{\{p t\}}\right)=(V, \alpha)$, this completes the proof.

## 8 Structured spaces II

### 8.1 Structured spaces of type (V, $\alpha$ )

Definition 8.1 Let $F$ be a field containing $k$ and let ( $V=k^{n}, \alpha$ ) be a structured space. We shall say that $\left(W=F^{n}, \beta\right)$ is a structured space of type $(V, \alpha)$ if

$$
\begin{equation*}
(W, \beta) \otimes_{F} E \simeq(V, \alpha) \otimes_{k} E \tag{9}
\end{equation*}
$$

for some field extension $E / F$.
Remark 8.2 A structured space ( $W=F^{n}, \beta$ ) of type $(V, \alpha)$ is usually called "an $F / k$ form of ( $V, \alpha$ )"; see e.g., [Se , III.1]. We will use the term "structured space" in order to avoid linguistic confusion in those cases where $\alpha$ and $\beta$ are themselves multilinear forms, as in, e.g., Example 6.3(b), (c), (d), (j).

Remark 8.3 If ( $W, \beta$ ) is a structured space of type ( $V, \alpha$ ) then we can always choose the field extension $E / F$ in Definition 8.1 to be finitely generated. Indeed, if $v_{1}, \ldots, v_{n}$ is a $k$-basis of $V, w_{1}, \ldots, w_{n}$ is an $F$-basis of $W$ and the isomorphism (9) is given by $w_{i} \mapsto \sum_{j} a_{i j} v_{i j}$ with $a_{i j} \in E$ then we can replace $E$ by $F\left(a_{i j}\right)$.

Example 8.4 (a) Suppose $V=k^{n}$ and $\alpha$ is a volume (resp. non-degenerate symmetric bilinear) form on $V$. Then $(W, \beta)$ is of type $(V, \alpha)$ if and only if $W=F^{n}$ and $\beta$ is a a volume (resp. non-degenerate symmetric bilinear) form on $W$.
(b) Let $V=k^{n}$, let $e_{1}, \ldots, e_{n}$ be a $k$-basis of $V, \alpha_{1}=e_{1}^{*} \wedge \ldots \wedge e_{n}^{*} \in \Lambda^{n}(V)$, $\alpha_{2}=\left(e_{1}^{*}\right)^{2}+\ldots+\left(e_{n}^{*}\right)^{2} \in S^{2}(V)$, and $\alpha=\alpha_{1}+\alpha_{2}$. Recall that $\operatorname{Aut}_{k}(V, \alpha)=S O_{n}$; see Example 6.3(e). Now let $F$ be a field extension of $k$ and $W=F^{n}$. Then an easy linear algebra argument shows that $\beta_{2} \mapsto\left(W, \alpha_{1} \otimes_{k} F+\beta_{2}\right)$ defines a bijection between equivalence classes of bilinear forms of determinant 1 on $W$ (modulo SL( $W$ )) and isomorphism classes of structured spaces ( $W, \beta$ ) of type ( $V, \alpha$ ); cf. [KMRT, (29.29), p. 407].
(c) Suppose ( $V, \alpha$ ) is the algebra $k \oplus \ldots \oplus k$ ( n times), as in Example 6.3(f). Then ( $W, \beta$ ) is of type ( $V, \alpha$ ) if and only if $\beta$ defines the structure of an $n$-dimensional etale algebra over $F$, i.e., a direct sum of field extensions of $F$; cf. [KMRT, (29.9), p. 395].
(d) Suppose $\left(V=k^{27}, \alpha\right)=\mathbf{A}$, where $\mathbf{A}$ is the (split) Albert algebra; see Section 11.3. Then ( $W, \beta$ ) is structured spaces of type ( $V, \alpha$ ) if and only if $W=F^{27}$ and $\beta$ defines the structure of a central simple exceptional Jordan algebra on $W$; see [J, Sect V.7].

For other examples see Lemma 9.1 and Remark 11.4.
Lemma 8.5 Suppose ( $V=k^{n}, \alpha$ ) is a structured space with automorphism group $G=$ $\operatorname{Aut}_{k}(V, \alpha)$. Let $H$ be an algebraic subgroup of $G$, and let $X$ be a primitive generically free $H$-variety. Then $\left(\right.$ RMaps $\left._{H}(X, V), \alpha_{H}^{X}\right)$ is a structured space of type $(V, \alpha)$.

Proof. We may assume without loss of generality that $X$ is irreducible. Indeed, otherwise an irreducible component $X_{1}$ of $X$ is an ( $H, H_{1}$ )-section (see Example 2.10); we can thus replace $X$ by $X_{1}, H$ by $H_{1}$ and then appeal to Lemma $7.6(\mathrm{~b})$.

For irreducible $X$, consider the $H$-compression $f: H \times X \rightarrow X$, where $H$ acts on $H \times X$ by $h^{\prime}(h, x)=\left(h^{\prime} h, x\right)$ and $f(h, x)=h x$. Now apply Lemma 7.6(a) to this compression; the desired conclusion then follows from Lemma 7.6(c).

Proposition 8.6 Let $\left(V=k^{n}, \alpha\right)$ be a structured space with automorphism group $G=\operatorname{Aut}_{k}(V, \alpha)$, and let $F$ be a finitely generated field extension of $k$. Then
(a) a structured space $\left(W=F^{n}, \beta\right)$ is of type $(V, \alpha)$ if and only if there exists a generically free primitive $G$-variety $X$ such that $(W, \beta) \simeq\left(\right.$ RMaps $\left._{G}(X, V), \alpha_{G}^{X}\right)$.
(b) Suppose $X$ and $X^{\prime}$ are generically free primitive $G$-varieties such that $k(X)^{G}=$ $k\left(X^{\prime}\right)^{G}$ and $\left(R M a p s_{G}(X, V), \alpha_{G}^{X}\right) \simeq\left(R M a p s_{G}\left(X^{\prime}, V\right), \alpha_{G}^{X^{\prime}}\right)$ (as structured spaces). Then $X \simeq X^{\prime}$ (as $G$-varieties).

Proof. (a) The "if" assertion follows from Lemma 8.5. To prove the converse, assume ( $W=F^{n}, \beta$ ) is a structured space of type $\left(V=k^{n}, \alpha\right)$. Choose $E / F$ as in Definition 8.1. By Remark 8.3 we may assume that $E$ is finitely generated over $F$ and, hence, over $k$. Thus there exist irreducible varieties $X_{0}$ and $Y_{0}$ such that $E=k\left(Y_{0}\right), F=k\left(X_{0}\right)$, and the field extension $E / F$ is induced by a dominant rational morphism $\phi: Y_{0} \rightarrow X_{0}$.

Let $B=\left\{b_{1}, \ldots, b_{n}\right\}$ be an $F$-basis of $W$. Then $B$ is also an $E$-basis of $V \otimes_{k} E$; see (9). Recall that $(V, \alpha) \otimes_{k} E \simeq\left(\operatorname{RMaps}_{G}(Y, V), \alpha_{G}^{Y}\right)$, where $Y=G \times Y_{0}$; see Lemma 7.6(c). Thus we can view $b_{1}, \ldots, b_{n}$ as $G$-equivariant rational maps $Y \rightarrow V$; together they define a $G$-equivariant rational map $b: Y \rightarrow V^{n}$. Let $Z$ be the closure of $b(Y)$ in $V^{n}$. Note that $Z \cap\left(V^{n}\right)^{\prime} \neq \emptyset$ (see Lemma 7.4) and thus the rational map $\bar{b}: Y_{0} \rightarrow Z_{0}=Z / G$ induced by $b: Y \rightarrow Z$ is given by $\left(\alpha_{I}^{J}\right)_{Z} \mapsto \beta_{I}^{J}(B)$; see Remark 2.4 and Lemma 6.4(b). Since every $\beta_{I}^{J}(B)$ lies in $F, \bar{b}$ factors through $X_{0}$ (via $\phi$ ). Consequently $b$ factors through $X=X_{0} \times_{Z_{0}} Z$. These maps are shown in the following diagram.


Here the vertical arrows represent quotient maps and $c$ (resp. $\bar{c}$ ) is the map $X \rightarrow \rightarrow$ $Z \subset V^{n}\left(\right.$ resp. $\left.X_{0} \rightarrow Z_{0} \subset V^{n} / G\right)$ induced by $b$ (resp. $\left.\bar{b}\right)$. Let $c=\left(c_{1}, \ldots, c_{n}\right)$, where each $c_{i}$ is a $G$-equivariant rational map $X \rightarrow V$ and let $C=\left\{c_{1}, \ldots, c_{n}\right\}$. Since $Z \cap\left(V^{n}\right)^{\prime} \neq \emptyset, C$ is an $F$-basis of $R M a p s_{G}(X, V)$; see Lemma 7.4. By our construction $\left(\alpha_{G}^{X}\right)_{I}^{J}(C)=\beta_{I}^{J}(B)$ for every $I$ and $J$. This means that the structured spaces ( $W, \beta$ ) and $R M a p s_{G}(X, V)$ are isomorphic via the linear map $W \longrightarrow R M a p s_{G}(X, V)$ which takes $b_{i}$ to $c_{i}$.
(b) Denote the structured space $\left(R M a p s_{G}(X, V), \alpha_{G}^{X}\right)$ by $(W, \beta)$. We want to show that $X$ can be uniquely recovered from $(W, \beta)$. Let $C=\left\{c_{1}, \ldots, c_{n}\right\}$ be a basis of $W=R M a p s_{G}(X, V), c=\left(c_{1}, \ldots, c_{n}\right)$ be the rational map $X \rightarrow V^{n}$ defined by $C$, and $Z$ be the closure of $c(X)$ in $V^{n}$. Then by Lemma $7.4, Z \cap\left(V^{n}\right)^{\prime} \neq \emptyset$; thus $c$ induces a rational map $\bar{c}: X_{0} \longrightarrow Z_{0}$, where $Z_{0}=\pi(Z) \simeq Z / G$ and $\pi: V^{n} \longrightarrow V^{n} / G$ is the quotient map. The maps $c$ and $\bar{c}$ are shown in the diagram below.


Here the vertical maps are quotient maps and $\bar{c}$ is given by $\alpha_{I}^{J} \mapsto \beta_{I}^{J}(C)$. (Recall that $k\left(V^{n} / G\right)=k\left(\alpha_{I}^{J}\right)$; see Lemma 6.4(b).) Now note that $X \simeq X_{0} \times_{Z_{0}} Z$ (see Lemma 2.16(b)) and $Z=\pi^{-1}\left(Z_{0}\right)$.

This shows that $X$ can be uniquely recovered from ( $W, \beta$ ). More precisely, suppose

$$
\left(R M \operatorname{aps}_{G}(X, V), \alpha_{G}^{X}\right) \stackrel{\text { def }}{=}(W, \beta) \simeq\left(\operatorname{RMaps}_{G}\left(X^{\prime}, V\right), \alpha_{G}^{X^{\prime}}\right)
$$

via a map which sends the basis $C$ of $(W, \beta)$ to a basis $D$ of ( $\left.\operatorname{RMaps}_{G}\left(X^{\prime}, V\right), \alpha_{G}^{X^{\prime}}\right)$. Denote the $G$-equivariant rational morphism $\left(d_{1}, \ldots, d_{n}\right): X^{\prime} \rightarrow V^{n}$ by $d$. Then the induced map $\bar{d}: X^{\prime} / G=X_{0} \rightarrow V^{n} / G$ of quotient spaces is the same as $\bar{c}$; thus $X^{\prime} \simeq X_{0} \times_{Z_{0}} Z \simeq X$, as claimed .

### 8.2 Structured spaces and essential dimension

Definition 8.7 Let $F$ be a field, $\left(W=F^{n}, \beta\right)$ be a structured space and $F_{0}$ be a subfield of $F$.
(a) We say that ( $W, \beta$ ) is defined over $F_{0}$ if there exists an $F$-basis $B$ of $W$ such that every coefficient $\beta_{I}^{J}(B)$ of $\beta$ with respect to this basis lies in $F_{0}$. Equivalently, $(W, \beta)$ is defined over $F_{0}$ if and only if it is isomorphic to $\left(W_{0}, \beta_{0}\right) \otimes_{F_{0}} F$ for some structured space ( $W_{0}=F_{0}^{n}, \beta_{0}$ ).
(b) $\tau(W, \beta) \stackrel{\text { def }}{=} \min \left\{\operatorname{trdeg}_{k}\left(F_{0}\right)\right\}$, where $(W, \beta)$ is defined over $F_{0}$. Equivalently, $\tau(W, \beta)=\min \left\{\operatorname{trdeg}_{k} k\left(\beta_{I}^{J}(B) \mid I, J\right)\right\}$, as $B$ ranges over all $F$-bases of $W$.

