Lecture 3: The Yang–Mills equations

In this lecture we will introduce the Yang–Mills action functional on the space of connections and the corresponding Yang–Mills equations. The strategy will be to work locally with the gauge fields and ensure that the objects we contruct are gauge-invariant.

Throughout this lecture $P \to M$ will denote a principal G-bundle and $H \subset TP$ a connection with connection one-form ω and curvature two-form Ω. We will let $s_\alpha : U_\alpha \to P$ denote the canonical sections associated to a trivialisation. We will let $A_\alpha = s_\alpha^*$ and $F_\alpha = s_\alpha^* \Omega$ denote the corresponding gauge field and field-strength. On overlaps, the field-strengths are related as in equation (11).

3.1 Some geometry

Until now we have imposed no conditions on M or on G, but this will now change. From now on M will be an oriented pseudo-riemannian n-dimensional manifold with metric g. The orientation on M is given by a nowhere-vanishing n-form, which we will take to be the volume form of the metric.

3.1.1 The volume form

By passing to a refinement, if necessary, we will assume that our trivialising cover $\{U_\alpha\}$ is such that on each U_α the tangent bundle too is trivial. This represents no loss of generality. Then on each U_α we can find one-forms $\theta_i \in \Omega^1(U_\alpha)$ such that the metric takes the form

$$g = \sum_{i=1}^{n} \varepsilon_i \theta_i^2 ,$$

for some signs ε_i. Let there be s positive and t negative signs. On overlaps, the θ_i will transform by local (special, since M is orientable) orthogonal transformations, but the numbers s and t will not change (Sylvester's law of inertia). We say that M has signature (s, t). Let us define an n-form

$$\theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_n \in \Omega^n(U_\alpha) .$$

on each U_α. The orientability of M implies that these forms agree on overlaps and hence define an n-form $\mathrm{dvol} \in \Omega^n(M)$ called the **volume form** of the metric g. We will assume that dvol gives M its orientation. The volume form allows us to integrate (e.g., compactly supported) functions on M: $\int_M f\, \mathrm{dvol}$ invariantly.

3.1.2 The Hodge $\star$ operator

The metric g defines an inner product $\langle -, - \rangle$ on one-forms by declaring the θ_i to be orthonormal:

$$\langle \theta_i, \theta_j \rangle = \begin{cases} \varepsilon_i , & \text{if } i = j, \\ 0 , & \text{otherwise,} \end{cases}$$

and extending bilinearly to arbitrary one-forms on U_α. Since on overlaps the θ_i transform by (special) orthogonal transformations, the inner product is well-defined on one-forms on M. Similarly, the metric defines an inner product on k-forms, but to define it, we need to introduce some notation.

A sequence $I = (i_1, \ldots, i_k)$, where $1 \le i_1 < i_2 < \cdots < i_k \le n$, is called a **multi-index of length** $|I| = k$. Let us define $\theta_I := \theta_{i_1} \wedge \theta_{i_2} \wedge \cdots \wedge \theta_{i_k}$. Then every k-form on $\Omega^k(U_\alpha)$ can be written as a linear combination of the $(\theta_I)_{|I|=k}$ with coefficients which are functions on U_α. The inner product on $\Omega^k(U_\alpha)$ is defined by

$$\langle \theta_I, \theta_J \rangle = \begin{cases} \varepsilon(I) , & \text{if } I = J, \\ 0 , & \text{otherwise,} \end{cases}$$

where $\varepsilon(I) = \varepsilon(i_1)\varepsilon(i_2)\cdots\varepsilon(i_k)$ for $I = (i_1, \ldots, i_k)$, and extending it bilinearly to all of $\Omega^k(U_\alpha)$. As before, the inner product so defined agrees on overlaps and hence extends to an inner product on $\Omega^k(M)$.

We can now define the **Hodge $\star$ operator**: $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ by

$$\alpha \wedge \star\beta = \langle\alpha,\beta\rangle\,\mathrm{dvol}\,,$$

where $\alpha,\beta \in \Omega^k(M)$. We can be more explicit, by showing what the Hodge $\star$ operator does to the θ_I. By definition,

$$\theta_I \wedge \star\theta_I = \varepsilon(I)\,\mathrm{dvol}\,,$$

whence

$$\star\theta_I = \varepsilon(I)\varsigma(I)\theta_{\bar{I}}\,,$$

where $\bar{I}$ is the complementary multi-index to I; that is, the unique multi-index of length $|\bar{I}| = n-k$ such that $I \cup \bar{I} = \{1,2,\ldots,n\}$ (as sets), and $\varsigma(I)$ is the sign of the permutation of $(1,2,\ldots,n)$ given by concatenating $I \sqcup \bar{I}$.

Done? ❑ **Exercise 3.1.** Let $n = 4$ and let g have positive-definite signature $(4,0)$. Calculate the Hodge $\star$ acting on all θ_I. Show that $\star^2 = \mathrm{id}$ on 2-forms. Now do the same for lorentzian signature $(3,1)$ and show that $\star^2 = -\mathrm{id}$ on 2-forms. Can you guess what happens in split signature $(2,2)$?