Theorem 8.8 Let $(V, \alpha)$ be a structured space and $G=\operatorname{Aut}_{k}(V, \alpha)$.
(a) If $X$ is a generically free primitive $G$-variety then

$$
\operatorname{ed}(X, G)=\tau\left(R M a p s_{G}(X, V), \alpha_{G}^{X}\right)
$$

(b) $\operatorname{ed}(G)=\max \{\tau(W, \beta)\}$, where the maximum is taken over all structured spaces ( $W, \beta$ ) of type $(V, \alpha)$.

Proof. (a) Denote the structured space ( RMaps $_{G}(X, V), \alpha_{G}^{X}$ ) by $(W, \beta)$.
If $X \rightarrow Y$ is a $G$-compression then by Lemma 7.6(a) the structured space ( $W, \beta$ ) is defined over $k(Y)^{G}$. If we choose $Y$ to be of minimal possible dimension, then $\operatorname{trdeg}_{k} k(Y)^{G}=\operatorname{dim}(Y / G)=\operatorname{ed}(X, G)$. This proves that $\tau(W, \beta) \leq \operatorname{ed}(X, G)$.

To prove the opposite inequality, write $(W, \beta) \simeq(\tilde{W}, \tilde{\beta}) \otimes_{\tilde{F}} \bar{F}$, where $(\tilde{W}, \tilde{\beta})$ is a structured space over $\tilde{F}$ and $\operatorname{trdeg}_{k}(\tilde{F})=\tau(W, \beta)$. Note that since $(W, \beta)$ is a structured space of type ( $V, \alpha$ ), so is $(\tilde{W}, \tilde{\beta})$. Thus by Proposition 8.6(a),

$$
(\tilde{W}, \tilde{\beta}) \simeq\left(\operatorname{RMaps}_{G}(\tilde{X}), \alpha_{G}^{\tilde{X}}\right)
$$

for some primitive generically free $G$-variety $\tilde{X}$ with $k(\tilde{X})^{G}=\tilde{F}$. On the other hand, the extension $F / \tilde{F}$ is given by a rational morphism $X / G \rightarrow \tilde{X} / G$ of algebraic varieties. Using this morphism we can construct the fiber product $Y=X / G \times_{\tilde{X} / G} \tilde{X}$. By Lemma 7.6(a),

$$
\begin{gathered}
\left(\text { RMaps }_{G}(Y, V), \alpha_{G}^{Y}\right) \simeq\left(\text { RMaps }_{G}(\tilde{X}, V), \alpha_{G}^{\tilde{X}}\right) \otimes k(X)^{G} \simeq \\
(\tilde{W}, \tilde{\beta}) \otimes_{\tilde{F}} F=(W, \beta)=\left(\text { RMaps }_{G}(X, V), \alpha_{G}^{X}\right) .
\end{gathered}
$$

Hence, by Proposition 8.6(b), $X$ is birationally isomorphic to $Y=X / G \times{ }_{\tilde{X} / G} \tilde{X}$. Since the natural projection $Y \longrightarrow \tilde{X}$ is a $G$-compression, we conclude that

$$
\operatorname{ed}(X, G)=\operatorname{ed}(Y, G) \leq \operatorname{dim}(\tilde{X} / G)=\operatorname{trdeg}_{k} k(\tilde{X} / G)=\operatorname{trdeg}_{k} \tilde{F}=\tau(W, \beta),
$$

as claimed.
(b) Recall that $\operatorname{ed}(G)=\max \{\operatorname{ed}(X, G)\}$, as $X$ ranges over all generically free primitive $G$-varieties; see Definition 3.5. The desired conclusion now follows from part (a) and Proposition 8.6.

We conclude this section with two simple applications of Theorem 8.8. Other applications will be given in Sections 9-11.

Example 8.9 (a) Suppose $V=k^{n}$ and $\alpha$ be a volume form on $V$, so that $\operatorname{Aut}_{k}(V, \alpha)=$ $\mathrm{SL}_{n}$. Then every volume $\beta$ form on $W=F^{n}$ can be written as $b_{1} \wedge \ldots \wedge b_{n}$ for some basis $b_{1}, \ldots, b_{n}$ of $W$. Thus every structured space ( $W, \beta$ ) of type $(V, \alpha)$ is isomorphic to $(V, \alpha) \otimes_{k} F$, and, consequently, $\tau(W, \alpha)=0$. Applying Theorem 8.8 , we conclude that ed $\left(\mathrm{SL}_{n}\right)=0$; cf. Example 3.9(c) and Theorem 5.4.
(b) (cf. [Se 4, III.1.2(a)]) Suppose $V=k^{2 n}$ and $\alpha \in \Lambda^{2}\left(V^{*}\right)$ is symplectic form on $V$. Then every symplectic form $\beta$ form on $W=F^{n}$ can be written as

$$
\beta=b_{1}^{*} \wedge b_{2}^{*}+\ldots+b_{2 n-1}^{*} \wedge b_{2 n}^{*}
$$

in some basis $b_{1}, \ldots, b_{n}$ of $W$. This means that every structured space ( $W, \beta$ ) of type ( $V, \alpha$ ) is isomorphic to $(V, \alpha) \otimes_{k} F$ and thus $\tau(W, \alpha)=0$. By Theorem 8.8, we conclude that ed $\left(\mathrm{Sp}_{2 n}\right)=0$; cf. Theorem 5.4.

## 9 Projective linear groups

### 9.1 Central simple algebras

Recall that $\mathrm{PGL}_{n}=\operatorname{Aut}_{k}\left(\mathrm{M}_{n}\right)$, where we view the matrix algebra $\mathrm{M}_{n}=\mathrm{M}_{n}(k)$ as a structured space ( $k^{n^{2}}, \alpha$ ); see Definition 6.1(c).

Lemma 9.1 (Wedderburn; see e.g., [Se $2_{2}$, X. 5 Proposition 7] or [KMRT, 1.1]) Let F be a field containing $k$ and let $W=F^{n^{2}}$. Then $(W, \beta)$ is a structured space of type $\mathrm{M}_{n}$ if and only if $\beta \in W^{*} \otimes W^{*} \otimes W$ defines the structure of a central simple $F$-algebra on $W$.

If $A$ is a central simple algebra, we will denote the center of $A$ by $Z(A)$. Division algebras will usually be denoted by $D$. Recall that $\operatorname{UD}(m, n)$ is the division algebra generated by $m$ generic $n \times n$ matrices $X_{1}=\left(x_{i j}^{(1)}\right), \ldots, X_{m}=\left(x_{i j}^{(m)}\right)$. For a more detailed discussion of these algebras we refer the reader to $\left[\mathrm{Pr}_{1}\right.$, Part II] or [ $\left.\mathrm{Pi}, 20.8\right]$.

Lemma $9.2($ a $) \operatorname{ed}\left(\mathrm{PGL}_{n}\right)=\max \{\tau(A)\}$, as $A$ ranges over all central simple algebras of degree $n$ whose center contains $k$.
(b) $\operatorname{ed}\left(\mathrm{PGL}_{n}\right)=\tau(\mathrm{UD}(m, n))$ for any $m \geq 2$.
(c) $\operatorname{ed}\left(\mathrm{PGL}_{n}\right)=\max \{\tau(D)\}$, as $D$ ranges over all division algebras of degree $n$ whose center contains $k$.

Proof. Part (a) follows from Lemma 9.1 and Theorem 8.8(b). To prove part (b), consider the representation of $\mathrm{PGL}_{n}$ on $V_{m}=\left(\mathrm{M}_{n}\right)^{m}$ given by

$$
g\left(a_{1}, \ldots, a_{m}\right)=\left(g a_{1} g^{-1}, \ldots, g a_{m} g^{-1}\right) .
$$

The structured space ( $R M a p s_{\mathrm{PGL}_{n}}\left(V_{m}, V_{1}\right), \alpha_{\mathrm{PGL}_{n}}^{V_{m}}$ ) is the algebra of rational $\mathrm{PGL}_{n}$ equivariant maps $V_{m} \rightarrow V_{1}=M_{n}$; see Example 7.5 . By a theorem of Procesi this algebra is isomorphic to $\mathrm{UD}(m, n)$; see $\left[\mathrm{Pr}_{2}, \mathrm{Thm}\right.$. 2.1]. (The isomorphism identifies $X_{i} \in \mathrm{UD}(m, n)$ with the projection map $\pi_{i}: V_{m}=\left(M_{n}\right)^{m} \rightarrow M_{n}$ given by $\pi_{i}\left(a_{1}, \ldots, a_{m}\right)=a_{i}$ ). Now part (b) follows from Theorem 8.8(a). Part (c) is an immediate consequence of (a) and (b).

Theorem 9.3 ed $\left(\mathrm{PGL}_{n^{r}}\right) \geq 2 r$ for any $n \geq 2$ and $r \geq 1$.
Proof. [Re, Theorem 16.1(b)] shows that $\tau\left(\mathrm{UD}\left(2, n^{r}\right)\right) \geq 2 r$. The desired conclusion now follows from Lemma 9.2(b).

### 9.2 Cyclic algebras

Lemma 9.4 (a) If $D$ is a division algebra of degree $n \geq 2$ then $\tau(D) \geq 2$.
(b) If $D$ is a cyclic division algebra then $\tau(D)=2$.
(c) $\operatorname{ed}\left(\mathrm{PGL}_{n}\right)=2$ if $n=2,3$, and 6 .

Proof. (a) Assume the contrary: $D=D_{0} \otimes_{K_{0}} K$, where $K$ is the center of $D$ and $K_{0}$ is the center of $D_{0}$ and $\operatorname{trdeg}_{k}\left(K_{0}\right) \leq 1$. Then $D_{0}$ cannot be a division algebra by Tsen's theorem (see e.g., [Pi, 19.4]), a contradiction. Another proof of part (a) can be deduced from Corollary 5.7(a) (with $G=\mathrm{PGL}_{n}$ ).
(b) Since $D$ is cyclic, there are elements $x, y \in D$ such that $x^{n}=u \in K, y^{n}=v \in K$, $x y=\zeta y x$ where $\zeta$ is a primitive $n$-th root of unity. (Recall that we are assuming $k$ is algebraically closed and of characteristic 0 ; in particular, $\zeta \in k$.) Examining the structure constants of $D$ in the basis $B=\left\{x^{i} y^{j} \mid i, j=0, \ldots, n-1\right\}$, we see that $D$ is defined over the field $k(u, v)$. Thus $\tau(D) \leq \operatorname{trdeg}_{k}(u, v) \leq 2$. Part (b) follows from this inequality and part (a).
(c) Follows from part (b) and Lemma 9.2(c), since every division algebra of degree 2,3 , or 6 is cyclic; see e.g., $[\mathrm{Pi}, 15.6]$.

### 9.3 Algebras of odd degree

In this section we use a theorem of Rowen [Row] to give an upper bound on $\operatorname{ed}\left(\mathrm{PGL}_{n}\right)$ for odd $n$.

Recall that by a theorem of Wedderburn every central simple algebra $A$ with center $F$ is of the form $A=\mathrm{M}_{d}(D)$, where $D$ is a division algebra with center $F$. The algebra $D$ is unique up to isomorphism (of $F$-algebras); we shall refer to it as the underlying
division algebra of $A$. The degree of $D$ is called the index of $A$ and is denoted by $\operatorname{Ind}(A)$. Two central simple algebras with center $F$ are called Brauer equivalent if they have $F$-isomorphic underlying division algebras. Brauer equivalence classes of central simple algebras form an abelian group under tensor products; this group is called the Brauer group of $F$ and is denoted by $\operatorname{Br}(F)$. For a more detailed discussion of the structure theory of central simple algebras and the Brauer group we refer the reader to [Pi].

Lemma 9.5 Let $D_{1}$ and $D_{2}$ be division algebras with center $F$. Suppose $D_{1}$ and $D_{2}$ generate the same cyclic subgroup of $\operatorname{Br}(F)$. Then $\tau\left(D_{1}\right)=\tau\left(D_{2}\right)$.

Proof. By symmetry it is enough to show

$$
\begin{equation*}
\tau\left(D_{1}\right) \geq \tau\left(D_{2}\right) \tag{10}
\end{equation*}
$$

Note that our assumption forces $D_{1}$ and $D_{2}$ to have the same degree, which we shall denote by $n$; see [Pi, Proposition $13.4\left(\right.$ viii)]. Write $D_{1}=E_{1} \otimes_{K} F$, where $E_{1}$ is a subalgebra of $D_{1}$ of degree $n, K$ is the center of $E_{1}$ and $\operatorname{trdeg}_{k}(K)=\tau\left(D_{1}\right)$. Suppose $D_{1}^{\otimes i}=\mathrm{M}_{n^{i}-1}\left(D_{2}\right)$, where $i$ relatively prime to $n$. We claim that the central simple algebra $E_{1}^{\otimes i}$ has index $n$. Indeed, on the one hand, $\operatorname{Ind}\left(E_{1}^{\otimes i}\right) \leq \operatorname{Ind}\left(E_{1}\right)=n$ and, on the other hand, $\operatorname{Ind}\left(E_{1}^{\otimes i}\right) \geq \operatorname{Ind}\left(D_{1}^{\otimes i}\right)=n$. Thus we can write $E_{1}^{\otimes i}=\mathrm{M}_{n^{i-1}}\left(E_{2}\right)$ for some division algebra $E_{2}$ of degree $n$, with center $K$. We now claim that

$$
\begin{equation*}
D_{2}=E_{2} \otimes_{K} F . \tag{11}
\end{equation*}
$$

Note that (11) immediately implies (10), since it says that $D_{2}$ is defined over $K$. To prove (11), observe that $\mathrm{M}_{n^{i-1}}\left(E_{2}\right) \otimes_{K} F=E_{1}^{\otimes i} \otimes_{K} F=D_{1}^{\otimes i}=\mathrm{M}_{n^{i-1}}\left(D_{2}\right)$. Now (11) follows from the uniqueness part of Wedderburn's theorem; see [Pi, Section 3.5].

Theorem 9.6 Let $n \geq 3$ be an odd integer. Then $\operatorname{ed}\left(\mathrm{PGL}_{n}\right) \leq n+\frac{(n-1)(n-2)}{2}$.
Proof. Let $D_{1}=\mathrm{UD}(2, n)$ and $D_{2}$ be the underlying division algebra of $D_{1}^{\otimes 2}$. By a theorem of Rowen [Row], $\tau\left(D_{2}\right) \leq n+\frac{(n-1)(n-2)}{2}$. Combining Lemma 9.5 with Lemma 9.2(b), we obtain ed $\left(\mathrm{PGL}_{n}\right)=\tau\left(D_{1}\right)=\tau\left(D_{2}\right) \leq n+\frac{(n-1)(n-2)}{2}$, as claimed.

### 9.4 Algebras of composite degree

Lemma 9.7 Suppose $n_{1}$ and $n_{2}$ are relatively prime positive integers. For $i=1,2$ let $D_{i}$ be a division algebra of degree $n_{i}$ with center $F$ and let $D=D_{1} \otimes_{F} D_{2}$. Then
(a) $\tau\left(D_{i}\right) \leq \tau(D)$ for $i=1,2$.
(b) $\tau(D) \leq \tau\left(D_{1}\right)+\tau\left(D_{2}\right)$.

Proof. (a) Let $E$ be a division subalgebra of $D$ of degree $n_{1} n_{2}$ such that $\operatorname{trdeg}_{k} Z(E)$ is as small as possible, i.e., is equal to $\tau(D)$. Then we can write $E$ as $E_{1} \otimes_{Z(E)} E_{2}$, where $\operatorname{deg}\left(E_{i}\right)=n_{i}$ for $i=1,2$. Thus $D=E \otimes_{Z(E)} F=\left(E_{1} \otimes_{Z(E)} F\right) \otimes_{Z(D)}\left(E_{2} \otimes_{Z(E)} F\right)$. Since the decomposition $D=D_{1} \otimes_{Z(D)} D_{2}$ is unique, we have $D_{1} \simeq E_{1} \otimes_{Z(E)} F$ and $D_{2} \simeq E_{2} \otimes_{Z(E)} F$. This means that $D_{1}$ and $D_{2}$ are both defined over $Z(E)$; consequently, $\tau\left(D_{i}\right) \leq \tau(D)$ for $i=1,2$.
(b) Suppose the algebra $D_{i}$ is defined over $F_{i} \subset F$ with $\operatorname{trdeg}_{k}\left(F_{i}\right)=\tau\left(D_{i}\right)$; here $i=1,2$. In other words, $D_{i}$ has a subalgebra $E_{i}$ of degree $n_{i}$ whose center is $F_{i}$. Let $E=E_{1} E_{2}$ be the division subalgebra generated by $E_{1}$ and $E_{2}$ inside $D$. Then $D=E \otimes_{Z(E)} F$ and therefore $D$ is defined over $Z(E)$. Note that $Z(E)$ is the subfield of $F$ generated (over $k$ ) by $F_{1}$ and $F_{2}$. Hence, $\operatorname{trdeg}_{k} Z(E) \leq \operatorname{trdeg}_{k} F_{1}+\operatorname{trdeg}{ }_{k} F_{2} \leq$ $\tau\left(D_{1}\right)+\tau\left(D_{2}\right)$, as claimed.

Proposition 9.8 Let $n_{1}, n_{2} \geq 2$ be relatively prime integers. Then
(a) $\mathrm{ed}\left(\mathrm{PGL}_{n_{i}}\right) \leq \mathrm{ed}\left(\mathrm{PGL}_{n_{1} n_{2}}\right)$ for $i=1,2$.
(b) $\operatorname{ed}\left(\mathrm{PGL}_{n_{1} n_{2}}\right) \leq \operatorname{ed}\left(\mathrm{PGL}_{n_{1}}\right)+\operatorname{ed}\left(\mathrm{PGL}_{n_{2}}\right)$.

Proof. (a) It is enough to show that $\operatorname{ed}\left(\mathrm{PGL}_{n_{1}}\right) \leq \operatorname{ed}\left(\mathrm{PGL}_{n_{1} n_{2}}\right)$. Let $D_{1}=$ $\mathrm{UD}\left(Z\left(D_{1}\right)\right), D_{2}=\mathrm{M}_{n_{2}}\left(Z\left(D_{1}\right)\right)$, and $D=D_{1} \otimes_{Z\left(D_{1}\right)} D_{2}=M_{n_{2}}\left(D_{1}\right)$. The desired inequality now follows from Lemmas 9.2 and 9.7 (a).
(b) The algebra $D=\mathrm{UD}\left(2, n_{1} n_{2}\right)$ can be written as $D_{1} \otimes_{Z(D)} D_{2}$, where $D_{1}$ and $D_{2}$ are subalgebras of $D$ of degrees $n_{1}$ and $n_{2}$ respectively; see, e.g., [Pi, 14.4]. Applying Lemmas 9.2 and 9.7 (b), we obtain

$$
\mathrm{ed}\left(\mathrm{PGL}_{n_{1} n_{2}}\right)=\tau(D) \leq \tau\left(D_{1}\right)+\tau\left(D_{2}\right) \leq \mathrm{ed}\left(\mathrm{PGL}_{n_{1}}\right)+\mathrm{ed}\left(\mathrm{PGL}_{n_{2}}\right)
$$

as claimed.
Remark 9.9 Proposition $9.8(\mathrm{~b})$ can often be used to strengthen the upper bounds on ed $\left(\mathrm{PGL}_{n}\right)$ given by Theorem 4.5, Lemma $9.4(\mathrm{c})$ and Theorem 9.6. For example, since $\operatorname{ed}\left(\mathrm{PGL}_{5}\right) \leq 11$ (by Theorem 9.6 ) and $\operatorname{ed}\left(\mathrm{PGL}_{6}\right)=2$ (by Lemma $9.4(\mathrm{c})$ ), we have

$$
\mathrm{ed}\left(\mathrm{PGL}_{30}\right) \leq \mathrm{ed}\left(\mathrm{PGL}_{6}\right)+\operatorname{ed}\left(\mathrm{PGL}_{5}\right) \leq 13
$$

On the other hand, a direct application of Theorem 4.5 with $n=30$ gives a much weaker bound ed $\left(\mathrm{PGL}_{30}\right) \leq 840$.

More generally, Proposition 9.8 , in combination with Theorem 9.6, allows us to strengthen the bound given by Theorem 4.5 for every $n$ which is not a power of 2 .

## 10 Orthogonal groups

The main results of the next two sections (namely, Theorems 10.3, 10.4, 11.2, and 11.5) were communicated to us by J.-P. Serre. The proofs we present here are based on TsenLang theory; they are somewhat different from Serre's original proofs, which will be given in Section 12.

### 10.1 Pfister forms

Let $F$ be a field. As usual, we shall denote the quadratic form $\sum_{i=1}^{n} a_{i} x_{i}^{2}$ on $F^{n}$ by $\left.<a_{1}, \ldots, a_{n}\right\rangle$; the $n$-fold Pfister form $<1, a_{1}>\otimes \ldots \otimes<1, a_{n}>$ will be denoted by $\ll a_{1}, \ldots, a_{n} \gg$.

Lemma 10.1 (see, e.g., [Pf, Ch. 8, Ex. 1.2.6]) Suppose $a_{1}, \ldots, a_{n}$ are independent indeterminates over $k$ and $F=k\left(a_{1}, \ldots, a_{n}\right)$. Then the Pfister form $\left.\ll a_{1}, \ldots, a_{n}\right\rangle>$ is anisotropic over $F$.

Remark 10.2 Pfister forms naturally arise in the following context. Suppose $\beta$ is a symmetric bilinear form on $W$. Recall that $\beta$ gives rise to the symmetric bilinear $\Lambda^{i}(\beta)$ on $\Lambda^{i}(W)$ given by

$$
\begin{equation*}
\Lambda^{i}(\beta)\left(w_{1} \otimes \ldots \otimes w_{i}, w_{1}^{\prime} \otimes \ldots \otimes w_{i}^{\prime}\right)=\sum_{\sigma \in S_{n}}(-1)^{\sigma} \beta\left(w_{1}, w_{\sigma(1)}^{\prime}\right) \ldots \beta\left(w_{i}, w_{\sigma(i)}^{\prime}\right) . \tag{12}
\end{equation*}
$$

Then $\Lambda(\beta) \stackrel{\text { def }}{=} \Lambda^{1}(\beta) \oplus \ldots \oplus \Lambda^{n}(\beta)$ is a symmetric bilinear form on $\Lambda(W)=\Lambda^{1}(W) \oplus$ $\ldots \oplus \Lambda^{n}(W)$. An easy direct computation shows that if $\beta(x, x)=<a_{1}, \ldots, a_{n}>$ then $<1>\oplus \Lambda(\beta)(x, x)=\ll a_{1}, \ldots, a_{n} \gg$.

### 10.2 The essential dimension of $O_{n}$

Theorem 10.3 ed $\left(O_{n}\right)=n$ for every $n \geq 1$.
Proof. We proved the inequality $\operatorname{ed}\left(O_{n}\right) \leq n$ in Example 3.10(a) (and then again in Example 4.2); thus we only need to show ed $\left(O_{n}\right) \geq n$. Suppose $\alpha$ is a non-degenerate symmetric bilinear form on $V=k^{n}$, so that $\operatorname{Aut}_{k}(V, \alpha)=O_{n}$. A structured space ( $W=F^{m}, \beta$ ) is of type ( $V, \alpha$ ) if and only if $\beta$ is a non-degenerate symmetric bilinear form on $W$. Thus in view of Theorem 8.8 it suffices to construct a field extension $F / k$ and a symmetric bilinear form $\beta$ on $W=F^{n}$ such that $\tau(W, \beta) \geq n$.

We now proceed to construct $F$ and $\beta$. Let $a_{1}, \ldots, a_{n}$ be algebraically independent indeterminates over $k, F=k\left(a_{1}, \ldots, a_{n}\right), W=F^{n}, e_{1}, \ldots, e_{n}$ be a basis of $W$ and $\beta=a_{1}\left(e_{1}^{*}\right)^{2}+\ldots+a_{n}\left(e_{n}^{*}\right)^{2}$ be a symmetric bilinear form on $W$. We claim that $\tau(W, \beta) \geq n$ (or, equivalently, $=n$, since $\operatorname{trdeg}_{k}(F)=n$ ). Indeed, extend $\beta$ to a symmetric bilinear form $\Lambda(\beta)$ to $\Lambda(V)$. Recall that by Remark 10.2

$$
<1>\oplus \Lambda(\beta)(x, x)=\ll a_{1}, \ldots, a_{n} \gg ;
$$

hence, $\langle 1\rangle \oplus \Lambda(\beta)(x, x)$ is an anisotropic quadratic form over $F$; see Lemma 10.1. On the other hand, suppose $\beta=\beta_{0} \otimes_{F_{0}} F$ where $\beta_{0}$ is a symmetric bilinear form on $W_{0}=\left(F_{0}\right)^{n}$ and $\operatorname{trdeg}_{k}\left(F_{0}\right)=n-1$. By the Tsen-Lang theorem the quadratic form $<1>\oplus \Lambda\left(\beta_{0}\right)(x, x)$ defined on the $2^{n}$-dimensional space $F_{0} \oplus \Lambda\left(W_{0}\right)$ is isotropic; see e.g., [Pf, Sect. 5.1]. Since, $\Lambda(\beta)=\Lambda\left(\beta_{0}\right) \otimes_{F_{0}} F$, we conclude that the quadratic form $<1>\oplus \Lambda(\beta)(x, x)$ is isotropic as well, a contradiction.