Iterating the Hodge $\star$ operator yields a map $\star^2 : \Omega^k(M) \to \Omega^k(M)$. To recognise it, we act on θ_I:

$$\star^2\theta_I = \varepsilon(I)\varsigma(I)\star\theta_{\bar{I}} = \varepsilon(I)\varepsilon(\bar{I})\varsigma(I)\varsigma(\bar{I})\theta_I\,,$$

whence $\star^2$ is a scalar operator, acting as a sign. To work out the sign, notice that $\varepsilon(I)\varepsilon(\bar{I}) = (-1)^t$ and that $\varsigma(I)\varsigma(\bar{I}) = (-1)^{|I||\bar{I}|}$,

$$\star^2 = (-1)^t(-1)^{k(n-k)}\,\mathrm{id} \qquad \text{on } \Omega^k(M).$$

Done? ❑ **Exercise 3.2.** Let M be even-dimensional. Show how the Hodge $\star$ operator transforms under a conformal transformation and show that it is conformally invariant acting on middle-dimensional forms. In other words, rescale the metric on M to $\tilde{g} = e^{2f}g$, and work out the relation between the Hodge operators $\star_g$ and $\star_{\tilde{g}}$. In particular, show that they agree on middle-dimensional forms.

3.1.3 Inner product on bundle-valued forms

We would also like to define inner products on forms with values in an associated vector bundle $P \times_G V$. Locally, on each U_α, we view such forms as forms with values in V. To define an inner product on such locally defined forms, all we need an inner product on V; but if we want this inner product to glue well on overlaps, we must require that it be G-invariant, so that for all $g \in G$, $\boldsymbol{v}, \boldsymbol{w} \in V$,

$$\langle\varrho(g)\boldsymbol{v}, \varrho(g)\boldsymbol{w}\rangle = \langle\boldsymbol{v},\boldsymbol{w}\rangle\,.$$

Indeed, if $\zeta \in \Omega^k(M; P \times_G V)$ is represented locally by $\zeta_\alpha \in \Omega^k(U_\alpha; V)$, consider the function $\langle\zeta_\alpha,\zeta_\alpha\rangle \in C^\infty(U_\alpha)$, where $\langle -,-\rangle$ denotes both the inner product on V and the inner product on forms. On a nonempty overlap $U_{\alpha\beta}$,

$$\langle\zeta_\alpha,\zeta_\alpha\rangle = \langle\varrho(g_{\alpha\beta})\zeta_\beta, \varrho(g_{\alpha\beta})\zeta_\beta\rangle = \langle\zeta_\beta,\zeta_\beta\rangle\,,$$

whence it defines a global function $\langle\zeta,\zeta\rangle \in C^\infty(M)$.

The existence of a G-invariant inner product on V is of course not guaranteed, but if G is compact, for example, then we may always construct one by departing from any positive-definite inner product and averaging over the group with respect to the Haar measure.

In the case of the adjoint bundle ad P, we require an inner product on the Lie algebra $\mathfrak{g}$ which is invariant under the adjoint action of G. For example, if $\mathfrak{g}$ is semisimple then the Killing form κ, defined by

$$\kappa(X,Y) = \mathrm{Tr}\,\mathrm{ad}_X\,\mathrm{ad}_Y$$

where $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{ad}_X Y = [X,Y]$, is a possible such inner product. Of course, there are nonsemisimple (even nonreductive) Lie algebras admitting an ad-invariant inner product; although for a positive-definite inner product $\mathfrak{g}$ must be the Lie algebra of a compact group, hence reductive. In any case we will assume in what follows that $\mathfrak{g}$ has such an inner product.

3.2 The variational problem

3.2.1 The action functional

The gauge field-strengths F_α define a 2-form $F_A \in \Omega^2(M; \operatorname{ad} P)$ whose norm defines a function on M:

$$|F_A|^2 = \langle F_A, F_A \rangle \ .$$

Notation

We may at times use the notation

$$\operatorname{Tr}(F_A \wedge \star F_A) := |F_A|^2 \, \mathrm{dvol} \in \Omega^n(M) \ .$$

We will define the **Yang–Mills action** to be

$$\boxed{S_{YM} = \int_M |F_A|^2 \, \mathrm{dvol} \ ,} \tag{15}$$

provided that the integral exists. This will be the case for M compact, for example.