### 10.3 The essential dimension of $S O_{n}$

Theorem $10.4 \operatorname{ed}\left(S O_{n}\right)=\left\{\begin{array}{l}0 \text { if } n=2 \\ n-1, \text { if } n \neq 2\end{array}\right.$
Note that Theorem 10.4 says, in particular, that $\operatorname{ed}\left(S O_{n}\right) \neq 1$ for any $n$. This is consistent with Corollary 5.7(b).
Proof. Recall that we proved the inequality

$$
\begin{equation*}
\operatorname{ed}\left(S O_{n}\right) \leq n-1 \tag{13}
\end{equation*}
$$

for every $n \geq 1$ in Example 3.10(b) (and then again in Example 4.2). If $n$ is odd then $O_{n} \simeq S O_{n} \times \mathbb{Z} / 2 \mathbb{Z}$ and thus $\operatorname{ed}\left(O_{n}\right) \leq \operatorname{ed}\left(S O_{n}\right)+\operatorname{ed}(\mathbb{Z} / 2 \mathbb{Z})$. Since ed $\left(O_{n}\right)=n$ (see Theorem 10.3) and $\operatorname{ed}(\mathbb{Z} / 2 \mathbb{Z})=1$, this means $n-1 \leq \operatorname{ed}\left(S O_{n}\right)$. Combining this inequality with (13), we conclude that $\operatorname{ed}\left(S O_{n}\right)=n-1$ for every odd integer $n \geq 1$.

We now assume that $n$ is even. If $n=2$ then $S O_{2}$ is isomorphic to the 1 -dimensional torus $k^{*}$ and thus ed $\left(\mathrm{SO}_{2}\right)=0$; see Example 3.9(a).

It remains to show that $\operatorname{ed}\left(S O_{n}\right) \geq n-1$ for $n=2 m$, where $m \geq 2$. Let $V=k^{n}$, let $\epsilon_{1}, \ldots, e_{n}$ be a $k$-basis of $V, \alpha_{1}=e_{1}^{*} \wedge \ldots \wedge e_{n}^{*}, \alpha_{2}=\left(e_{1}^{*}\right)^{2}+\ldots+\left(e_{n}^{*}\right)^{2}$, and $\alpha=\alpha_{1}+\alpha_{2}$, as in Example 8.4(b). Recall that $\operatorname{Aut}_{k}(V, \alpha)=S O_{n}$; see Example 6.3(e). By Theorem 8.8 it is enough to construct a structured space ( $W=F^{n}, \beta$ ) of type ( $V, \alpha$ ) such that $\tau(W, \beta) \geq n-1$.

We now proceed to construct $F, W$, and $\beta$. Let $F=k\left(a_{1}, \ldots, a_{n-1}\right)$, where $a_{1}, \ldots, a_{n-1}$ are independent indeterminates over $k, W=V \otimes_{k} F$ and $\beta=\beta_{1}+\beta_{2}$, where $\beta_{1}=e_{1}^{*} \wedge \ldots \wedge e_{n}^{*}=\alpha_{1} \otimes_{k} F \in \Lambda^{n}\left(W^{*}\right)$ and

$$
\beta_{2}=a_{1}\left(e_{1}^{*}\right)^{2}+\ldots+a_{n-1}\left(e_{n-1}^{*}\right)^{2}+\left(a_{1} \ldots a_{n-1}\right)^{-1}\left(e_{n}^{*}\right)^{2} \in S^{2}\left(W^{*}\right) .
$$

Since $\operatorname{det}\left(\beta_{2}\right)=1,(W, \beta)$ is a structured space of type ( $V, \alpha$ ); see Example 8.4(b).
It is enough to show that bilinear form $\beta_{2}$ cannot be defined over a field $F_{0}$ with $\operatorname{trdeg}_{k}\left(F_{0}\right)<n-1$. Indeed, this will immediately imply that $\beta$ cannot be defined over $F_{0}$, thus proving $\tau(W, \beta) \geq n-1$. Note that $\beta_{2}$ is equivalent to the bilinear form

$$
\gamma=a_{1}\left(e_{1}^{*}\right)^{2}+\ldots+a_{n-1}\left(e_{n-1}^{*}\right)^{2}+\left(a_{1} \ldots a_{n-1}\right)\left(e_{n}^{*}\right)^{2} \in S^{2}\left(V^{*}\right) ;
$$

thus it is enough to show that $\gamma$ cannot be defined over any field $F_{0}$ with $\operatorname{trdeg}_{k}\left(F_{0}\right)<$ $n-1$.

We now argue as in the proof of Theorem 10.3. Let $\Lambda^{i}(\gamma)$ be the symmetric bilinear form on $\Lambda^{i}(W)$ defined in Remark 10.2. An easy computation shows that

$$
\begin{align*}
\Lambda^{i}(\gamma)(x, x)=\oplus_{1 \leq j_{1}<\ldots<j_{i} \leq n-1} & <a_{j_{1}}, \ldots, a_{j_{i}}>  \tag{14}\\
& \oplus_{1 \leq h_{1}<\ldots<h_{n-1-i} \leq n-1}<a_{h_{1}}, \ldots, a_{h_{n-1-i}}>
\end{align*} .
$$

Consider the form $\lambda(\gamma)=\Lambda^{1}(\gamma) \oplus \ldots \Lambda^{m-1}(\gamma)$ on $\Lambda^{1}(W) \oplus \ldots \Lambda^{m-1}(W)$. (Recall that we are assuming $n=2 m$ and $m \geq 2$.) Adding up the forms (14), we see that
$<1>\oplus \lambda(\gamma)(x, x)$ is equivalent to $\ll a_{1}, \ldots, a_{n-1} \gg$ and therefore, is anisotropic over $F$; see Lemma 10.1.

On the other hand, suppose $\gamma=\delta \otimes_{F_{0}} F$, where $\delta$ is a symmetric bilinear form on $\left(F_{0}\right)^{n}$ and $\operatorname{trdeg}_{k}\left(F_{0}\right)<n-1$. Then $\lambda(\gamma)=\lambda(\delta) \otimes_{F_{0}} F$, where

$$
\lambda(\delta)=\Lambda^{1}(\delta) \oplus \ldots \Lambda^{m-1}(\delta)
$$

The quadratic form $<1>\oplus \lambda(\delta)(x, x)$ is then defined on $F_{0} \oplus \Lambda^{1}\left(W_{0}\right) \oplus \ldots \oplus$ $\Lambda^{m-1}\left(W_{0}\right) \simeq F_{0}^{2^{n-1}}$. Thus the Tsen-Lang theorem says that this form is isotropic over $F_{0}$; see [Pf, Sect. 5.1]. Hence, so is $\langle 1\rangle \oplus \lambda(\gamma)(x, x)$, a contradiction. This completes the proof of the theorem.

## 11 Some exceptional groups

### 11.1 Octonion algebras

Let $F$ be a field. Recall that for any $0 \neq a, b, c \in F$, the octonion (or Cayley-Dickson) algebra $\mathbf{O}_{F}(a, b, c)$ is defined as follows. Let

$$
Q=(a, b)_{2}=F\{i, j\} /\left(i^{2}=a, j^{2}=b, j i=-i j\right)
$$

be a 4 -dimensional (associative) quaternion algebra over $F$. This algebra is equipped with an involution $x \rightarrow \bar{x}$ given by

$$
\begin{equation*}
\overline{x_{0}+x_{1} i+x_{2} j+x_{3} i j}=x_{0}-x_{1} i-x_{2} j-x_{3} i j . \tag{15}
\end{equation*}
$$

(Here $x_{0}, \ldots, x_{3} \in F$.) Now $\mathbf{O}_{F}(a, b, c) \stackrel{\text { def }}{=} Q+Q l$ is an 8 -dimensional $F$-algebra with (non-associative) multiplication given by $(x+y l)(z+w l)=(x z+c \bar{w} y)+(w x+y \bar{z}) l$. The involution (15) extends from $Q$ to $\mathbf{O}_{F}(a, b, c)$ via $\overline{x+y l}=\bar{x}-y l$. The algebra $\mathbf{O}_{F}(a, b, c)$ is also equipped with $F$-valued trace and norm functions given by $t(x)=x+\bar{x}$ and $n(x)=x \bar{x}=\bar{x} x$; moreover, $x^{2}-\operatorname{tr}(x) x+n(x)=0$ for any $x \in \mathbf{O}_{F}(a, b, c)$. Using this identity, it is easy to show that every automorphism of $\mathbf{O}_{F}(a, b, c)$ preserves the trace and the norm. For a more detailed description of octonion algebras we refer the reader to [J, I.5], [Sc, III.4] or [SSSZ, 2.2].

If $F=k$, we shall write $\mathbf{O}$ for $\mathbf{O}_{F}(1,1,1)$.

### 11.2 The essential dimension of $G_{2}$

Recall that $\operatorname{Aut}_{k}(\mathbf{O})$ is the 14 -dimensional exceptional group $G_{2}$.
Lemma 11.1 Suppose $F$ is a field containing $k$. Then for any $0 \neq a, b, c \in F$, we have

$$
\mathbf{O}_{F}(a, b, c) \otimes_{F} E=\mathbf{O} \otimes_{k} E,
$$

where $E=F(\sqrt{a}, \sqrt{b}, \sqrt{c})$. In particular, $\mathbf{O}_{F}(a, b, c)$ is an algebra of type $\mathbf{O}$.

Proof. The $E$-linear map $\mathbf{O}_{F}(a, b, c) \otimes_{k} E \longrightarrow \mathbf{O} \otimes_{F} E$ given by

$$
\begin{aligned}
& i \mapsto \sqrt{a} i \\
& j \mapsto \sqrt{b} j \\
& l \mapsto \sqrt{c} l
\end{aligned}
$$

is an isomorphism of $E$-algebras.
Theorem 11.2 ed $\left(G_{2}\right)=3$.
Proof. We first prove that $\operatorname{ed}\left(G_{2}\right) \geq 3$. Let $F=k\left(a_{1}, a_{2}, a_{3}\right)$, where $a_{1}, a_{2}$ and $a_{3}$ are algebraically independent indeterminates over $k$. In view of Lemma 11.1 and Theorem $8.8(\mathrm{~b})$ it is sufficient to show that $\tau\left(\mathbf{O}_{F}\left(a_{1}, a_{2}, a_{3}\right)\right) \geq 3$.

Expressing the trace form $\operatorname{tr}\left(x^{2}\right)$ in the basis $\left\{\left(i^{\epsilon_{1}} j^{\epsilon_{2}}\right) l^{\epsilon_{3}} \mid \epsilon_{1}, \epsilon_{2}, \epsilon_{3}=0,1\right\}$, we see that this form is equivalent to the 3 -fold Pfister form $\ll a_{1}, a_{2}, a_{3} \gg$ and thus is anisotropic over $F$; see Lemma 10.1. Thus by the Tsen-Lang theorem the form $\operatorname{tr}\left(x^{2}\right)$ (and, hence, the $F$-algebra $\left.\mathbf{O}_{F}\left(a_{1}, a_{2}, a_{3}\right)\right)$ cannot be defined over a field $F_{0}$ with $\operatorname{trdeg}_{k}\left(F_{0}\right) \leq 2$. This proves that $\tau\left(\mathbf{O}_{F}\left(a_{1}, a_{2}, a_{3}\right)\right)$ and therefore $\operatorname{ed}\left(G_{2}\right) \geq 3$.

Our proof of the opposite inequality will rely on an argument analogous to the one we used in Example 3.10. Let $V=k^{8}$ be the underlying vector space of the split octonion algebra $\mathbf{O}$. Since $\mathbf{O}$ is generated by three elements, the natural linear representation of $G_{2}$ on $V^{3}=k^{24}$ is generically free. Let

$$
Y=\left\{\left(x_{1}, x_{2}, x_{3}\right) \mid \operatorname{tr}\left(x_{r}\right)=\operatorname{tr}\left(x_{r} x_{s}\right)=\operatorname{tr}\left(\left(x_{1} x_{2}\right) x_{3}\right)=0, \operatorname{tr}\left(x_{r}^{2}\right) \neq 0,1 \leq r<s \leq 3\right\} .
$$

Then $Y$ is a (locally closed) $G_{2}$-invariant subvariety of $V^{3}$. Moreover, $Y$ irreducible and $\operatorname{dim}(Y)=17$; this can be proved by considering the projection $Y \longrightarrow V^{2}$ to the first two components. Since $Y$ contains a triple of generators of $\mathbf{O}$ (e.g., $(i, j, l)$ ), the action of $G_{2}$ on $Y$ is generically free.

Now let $f: V \rightarrow Y$ be the $G_{2}$-equivariant rational map given by $f\left(x_{1}, x_{2}, x_{3}\right)=$ $\left(y_{1}, y_{2}, y_{3}\right)$, where

$$
\begin{aligned}
& y_{1}=P\left(x_{1} ; 1\right) \\
& y_{2}=P\left(x_{2} ; 1, y_{1}\right) \\
& y_{3}=P\left(x_{3} ; 1, y_{1}, y_{2}, y_{1} y_{2}\right) ;
\end{aligned}
$$

here $P\left(v ; v_{1}, \ldots, v_{m}\right)$ is defined by equation (4) of Example 3.10 with $q(x, y)=\operatorname{tr}(x y)$. Since $f_{\mid Y}=i d_{\mid Y}, f$ is a compression. Thus ed $\left(G_{2}\right) \leq \operatorname{dim}(Y)-\operatorname{dim}\left(G_{2}\right)=17-14=3$, as claimed.

Remark 11.3 (a) By [Se ${ }_{3}$, Sect 8.2] $G_{2}$ is 2 -special; see Definition 5.5. In view of Proposition 5.6(b) this gives an alternative proof of the inequality ed $\left(G_{2}\right) \geq 3$. Moreover, by Proposition 5.6(a), ed $\left(X, G_{2}\right)=0$ or 3 for every primitive (or, equivalently, irreducible) generically free $G_{2}$-variety $X$.

Remark 11.4 The converse of Lemma 11.1 is also true, namely
$\left(W=F^{8}, \beta\right)$ is structured space of type $\mathbf{O}$ if and only if $(W, \beta)=\mathbf{O}_{F}(a, b, c)$ for some $a, b, c \in F$. This gives an alternative proof of the inequality $\operatorname{ed}\left(G_{2}\right) \leq 3$.

The first assertion is a consequence of a theorem of Zorn [Sc, III.3.17] (see also [SSSZ, 7.3] for a more general result of Kleinfeld). To prove the second assertion, note that $\mathbf{O}_{F}(a, b, c)$ is defined over the field $k(a, b, c)$, which has transcendence degree $\leq 3$ over $k$. Thus $\tau\left(\mathbf{O}_{F}(a, b, c)\right) \leq 3$ for every $F$ and every $a, b, c \in F^{*}$. By Theorem 8.8(b) this implies ed $\left(G_{2}\right) \leq 3$.

### 11.3 The Albert algebra

Recall that the (split) Albert algebra $\mathbf{A}=H_{3}(\mathbf{O})$ is the Jordan algebra of $3 \times 3$ hermitian matrices over the octonion algebra $\mathbf{O}$. Elements of $\mathbf{A}$ are of the form

$$
X=\left(\begin{array}{lll}
a & x & y  \tag{16}\\
\bar{x} & b & z \\
\bar{y} & \bar{z} & c
\end{array}\right),
$$

where $a, b, c \in k$ and $x, y, z \in \mathbf{O}$, and multiplication is given by $X \cdot Y=1 / 2(X Y+Y X)$, where $X Y$ is the usual matrix product of $X$ and $Y$. Every element of $X$ of A satisfies a "Cayley-Hamilton identity" of the form

$$
X^{3}-\operatorname{tr}(X) X^{2}+\frac{\operatorname{tr}(X)^{2}-\operatorname{tr}\left(X^{2}\right)}{2} X-n(X)=0,
$$

where $\operatorname{tr}(X)=a+b+c \in k$ is the trace of $X$ and $n(X)$ is the cubic norm of $X$; see [J, p. 233]. Both the trace and the norm are preserved under automorphisms of $\mathbf{A}$.

Let $\epsilon_{\alpha \beta}$ be the $(\alpha, \beta)$ matrix unit in $\mathrm{M}_{3}(k)$. We shall denote the element $x e_{\alpha \beta}+\bar{x} e_{\alpha \beta}$ by $[x]_{\alpha \beta}$. Then the 27 elements $\epsilon_{11}, e_{22}, \epsilon_{33}$ and $\left[\left(i^{\epsilon_{1}} j^{\epsilon_{2}}\right) l^{\epsilon_{3}}\right]_{\alpha \beta}$ form a $k$-basis of $\mathbf{A}$; here $\epsilon_{1}, \epsilon_{2}, \epsilon_{3}=0,1$ and $1 \leq \alpha<\beta \leq 3$.

### 11.4 The essential dimension of $F_{4}$

Recall that $\operatorname{Aut}_{k}(\mathbf{A})$ is the exceptional group $F_{4}$.
Theorem $11.5 \operatorname{ed}\left(F_{4}\right) \geq 5$.
Proof. By Theorem 8.8 it is sufficient to construct a field $F$ containing $k$ and an $F$-algebra $B$ of type A with $\tau(B) \geq 5$. We now proceed to construct such an algebra. Let $s_{1}, s_{2}, s_{3}, t_{1}, t_{2}$ be five independent variables over $k$,

$$
\begin{equation*}
t_{3} \stackrel{\text { def }}{=} t_{1}^{-1} t_{2}^{-1}, \tag{17}
\end{equation*}
$$

$S_{\alpha}=s_{\alpha}^{2}$ and $T_{\alpha}=t_{\alpha}^{2}$ for $\alpha=1,2,3$. Let $E$ and $F$ be the fields given by $F=$ $k\left(S_{1}, S_{2}, S_{3}, T_{1}, T_{2}\right)$ and $E=k\left(s_{1}, s_{2}, s_{3}, t_{1}, t_{2}\right)$, and let $B$ be the $F$-subspace of $\mathbf{A} \otimes_{k} E$ spanned by the 27 elements

$$
\begin{equation*}
\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \text { and } b_{\alpha}^{\epsilon_{1}, \epsilon_{2}, \epsilon_{3}}=t_{\alpha} s_{1}{ }_{1}^{\epsilon_{1}} s_{2}^{\epsilon_{2}} s_{3}{ }^{\epsilon_{3}}\left[\left(i^{\epsilon_{1}} j^{\epsilon_{2}}\right) t^{\epsilon_{3}}\right]_{\beta \gamma}, \tag{18}
\end{equation*}
$$

where $\epsilon_{1}, \epsilon_{2}, \epsilon_{3}=0,1,\{\alpha, \beta, \gamma\}=\{1,2,3\}$ and $\beta<\gamma$. We claim that $B$ is an $F$ subalgebra of of $\mathbf{A} \otimes_{k} E$. Since the elements (18) form an $E$-basis of $\mathbf{A} \otimes_{k} E$, it is sufficient to check that all structure constants of $\mathbf{A} \otimes_{k} E$ in this basis lie in $F$. Indeed,

$$
\begin{aligned}
& \epsilon_{\alpha \alpha} \cdot \epsilon_{\alpha \alpha}=\epsilon_{\alpha \alpha} \\
& \epsilon_{\alpha \alpha} \cdot e_{\beta \beta}=0 \\
& \epsilon_{\alpha \alpha} \cdot b_{\alpha}^{\epsilon_{2}, \epsilon_{2}, \epsilon_{3}}=0 \\
& e_{\alpha \alpha} \cdot b_{\alpha}^{\epsilon_{1}, \epsilon_{2}, \epsilon_{3}}=\frac{1}{2} b_{\beta}^{\epsilon_{1}, \epsilon_{2}, \epsilon_{3}} \\
& b_{\alpha}^{\epsilon_{1}, \epsilon_{2}, \epsilon_{3}} \cdot b_{\alpha}^{\delta_{1}, \delta_{2}, \delta_{3}}=\left\{\begin{array}{l} 
\pm T_{\alpha} S_{1}^{\epsilon_{1}} S_{2}^{\epsilon_{2}} S_{3}^{\epsilon_{3}}\left(\epsilon_{\beta \beta}+\epsilon_{\gamma \gamma}\right) \text { if }\left(\epsilon_{1}, \epsilon_{2}, \epsilon_{3}\right)=\left(\delta_{1}, \delta_{2}, \delta_{3}\right) \\
0 \text { otherwise } .
\end{array}\right. \\
& b_{\alpha}^{\epsilon_{1}, \epsilon_{2}, \epsilon_{3}} \cdot b_{\beta}^{\delta_{1}, \delta_{2}, \delta_{3}}= \pm T_{\gamma}^{-1} S_{1}^{\epsilon_{1} \delta_{1}} S_{2}^{\epsilon_{2} \delta_{2}} S_{3}^{\epsilon_{3} \delta_{3}} b_{\gamma}^{\epsilon_{1}+\delta_{1}, \epsilon_{2}+\delta_{2}, \epsilon_{3}+\delta_{3}} .
\end{aligned}
$$

(cf. [Sc, p. 105]). Here we always assume that $\{\alpha, \beta, \gamma\}=\{1,2,3\}$; in the last formula the subscripts $\epsilon_{n}$ and $\delta_{n}$ are added and multiplied modulo 2 . All structure constants appearing in this table are, indeed, elements of $F$. We have therefore verified that $B$ is a 27 -dimensional $F$-algebra. Since the elements (18) form an $E$-basis of $\mathbf{A} \otimes_{k} E$, we have $B \otimes_{F} E \simeq \mathbf{A} \otimes_{k} E$. Thus $B$ is an algebra of type $A$.

It remains to prove that $\tau(B) \geq 5$ (or, equivalently, $=5$, since $\operatorname{trdeg}_{k}(F)=5$ ). We shall do so by examining the trace form of $B$ and showing that it cannot be defined over a field $F_{0}$ with $\operatorname{trdeg}_{k}\left(F_{0}\right) \leq 4$ (and, consequently, $B$ cannot be defined over such a field).

Our multiplicaton table shows that the 27 basis elements (18) are mutually orthogonal with respect to the trace form. Moreover, taking the trace of the square of each element and remembering that (i) $\sqrt{2}, \sqrt{-1} \in k \subset F$ by our assumption on $k$ and (ii) $T_{3}=T_{1}^{-1} T_{2}^{-1}$ by (17), we obtain

$$
\begin{equation*}
\left.\operatorname{tr}\left(x^{2}\right) \simeq<1,1\right\rangle \oplus \phi, \tag{19}
\end{equation*}
$$

where
$\phi=<1>\oplus T_{1} \ll S_{1}, S_{2}, S_{3} \gg \oplus T_{2} \ll S_{1}, S_{2}, S_{3} \gg \oplus T_{1} T_{2} \ll S_{1}, S_{2}, S_{3} \gg$.
Note that

$$
\begin{equation*}
\phi \oplus<S_{1}, S_{2}, S_{3}, S_{1} S_{2}, S_{1} S_{3}, S_{2} S_{3}, S_{1} S_{2} S_{3}>\simeq \ll S_{1}, S_{2}, S_{3}, T_{1}, T_{2} \gg . \tag{20}
\end{equation*}
$$

Since the Pfister form $\ll S_{1}, S_{2}, S_{3}, T_{1}, T_{2} \gg$ is anisotropic over $F$ (see Lemma 10.1), so is $\phi$. Thus (19) gives the Witt decomposition of the quadratic form $\operatorname{tr}\left(x^{2}\right)$ over $F=k\left(S_{1}, S_{2}, S_{3}, T_{1}, T_{2}\right)$; the anisotropic part of this form equals $\phi$ and has dimension
25. On the other hand, if $B$ were defined over a field $F_{0}$ with $\operatorname{trdeg}_{k}\left(F_{0}\right) \leq 4$ then $\operatorname{tr}\left(x^{2}\right)$ would also be defined over $F_{0}$ and, consequently, by the Tsen-Lang theorem, its anisotropic part would have dimension $\leq 2^{4}=16$. This contradiction proves that $\tau(B) \geq 5$ and thus ed $\left(F_{4}\right) \geq 5$, as claimed.

### 11.5 The essential dimension of $E_{6}$

Recall that the simply connected exceptional group $E_{6}$ is isomorphic to $\operatorname{Aut}_{k}(V, \alpha)$, where $V=k^{27}$ is the underlying vector space of the Albert algebra $\mathbf{A}$ and $\alpha \in S^{3}\left(V^{*}\right)$ is the symmetric trilinear form associated to the cubic norm in $\mathbf{A}$.

Proposition 11.6 ed $\left(E_{6}\right) \geq 3$.
Proof. Let $F=k(a, b, c)$. By a theorem of Albert there exists an exceptional Jordan division $F$-algebra $B$; see [J, IX. 12 Thm. 21]. Let $W=F^{27}$ be the underlying vector space of $B$ and let $\beta \in S^{3}\left(V^{*}\right)$ be the symmetric trilinear form associated to the cubic norm in $B$. Since $B$ is an algebra of type $\mathbf{A},(W, \beta)$ is a structured space of type $(V, \alpha)$. (Here $V=k^{27}$ and $\alpha \in S^{3}\left(V^{*}\right)$ are as above.)

By Theorem 8.8 it suffices to show that $\tau(W, \beta) \geq 3$. Assume the contrary: $(W, \beta)$ is defined over a field $F_{0}$ with $\operatorname{trdeg}_{k}\left(F_{0}\right) \leq 2$. Then the cubic norm $n_{B}(x)=\beta(x, x, x)$ on $B$ can also be defined over $F_{0}$. Note that $n_{B}$ is a cubic form in 27 variables; thus by the Tsen-Lang theorem $n_{B}$ is isotropic, i.e. $n_{B}(x)=0$ for some $0 \neq x \in B$. This contradicts our assumption that $B$ is a division algebra; see [J, VI. 3 Cor. 3]. We therefore conclude that $\tau(W, \beta) \geq 3$, as claimed.

Proposition $11.7 \operatorname{ed}\left(E_{6}\right) \leq \operatorname{ed}\left(F_{4}\right)+1$.
Proof. $E_{6}$ is by definition a subgroup of $\mathrm{GL}_{27}=\mathrm{GL}(\mathbf{A})$; thus its linear representation on $V=\mathrm{A}^{27}$ is generically free. (Using [J, VI. 7 Thm .7 ], one can prove that the action of $E_{6}$ on $\mathbf{A}^{n}$ is generically free for every $n \geq 4$; however, we will not need this fact here.)

Let $S=\left\{\left(x_{1}, \ldots, x_{27}\right) \mid x_{1} \in k 1_{\mathbf{A}}\right\} \subset V$. Then the proof of $[I$, Thm. 1$]$ shows that $S$ is an $\left(E_{6}, H\right)$-section of $V$ (see Definition 2.9), where $H \simeq F_{4} \times \mathbb{Z}_{3}$ is the subgroup of $E_{6}$ generated by $F_{4}$ and multiplication by cube roots of 1 . Note that $S$ is a linear generically free $H$-variety. Thus by Lemma 4.1, ed $\left(E_{6}\right)=\mathrm{ed}\left(V, E_{6}\right) \leq \operatorname{ed}(S, H)=\operatorname{ed}(H)$ and by Lemma 3.8, $\operatorname{ed}(H)=\operatorname{ed}\left(F_{4} \times \mathbb{Z}_{3}\right) \leq \operatorname{ed}\left(F_{4}\right)+\operatorname{ed}\left(\mathbb{Z}_{3}\right)=\operatorname{ed}\left(F_{4}\right)+1$, as claimed.

## 12 Cohomological invariants

In this section we discuss the relationship between the essential dimension of an algebraic group $G$ and cohomological invariants associated to $G$. This material is based entirely on results communicated to us by J.-P. Serre.

### 12.1 Preliminaries

We begin by recalling several definitions and results related to Galois cohomology and cohomological invariants. For details we refer the reader to [Se $\left.{ }_{4}\right]$, [KMRT] and [B].

Let $H^{i}(F)=H^{i}\left(\Gamma_{F}, \mathbb{Z} / 2 Z\right)$, where $\Gamma_{F}=\operatorname{Gal}(\bar{F} / F), \bar{F}$ is the algebraic closure of $F$, and $\mathbb{Z} / 2 \mathbb{Z}$ is viewed as a trivial $\Gamma_{F}$-module.

By Kummer theory the first cohomology group $H^{1}(F)$ is canonically isomorphic to $F^{*} /\left(F^{*}\right)^{2}$; as usual, we will denote the class of $a \in F^{*}$ modulo $\left(F^{*}\right)^{2}$ by $(a)$.
Lemma 12.1 (see e.g., $\left[B R_{2}\right.$, Lemma 8.2]) Let $a_{1}, \ldots, a_{n}$ be independent variables over $k$ and let $F=k\left(a_{1}, \ldots, a_{n}\right)$. Then the cup product $\left(a_{1}\right) \ldots\left(a_{n}\right)$ is non-zero in $H^{n}(F)$.

Suppose $F$ is a field, $0 \neq a_{1}, \ldots, a_{n} \in F$ and $\phi=<a_{1}, \ldots, a_{n}>$ is a quadratic form. Then $r$-th Stiefel-Whitney class $w_{r}(\phi) \in H^{r}(F)$ is given by

$$
\begin{equation*}
w_{r}(\phi)=\sum_{1 \leq i_{1}<\ldots<i_{r} \leq n}\left(a_{i_{1}}\right)\left(a_{i_{2}}\right) \ldots\left(a_{i_{r}}\right) . \tag{21}
\end{equation*}
$$

By a theorem of Delzant $w_{r}(\phi)$ depends only on the equivalence class of $\phi$ and not on the specific presentation of $\phi$ as a sum of squares; see [De].

We shall now consider the following categories:
Fields - category of finitely generated field extensions of $k$,
Sets ${ }^{\prime}$ - category of pointed sets, i.e., sets with a marked element,
AGrps - category of abelian groups,
and the following covariant functors:
$H^{i}(\cdot)$, from Fields to AGrps.
$H^{1}(\cdot, G)$, from Fields to Sets ${ }^{\prime}$. Here $G$ is a (fixed) algebraic group.
$G-\operatorname{Var}(\cdot)$, from Fields to Sets'. Here again, $G$ is a fixed algebraic group. If $F \in$ Fields then $G-\operatorname{Var}(F)$ is the set of birational equivalence classes of generically free primitive $G$-varieties $X$ such that $k(X / G) \simeq F$ (as field extensions of $k$ ). Note that for each $F \in$ Fields the marked element in $G-\operatorname{Var}(F)$ is the $G$-variety $X_{0} \times G$; here the variety $X_{0}$ is chosen so that $k\left(X_{0}\right)=F$. An inclusion $F \hookrightarrow F^{\prime}$ of fields induces the map $G-\operatorname{Var}(F) \longrightarrow G-\operatorname{Var}\left(F^{\prime}\right)$ of pointed sets given by $X \mapsto X_{0}^{\prime} \times_{X / G} X$, where $k\left(X_{0}^{\prime}\right)=F^{\prime}$.

Spaces $_{(V, \alpha)}$, from Fields to Sets'. Here $\left(V=k^{n}, \alpha\right)$ is a (fixed) structured space. If $F \in$ Fields then $\operatorname{Spaces}_{(V, \alpha)}(F)$ is defined as the set of all structured spaces ( $W=$ $F^{n}, \beta$ ) of type ( $V, \alpha$ ); see Definition 8.1. The marked element in this set is $(V, \alpha) \otimes_{k} F$. An inclusion $F \hookrightarrow F^{\prime}$ of fields induces a morphism $\operatorname{Spaces}_{(V, \alpha)}(F) \longrightarrow \operatorname{Spaces}_{(V, \alpha)}\left(F^{\prime}\right)$ given by $(W, \beta) \mapsto(W, \beta) \otimes_{F} F^{\prime}$.

Definition 12.2 ([Se $\left.e_{3}, 6.1\right],[B, 4.1]$, [KMRT, 31.B]) Let $G$ be an algebraic group. A cohomological invariant of dimension $n$ associated to $G$ is a morphism of functors $f: H^{1}(\cdot, G) \longrightarrow H^{n}(\cdot)$.

### 12.2 Cohomological invariants and essential dimension

Lemma 12.3 (a) The functors $H^{1}(\cdot, G)$ and $G-\operatorname{Var}(\cdot)$ are isomorphic.
(b) If $(V, \alpha)$ is a structured space and $G=\operatorname{Aut}_{k}(V, \alpha)$ then the functors $H^{1}(\cdot, G)$, $G-\operatorname{Var}(\cdot)$ and $\operatorname{Spaces}_{(V, \alpha)}$ are isomorphic.

Proof. Part (a) is proved in [ $\left.\mathrm{Se}_{4}, \mathrm{I} .5 .2\right]$; see also [Po, 1.3] and Remark 2.17. To prove part (b), define

$$
f_{F}: G-\operatorname{Var}(\cdot)(F) \longrightarrow \operatorname{Spaces}_{(V, \alpha)}(F)
$$

by $f_{F}: X \mapsto\left(\right.$ RMaps $\left._{G}(X, V), \alpha_{G}^{X}\right)$ for every $F \in$ Fields. Now Lemma 7.6(c) shows that $f_{F}$ is a map of pointed sets for each $F$, Lemma 7.6(a) shows that the maps $f_{F}$ collectively define a morphism of functors $f$ from $G-\operatorname{Var}(\cdot)$ to $\operatorname{Spaces}_{(V, \alpha)}$, and Proposition 8.6 shows that $f$ is an isomorphism.

In view of the above lemma we can view a cohomological invariant of degree $n$ associated to $G$ as a morphism of functors $G-\operatorname{Var}(\cdot, G) \longrightarrow H^{n}(\cdot)$ or, if $G$ is of the form $\operatorname{Aut}(V, \alpha)$, as a morphism of functors $\operatorname{Spaces}_{(V, \alpha)} \longrightarrow H^{n}(\cdot)$.

Proposition 12.4 Let $f: G-V a r \longrightarrow H^{n}(\cdot)$ be a cohomological invariant associated to $G$ and let $X$ be a primitive generically free $G$-variety such that $f_{F}(X) \neq 0$. (Here $F=k(X)^{G}$.) Then
(a) $\operatorname{ed}(X, G) \geq n$.
(b) Let $V$ be a generically free linear representation of $G$ and let $K=k(V)^{G}$. Then $f_{K}(V) \neq 0$.

Proof. (a) Assume the contrary. Then there exists a $G$-compressions $X \rightarrow Y$ such that $\operatorname{dim}(Y / G) \leq n-1$, i.e.,

$$
\begin{equation*}
\operatorname{trdeg}_{k}(L) \leq n-1, \text { where } L=k(Y / G)^{G}=k(Y)^{G} . \tag{22}
\end{equation*}
$$

Since $X$ is birationally isomorphic to $X / G \times_{Y / G} Y$ (see Lemma 2.16(b)) and $f$ is a morphism of functors, we conclude that $f_{L}(Y) \xrightarrow{\text { res }} f_{F}(X)$ where res is the restriction map $H^{n}(L, C) \longrightarrow H^{n}(F, C)$. However, in view of (22), the cohomological dimension of $L$ is $\leq n-1$ (see [ $\mathrm{Se}_{4}$, II.4.2]) and thus $f_{L}(Y) \in H^{n}(L, C)=(0)$. Consequently, $f_{F}(X)=(0)$ in $H^{n}(F, C)$, a contradiction.
(b) By Corollary 2.20 , there exists a compression $X \times k^{d} \rightarrow V$, where $d=\operatorname{dim}(V)$. Then $f_{F^{\prime}}\left(X \times k^{d}\right)$ is a homomorphic image of $f_{K}(V)$; here $F^{\prime}=k\left(X \times k^{d}\right)^{G}$; hence it is enough to show that $f_{F^{\prime}}\left(X \times k^{d}\right) \neq 0$. Note that $f_{F^{\prime}}\left(X \times k^{d}\right)$ is the image of $f_{F}(X)$ under the restriction map $H^{n}(F) \rightarrow H^{n}\left(F^{\prime}\right)$. Since $F^{\prime}$ is a purely transcendental extension of $F$, this restriction map is injective; see $\left[\mathrm{Se}_{4}\right.$, Remark 1, p. 85]. Thus $f_{F^{\prime}}\left(X \times k^{d}\right) \neq 0$, as claimed.

Note that we can associate a trivial cohomological invariant to any group $G$ by setting $f_{F}$ to be the zero map for every $F \in$ Fields. We shall denote this invariant by $f=0$ and refer to it as the zero invariant.

Corollary 12.5 If there exists a non-zero cohomological invariant of dimension $n$ associated to $G$ then $\operatorname{ed}(G) \geq n$.

Proof. Immediate from Proposition 12.4; see Definition 3.5.

### 12.3 Examples

We now apply Corollary 12.5 to give alternative proofs of several lower bounds on $\operatorname{ed}(G)$, which we previously established by other methods. Every group $G$ we shall consider in Examples 12.6-12.10 is of the form $G=\operatorname{Aut}_{k}(V, \alpha)$; thus we will interpret a cohomological invariant of degree $n$ associated to $G$ as a morphism of functors $f$ : Spaces $_{(V, \alpha)}(\cdot) \longrightarrow H^{n}(\cdot)$; see Lemma 12.3.

Example 12.6 $G=O_{n}$. Let $V=k^{n}$ and $\alpha$ be a non-degenerate symmetric bilinear form on $V$ so that $\operatorname{Aut}_{k}(V, \alpha)=O_{n}$. Recall that for any $F \in$ Fields, $\left(W=F^{n}, \beta\right)$ is of type $F$ if and only if $\beta$ is a non-degenerate symmetric bilinear form on $W$. For each $i \geq 1$ we can now define a cohomological invariant $f_{i}: \operatorname{Spaces}_{(V, \alpha)}(\cdot) \longrightarrow H^{i}(\cdot)$ of degree $i$ associated to $O_{n}$ by $\left(f_{i}\right)_{F}(W, \beta)=w_{i}(\bar{\beta})$, where $\bar{\beta}$ is the quadratic form such that $\bar{\beta}(x)=\beta(x, x)$ and $w_{i}(\bar{\beta})$ is the Stiefel-Whitney class of $\bar{\beta}$.

It is easy to see that $f_{i}=0$ for all $i \geq n+1$. We claim that $f_{i} \neq 0$ for any $i=1, \ldots, n$. Indeed, suppose $F=k\left(a_{1}, \ldots, a_{i}\right)$ and $\bar{\beta}=\left\langle a_{1}, \ldots, a_{i}, 1, \ldots, 1\right\rangle$, where $a_{1}, \ldots, a_{i}$ are independent variables over $k$. Then $w_{i}(\bar{\beta})=\left(a_{1}\right) \cdot \ldots \cdot\left(a_{i}\right) \neq 0$ in $H^{i}(F)$; see Lemma 12.1. Applying Proposition 12.4 to $f_{n}$, we obtain an alternative proof the inequality $\operatorname{ed}\left(O_{n}\right) \geq n$ (cf. Theorem 10.3).

Example 12.7 $G=S O_{n}$. Let $V=k^{n}$ and let $\alpha=\alpha_{1}+\alpha_{2}$, where $\alpha_{1}$ is a volume form and $\alpha_{2}$ is a non-degenerate symmetric bilinear form on $V$, as in the beginning of Section 10.3. Recall that $\operatorname{Aut}_{k}(V, \alpha)=S O_{n}$ (see Example 6.3(e)) and that for any $F \in$ Fields, $\operatorname{Spaces}_{(V, \alpha)}(F)$ can be identified with the set of isomorphism classes of symmetric bilinear forms on $F^{n}$ of determinant 1 (see Example 8.4(b)). For every $i \geq 1$ we can define a cohomological invariant $f_{i}: \operatorname{Spaces}_{(V, \alpha)}(\cdot) \longrightarrow H^{i}(\cdot)$ of degree $i$ associated to $S O_{n}$ by $\left(f_{i}\right)_{F}\left(\beta_{2}\right)=w_{i}\left(\overline{\beta_{2}}\right)$, where $w_{i}$ is the $i$ th Stiefel-Whitney class, $\beta_{2}$ is a symmetric bilinear form of determinant 1 on $F^{n}$, and $\overline{\beta_{2}}(x)=\beta_{2}(x, x)$. Note that

$$
\overline{\beta_{2}} \simeq<a_{1}, \ldots, a_{n-1},\left(a_{1} \ldots a_{n-1}\right)^{-1}>\simeq<a_{1}, \ldots, a_{n-1}, a_{1} \ldots a_{n-1}>
$$

for some $a_{1}, \ldots, a_{n} \in F$. It is easy to see that $f_{i}=0$ for all $i \geq n$. If $n$ is odd then $w_{n-1}(\bar{\beta})=\left(a_{1}\right) \ldots\left(a_{n-1}\right)$, which is non-zero if $a_{1}, \ldots, a_{n}$ are independent variables over $k$ and $F=k\left(a_{1}, \ldots, a_{n}\right)$; see Lemma 12.1. This gives an alternative proof of the inequality ed $\left(S O_{n}\right) \geq n-1$ for $n$ odd; cf. Theorem 10.4.

If $n$ is even then $f_{n-1}=0$. Taking $a_{1}, \ldots, a_{n-2}$ to be independent variables over $k$ and $a_{n-1}=1$ and applying Lemma 12.1 , we see that $f_{n-2} \neq 0$. This proves the inequality $\operatorname{ed}\left(S O_{n}\right) \geq n-2$ for $n$ even. To recover the stronger inequality

$$
\begin{equation*}
\operatorname{ed}\left(S O_{n}\right) \geq n-1 \tag{23}
\end{equation*}
$$

of Theorem 10.4 for even integers $n \geq 4$, we consider the cohomological invariant

$$
g_{n-1}: \operatorname{Spaces}_{(V, \alpha)}(\cdot) \longrightarrow H^{n-1}(\cdot)
$$

given by $\left(a_{1}\right)\left(\underline{a_{2}}\right) \ldots\left(a_{n-1}\right)$; see $\left[\mathrm{Se}_{4}\right.$, Remark 2, p. 188]. Here $a_{1}, \ldots, a_{n-1}$ are the coefficients of $\beta_{2}$, as above. Taking $a_{1}, \ldots, a_{n-1}$ to be independent variables over k and applying Lemma 12.1 once again, we conclude that $g_{n-1} \neq 0$. This gives an alternative proof of the inequality (23).

Note that this argument is not as different from the proof of Theorem 10.4 as it may seem. The key step here is proving that $g_{n-1}$ is well-defined. One way of doing it involves passing to a Pfister form on $\oplus_{i=1}^{n / 2-1} \Lambda^{i}\left(k^{n}\right)$, in a manner similar to the argument we used in Section 10.3.

Example $12.8 G=S_{n}$. Let $A$ be the (split) etale $k$-algebra $k \oplus \ldots \oplus k$, so that $\operatorname{Aut}_{k}(A)=S_{n}$; see Example $6.3(\mathrm{f})$. Recall that $\left(W=F^{n}, \beta\right)$ is a structured space of type $A$ if and only if $\beta$ defines the structure of an $n$-dimensional etale $F$-algebra on $W$; see Example $8.4(\mathrm{c})$. Thus a cohomological invariant associated to $S_{n}$ is a cohomological invariant of etale algebras. These invariants were completely described by Serre; see [B, Section 4]. In particular, for each $i=1, \ldots, n$, one can define a cohomological invariant $f_{i}: \operatorname{Spaces}_{A}(\cdot) \rightarrow H^{i}(\cdot)$ by $\left(f_{i}\right)_{F}(B)=w_{i}\left(\operatorname{tr}_{B}\left(x^{2}\right)\right)$, where $B$ is an $n$-dimensional etale $F$-algebra, $\operatorname{tr}_{B}\left(x^{2}\right)$ is the trace form on $B$, and $w_{i}$ is the $i$-th Stiefel-Whitney class. Then $f_{i}=0$ for any $i>[n / 2]$ (see [B, 4.6]) but $f_{i} \neq 0$ for $i=[n / 2]$ (see [ $\mathrm{BR}_{2}$, Theorem 8.4]). Consequently, by Proposition 12.4 , ed $\left(S_{n}\right) \geq[n / 2]$; cf. [ $\mathrm{BR}_{1}$, Thm. 6.5].

Example $12.9 G=G_{2}$. A non-zero cohomological invariant $f_{3}$ of dimension 3 associated to $G_{2}$ is constructed in [ $\mathrm{Se}_{3}$, Sect. 8]. By Proposition 12.4 this construction gives an alternative proof of the inequality $\operatorname{ed}\left(G_{2}\right) \geq 3$; cf. Theorem 11.2 and Remark 11.3.

For the reader's convenience, we briefly recall the definition of $f_{3}$. Recall that $G_{2}=\operatorname{Aut}_{k}(\mathbf{O})$, where $\mathbf{O}$ is the split octonion algebra over $k$. Moreover, for $F \in$ Fields the structured space ( $W=F^{n}, \beta$ ) is of type $\mathbf{O}$ if and only if $(W, \beta)=\mathbf{O}_{F}(a, b, c)$ for some $a, b, c \in F$; see Remark 11.4. The trace form of $\mathbf{O}_{F}(a, b, c)$ is isomorphic to a 3fold Pfister form $\ll a, b, c \gg$. We now define $f_{3}$ as the Arason invariant of this form, i.e., $f_{F}: \operatorname{Spaces}_{\mathbf{O}}(F) \longrightarrow H^{3}(F)$ is given by $\mathbf{O}_{F}(a, b, c) \longrightarrow(a)(b)(c)$. This invariant is non-zero because $(a)(b)(c) \neq 0$ in $H^{3}(F)$ if $a, b, c$ are algebraically independent over $k$ and $F=k(a, b, c)$; see Lemma 12.1.

Example $12.10 G=F_{4}$. Non-zero cohomological invariants $f_{3}$ and $f_{5}$, of dimension, respectively 3 and 5 , associated to $F_{4}$ are constructed in [ $\mathrm{Se}_{3}$ ]; we briefly recall their definition below. The existence of $f_{5}$ gives an alternative proof of the inequality $\operatorname{ed}\left(F_{4}\right) \geq 5 ; c f$. Theorem 11.5.

Recall that $F_{4}=\operatorname{Aut}_{k}(\mathbf{A})$, where $\mathbf{A}$ is the split Albert algebra over $k$ and that structured spaces of type $\mathbf{A}$ are precisely 27 -dimensional exceptional simple Jordan
algebras; see Example 8.4(d). If $B$ is a $F$-algebra of type $\mathbf{A}$ than the trace form of $B$ satisfies the identity

$$
\operatorname{tr}_{B}\left(x^{2}\right) \oplus \ll a_{1}, a_{2}, a_{3} \gg \simeq<1,1,1>\oplus \ll a_{1}, a_{2}, a_{3}, b_{1}, b_{2} \gg
$$

for some $a_{1}, a_{2}, a_{3}, b_{1}, b_{2} \in F$, where the Pfister forms $\ll a_{1}, a_{2}, a_{3} \gg$ and $\ll$ $a_{1}, a_{2}, a_{3}, b_{1}, b_{2} \gg$ are uniquely defined by $B$; see [ $\mathrm{Se}_{3}$, Theorem 10]. (The proof of this assertion for reduced Jordan Algebras is, in fact, similar to our computation in Section 11.4.) Then $\left(f_{3}\right)_{F}(B) \stackrel{\text { def }}{=}\left(a_{1}\right)\left(a_{2}\right)\left(a_{3}\right) \in H^{3}(F)$ and $\left(f_{5}\right)_{F}(B) \stackrel{\text { def }}{=}$ $\left(a_{1}\right)\left(a_{2}\right)\left(a_{3}\right)\left(b_{1}\right)\left(b_{2}\right) \in H^{5}(F)$. To see that these invariants are non-zero, let $F=$ $k\left(S_{1}, S_{2}, S_{3}, T_{1}, T_{2}\right)$ and $B$ be as in the proof of Theorem 11.5. Then in view of (19) and (20), we have $\left(f_{3}\right)_{F}=\left(S_{1}\right)\left(S_{2}\right)\left(S_{3}\right) \neq 0$ in $H^{3}(F)$ and $\left(f_{5}\right)_{F}=\left(S_{1}\right)\left(S_{2}\right)\left(S_{3}\right)\left(T_{1}\right)\left(T_{2}\right) \neq$ 0 in $H^{5}(F)$; see Lemma 12.1. Thus $f_{3}, f_{5} \neq 0$, as claimed.

### 12.4 The Rost invariant and simply connected semisimple groups

The Rost invariant $H^{1}(\cdot, G) \longrightarrow H^{3}(\cdot)$ is defined in [KMRT, 31B]. This invariant is non-zero for every simply connected exceptional group $G$; see [KMRT, (31.40) and (31.47)]. For $G=G_{2}$ and $F_{4}$, this invariant is the invariant $f_{3}$ of (respectively) Examples 12.9 and 12.10. In the case $G=E_{6}$, the non-vanishing of the Rost invariant gives an alternative proof of the inequality $\operatorname{ed}\left(E_{6}\right) \geq 3$; cf. Proposition 11.6. For $G=E_{7}$ and $E_{8}$, we obtain the following result.

Proposition 12.11 (a) $\operatorname{ed}\left(E_{7}\right) \geq 3$, (b) $\operatorname{ed}\left(E_{8}\right) \geq 3$.
The next theorem may be viewed as a weak form of Serre's Conjecture II [ $\mathrm{Se}_{4}$, III.3, p. 139]. The conjecture says (in the case of function fields over $k$ ) that every simply connected semisimple group $G$ is 2-special; cf. Definition 5.5 and Remark 5.9. This is equivalent to the assertion that $\operatorname{ed}(X, G)=0$ or $\geq 3$ for every irreducible generically free $G$-variety $X$; cf. Proposition 5.6. On the other hand, Theorem 12.12 below says that ed $(V, G)=0$ or $\geq 3$, where $V$ is a vector space with a generically free linear action of $G$.

Theorem 12.12 Let $G$ be a simply connected semisimple group. Then either $G$ is special (and thus ed $(G)=0$ ) or $\operatorname{ed}(G) \geq 3$.

Proof. We may assume without loss of generality that $G$ is simple and simply connected. Indeed, if $\operatorname{ed}\left(G_{i}\right)=0$ for every simple factor $G_{i}$ of $G$ then $\operatorname{ed}(G)=0$ (see Lemma 3.8) and if $\operatorname{ed}\left(G_{i}\right) \geq 3$ for one of the factors $G_{i}$ then $\operatorname{ed}(G) \geq 3$ (see Lemma 3.7).

If $G=G_{2}, G=F_{4}$ or $G$ is of classical type then $G$ is known to be 2 -special (see Definition 5.5 and Remark 5.9) and thus the desired conclusion follows from Proposition 5.6. If $G=E_{6}$ then $\operatorname{ed}(G) \geq 3$ by Proposition 11.6. If $G=E_{7}$ or $E_{8}$ then $\operatorname{ed}(G) \geq 3$ by Proposition 12.11.

### 12.5 Mod p invariants

Definition 12.2 and Proposition 12.4 remain valid if $H^{i}(\cdot)$ is interpreted as $H^{i}(\cdot, C)$, where $C$ is a cyclic group of prime order (not necessarily $\mathbb{Z} / 2 \mathbb{Z}$ ). Moreover, Proposition 12.4(b) can be strengthened as follows.
Proposition 12.13 Suppose there exists a non-zero cohomological invariant $f: G-$ $\operatorname{Var}(\cdot) \longrightarrow H^{n}(\cdot, C)$. Let $V$ be a generically free linear representation of $G, X$ be a primitive generically free $G$-variety, and $X \rightarrow V$ be a $G$-compression which is generically $m: 1$. (In other words, $\left[k(X)^{G}: k(V)^{G}\right]=m$.) If $m$ is relatively prime to the order of $C$ then $\operatorname{ed}(X, G) \geq n$.
Proof. Since $m=\left[k(X)^{G}: k(V)^{G}\right]$ is relatively prime to $|C|$, the restriction map $H^{n}\left(k(V)^{G}, C\right) \longrightarrow H^{n}\left(k(X)^{G}, C\right)$ is injective; see [Se 4 , Sect. 2.4]. The desired conclusion now follows from Proposition 12.4(b).

Example 12.14 Applying Proposition 12.13 to the cohomological invariants we discussed above (all defined with $C=\mathbb{Z} / 2 \mathbb{Z}$ ), we arrive at the following result.

Suppose $V$ is a generically free linear representation of $G$ and, $X \rightarrow V$ is an $m: 1$-compression, where $m$ is an odd integer. Then
(a) $\operatorname{ed}(X, G)=n$, if $G=O_{n}$,
(b) $\operatorname{ed}(X, G)=n-1$, if $G=S O_{n}$ with $n \geq 3$,
(c) $\operatorname{ed}(X, G) \geq[n / 2]$ if $G=S_{n}$ (cf. [BR 2 , Thm. 7.1]),
(d) $\operatorname{ed}(X, G)=3$ if $G=G_{2}$,
(e) $\operatorname{ed}(X, G) \geq 5$ if $G=F_{4}$, and
(f) $\operatorname{ed}(X, G) \geq 3$ if $G=E_{6}, E_{7}$ or $E_{8}$.

Example 12.15 Let $V$ be a generically free linear representation of the exceptional group $F_{4}, X$ be a generically free $F_{4}$-variety, and $X \rightarrow V$ be an $F_{4}$-compression, which is generically $m: 1$, where $m$ is not divisible by 3 . Then $\operatorname{ed}\left(X, F_{4}\right) \geq 3$.

This follows from the non-vanishing of the Serre-Rost invariant

$$
H^{1}\left(\cdot, F_{4}\right) \simeq \operatorname{Spaces}_{\mathbf{A}}(\cdot) \longrightarrow H^{3}(\cdot, \mathbb{Z} / 3 \mathbb{Z}) ;
$$

see $\left[\mathrm{Se}_{3}\right.$, Sect. 9.3] and $\left[\mathrm{Rost}_{1}\right]$. (Here $\mathbf{A}$ is the Albert algebra and $\operatorname{Spaces}_{\mathbf{A}}(F)$ is the set of 27 -dimensional exceptional Jordan algebras over $F$, as in Examples 8.4(d) and 12.10.)

Remark 12.16 We do not know whether the inequality $\operatorname{ed}\left(P G L_{n^{r}}\right) \geq 2 r$ of Theorem 9.3 can be proved by the methods of this section. However, if $n=p$ is a prime then the above bound is "stable under prime to $p$-extensions" in the following sense. Let $V$ be a generically free linear representation of $\mathrm{PGL}_{p^{r}}$ and $X$ be a generically free
 tively prime to $p$, then $\operatorname{ed}\left(X, \mathrm{PGL}_{p^{r}}\right) \geq 2 r$. Equivalently, if $D$ is a prime-to- $p$ extension of a universal division algebra $\mathrm{UD}\left(m, p^{r}\right)$ then $\tau(D) \geq 2 r$; see [Re, Theorem 16.1(b)].

## Appendix

Since the time that this paper was first submitted for publication, a number of new results on essential dimension have been obtained by this and other authors. We summarize some of them in the table below.

| Group | EssentialDimension | Proof |
| :---: | :---: | :---: |
| $P O_{n}$ | $\geq n-1$ | [RY] |
|  | $=4$, if $n=7^{(*)}, 10$ | [ $\mathrm{Rost}_{2}$ ] |
|  | $=5$, if $n=8,9,11$ | [ $\mathrm{Rost}_{2}$ ] |
| $\operatorname{Spin}_{n}$ | $=6$, if $n=12,13$ | [ $\mathrm{Rost}_{2}$ ] |
|  | $=7$, if $n=14$ | [ $\mathrm{Rost}_{2}$ ] |
|  | ( $\geq$ [ $n / 2]+1$ if $n \equiv 0, \pm 1 \bmod 8$ | [RY] |
| $F_{4}$ | $=5$ | $\left[\mathrm{Ko}_{2}\right]$ |
| $E_{6}$ simply conn. | $\{\geq 4$ | $[\mathrm{RY}]^{(* *)}$ |
|  | $\left\{\begin{array}{l}\geq \\ \leq\end{array}\right.$ | $\left[\mathrm{Ko}_{2}\right]$ |
| $E_{7}$ simply conn. | $\{\geq 7$ | [RY] |
|  | $\left\{\begin{array}{l}\text { ¢ }\end{array}\right.$ | $\left[\mathrm{Ko}_{2}\right]$ |
| $E_{7}$ adjoint | $\geq 8$ | [RY] |
| $E_{8}$ | $\geq 9$ | [RY] |

${ }^{(*)}$ For $n=2, \ldots, 6$, the group $\operatorname{Spin}_{n}$ is special and, hence, has essential dimension 0 ; see Proposition 5.3.
${ }^{(* *)}$ Independent proofs of the inequality $\operatorname{ed}\left(E_{6}\right) \geq 4$ were communicated to us by R. S. Garibaldi and M. Rost.

## References

[B] E. Bayer-Fluckiger, Galois cohomology and the trace form, Jahresber. Deutsch. Math.-Verein 96(1994), no. 2, 35-55.
[BP] E. Bayer-Fluckiger, R. Parimala, Galois cohomology of classical groups over fields of cohomological dimension $\leq 2$, Invent. Math. 122 (1995), 195-229.
[BK] F. Bogomolov, P. Katsylo, Rationality of certain quotient varieties, Mat. Sbornik, 126(168) (1985), no. 4, 584-589.
[ $\left.\mathrm{BR}_{1}\right] \quad \mathrm{J}$. Buhler, Z. Reichstein, On the essential dimension of a finite group, Compositio Math. 106 (1997), 159-179.
[ $\left.\mathrm{BR}_{2}\right]$ J. Buhler, Z. Reichstein, On Tschirnhaus transformations, in "Number Theory", Proceedings of a conference held at Penn. State University, edited by S. Ahlgren, G. Andrews and K. Ono, Kluwer Acad. Publishers, 127-142, 1999. Preprint available at http://www.orst.edu/ reichstz/pub.html.
[De] A. Delzant, Définition des classes de Stiefel-Whitney d'un module quadratique sur un corps de caractéristique différente de 2, C. R. Acad. Sci. Paris 255 (1962), 1366-1368.
[Do] I. V. Dolgachev, Rationality of fields of invariants, in Algebraic Geometry, Bowdoin 1985, Proceedings of Symposia in Pure Mathematics, vol. 46, part 2, AMS, 1987, 3-16.
[Gi] Ph. Gille, Cohomologie galoisienne des groupes quasi-déployés sur des corps de dimension cohomologique $\leq 2$, preprint, 1998.
[G] A. Grothendieck, Torsion homologique et sections rationnelles, Exposé 5, Séminaire C. Chevalley, Anneaux de Chow et applications, 2nd année, IHP, 1958.
[H] J. E. Humphreys, Linear Algebraic Groups, Graduate Texts in Mathematics, Springer-Verlag, 1975.
[I] A. V. Iltyakov, On rational invariants of the group $E_{6}$, Proc. Amer. Math. Soc. 124, vol. 124, no. 12 (1996), 3637-3640.
[J] N. Jacobson, Structure and Representations of Jordan Algebras, AMS Colloq. Publ. 39, Amer. Math. Soc., Providence, Rhode Island, 1968.
[Ka ${ }_{1}$ ] P. I. Katsylo, Rationality of the orbit spaces of irreducible representations of the group SL $_{2}$, Izv. Acad. Nauk SSSR Ser. Mat. 47 (1983), 26-36.
English translation: Math. USSR-Izvestya, 22 (1984), no. 1, 23-32.
$\left[\mathrm{Ka}_{2}\right]$ P. I. Katsylo, Rationality of the module variety of mathematical instantons with $c_{2}=$ 5 , in Lie Groups, their discrete subgroups, and invariant theory, Advances in Soviet Math 8 (1992), American Math. Soc., Providence, RI., 65-68.
[KMRT] M.-A.Knus, A. Merkurjev, M. Rost, J.-P. Tignol, The Book of Involutions, AMS Colloquium Publications, vol. 44.
[ $\mathrm{Ko}_{1}$ ] V. E. Kordonsky, Essential dimensions of algebraic groups (in Russian), preprint.
[ $\left.\mathrm{Ko}_{2}\right]_{\text {V. E. Kordonsky, On the essential dimension and Serre's Conjecture II for excep- }}$ tional groups, (in Russian), preprint.
[Pf] A. Pfister, Quadratic Forms with Applications to Geometry and Topology, Cambridge University Press, 1995.
[Pi] R. S. Pierce, Associative Algebras, Springer, New York, 1982.
[Po] V. L. Popov, Sections in Invariant Theory, Proceedings of the Sophus Lie Memorial Conference, Scandinavian University Press, 1994, 315-362.
[PV] V. L. Popov, E. B. Vinberg, Invariant Theory, in Encyclopedia of Math. Sciences 55, Algebraic Geometry IV, edited by A. N. Parshin and I. R. Shafarevich, SpringerVerlag, 1994.
[ $\left.\mathrm{Pr}_{1}\right]$ C. Procesi, Non-commutative affine rings, Atti Acc. Naz. Lincei, S. VIII, v. VIII, fo. 6 (1967), 239-255.
[ $\left.\mathrm{Pr}_{2}\right] \quad$ C. Procesi, The invariant theory of $n \times n$-matrices, Adv. Math., 19 (1976), 306-381.
[Re] Z. Reichstein, On a theorem of Hermite and Joubert, Canadian J. Math., to appear. Preprint available at http://www.orst.edu/reichstz/pub.html.
[RY] Z. Reichstein, B. Youssin, Essential dimensions of algebraic groups and a resolution theorem for $G$-varieties, with an appendix by J. Kollár and E. Szabó, submitted for publication. Preprint available at http://www.orst.edu/reichstz/pub.html.
[Ros ${ }_{1}$ ] M. Rosenlicht, Some basic theorems on algebraic groups, American Journal of Math., 78 (1956), 401-443.
[ $\operatorname{Ros}_{2}$ ] M. Rosenlicht, A remark on quotient spaces, Anais da Academia Brasileira de Ciências 35 (1963), 487-489.
[Rost ${ }_{1}$ ] M. Rost, A (mod 3) invariant of exceptional Jordan algebras, C. R. Acad. Sci. Paris Sér. I Math. 313, no. 12 (1991), 823-827.
[Rost ${ }_{2}$ ] M. Rost, On Galois cohomology of Spin(14), preprint, March 1999. Available at http://www.physik.uni-regensburg.de/rom03516.
[Row] L. H. Rowen, Brauer factor sets and simple algebras, Trans. Amer. Math. Soc., 282, no. 2 (1984), 765-772.
[Sc] R. D. Schaefer, An Introduction to Non-associative Algebras, Academic Press, New York and London, 1966.
[ $\mathrm{Se}_{1}$ ] J.-P. Serre, Espaces fibrés algébriques, Exposé 5, Séminaire C. Chevalley, Anneaux de Chow et applications, 2nd année, IHP, 1958.
[Se 2 ] J.-P. Serre, Local Fields, Springer-Verlag, 1979.
[Se 3 ] J.-P. Serre, Cohomologie galoisienne: progrès et problèmes, in "Séminaire Bourbaki, Volume 1993/94, Exposés 775-789", Astérisque 227 (1995), 229-257.
[Se4] J.-P. Serre, Galois Cohomology, Springer, 1997.
[St] R. Steinberg, Regular elements of semisimple groups, Publ. Math. I.H.E.S. 25 (1965), 281-312. Reprinted in [Se ${ }_{4}$ ], pp. 155-186.
[SSSZ] I. P. Shestakov, A. I. Shirshov, A. M. Slin'ko, K. A. Zhevlakov, Rings that are Nearly Associative, Academic Press, 1982.


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