The above action does not depend on the choice of local sections used to pull back the curvature two-form to M. Indeed, let $\tilde{s}_\alpha : U_\alpha \to P$ be a different choice of local sections. Let $m \in U_\alpha$ and consider $\tilde{s}_\alpha(m)$ and $s_\alpha(m)$. Since they belong to the same fibre, there exists $h_\alpha(m) \in G$ such that

$$\tilde{s}_\alpha(m) = s_\alpha(m) h_\alpha(m) \ .$$

As m varies, this defines a function $h_\alpha : U_\alpha \to G$. Let $\tilde{F}_\alpha = \tilde{s}_\alpha^* \Omega$. Then for all $m \in U_\alpha$,

$$\begin{aligned}
\tilde{F}_\alpha(m) &= \tilde{s}_\alpha^* \Omega(\tilde{s}_\alpha(m)) \\
&= (R_{h_\alpha(m)} \circ s_\alpha)^* \Omega(s_\alpha(m) h_\alpha(m)) \\
&= s_\alpha^* R_{h_\alpha(m)}^* \Omega(s_\alpha(m) h_\alpha(m)) \\
&= s_\alpha^* \left(\operatorname{ad}_{h_\alpha(m)^{-1}} \circ \Omega(s_\alpha(m)) \right) && \text{(since } \Omega \text{ is invariant)} \\
&= \operatorname{ad}_{h_\alpha(m)^{-1}} \circ s_\alpha^* \Omega(s_\alpha(m)) \\
&= \operatorname{ad}_{h_\alpha(m)^{-1}} \circ F_\alpha(m)
\end{aligned}$$

whence, by the ad-invariance of the inner product, $|\tilde{F}|^2 = |F|^2$.

Similarly, the action does not depend on the choice of trivialisation. Indeed, given two trivialisations, we simply pass to a common refinement and use the independence on the choice of local section to show that the norm of the gauge field-strength does not change.

Therefore, if M is compact, then the Yang–Mills action defines a function on the space of connections: $S_{YM} : \mathscr{A} \to \mathbb{R}$. If M is not compact, then we must restrict to connections for which the integral exists. Moreover, the Yang–Mills action is gauge-invariant. Indeed, under a gauge transformation $\Phi \in \mathscr{G} \cong C^\infty(M; \operatorname{Ad} P)$

$$F_\alpha \mapsto F_\alpha^\Phi = \operatorname{ad}_{\phi_\alpha} \circ F_\alpha \ ,$$

whence $|F^\Phi|^2 = |F|^2$ due to the invariance of the inner product on $\mathfrak{g}$. This means that (for M compact) the Yang–Mills action descends to a function $\mathscr{A}/\mathscr{G} \to \mathbb{R}$.

3.2.2 The field equations

A connection A is said to be a **Yang–Mills connection** if it is a critical point of the Yang–Mills action. This means that all directional derivatives of S_{YM} vanish at A. We will now see that this condition turns into a second-order partial differential equation for A.

We recall that $\mathscr{A}$ is an affine space modelled on $\Omega^1(M; \operatorname{ad} P)$. This means that the tangent space to $\mathscr{A}$ at any point is isomorphic to $\Omega^1(M; \operatorname{ad} P)$. Given a connection $A \in \mathscr{A}$ and a one-form $\tau \in \Omega^1(M; \operatorname{ad} P)$, we consider the curve $A + t\tau$ in $\mathscr{A}$ whose tangent vector (at A) is precisely τ. The directional derivative of S_{YM} at A in the direction τ is given by

$$\frac{d}{dt} S_{YM}(A + t\tau)\Big|_{t=0}$$

and the Yang–Mills condition states that this vanishes for all τ. To see what this means, we first compute the curvature along the above curve. Working locally, but omitting the index α associated to the trivialisation, we have from the structure equation:

$$\begin{aligned} F_{A+t\tau} &= d(A + t\tau) + \tfrac{1}{2}[A + t\tau, A + t\tau] \\ &= F_A + t\left(d\tau + \tfrac{1}{2}[A,\tau] + \tfrac{1}{2}[\tau, A]\right) + \tfrac{1}{2}t^2[\tau,\tau] \\ &= F_A + t\,(d\tau + [A,\tau]) + \tfrac{1}{2}t^2[\tau,\tau] \\ &= F_A + t d_A\tau + \tfrac{1}{2}t^2[\tau,\tau]\,. \end{aligned}$$

Computing its norm,

$$\begin{aligned} |F_{A+t\tau}|^2 &= |F_A + t d_A\tau + \tfrac{1}{2}t^2[\tau,\tau]|^2 \\ &= \langle F_A + t d_A\tau + \tfrac{1}{2}t^2[\tau,\tau], F_A + t d_A\tau + \tfrac{1}{2}t^2[\tau,\tau]\rangle \\ &= |F_A|^2 + 2t\,\langle d_A\tau, F_A\rangle + t^2\left(|d_A\tau|^2 + \langle F_A, [\tau,\tau]\rangle\right) + t^3\,\langle d_A\tau, [\tau,\tau]\rangle + \tfrac{1}{4}t^4|[\tau,\tau]|^2\,. \end{aligned}$$

Therefore, the Yang–Mills condition is

$$0 = \frac{d}{dt} S_{YM}(A + t\tau)\Big|_{t=0} = 2\int_M \langle d_A\tau, F_A\rangle\,\mathrm{dvol} \qquad \text{for all } \tau \in \Omega^1(M; \operatorname{ad} P).$$

Let d_A^* denote the formal adjoint of d_A, so that

$$\int_M \langle d_A\tau, F_A\rangle\,\mathrm{dvol} = \int_M \langle \tau, d_A^* F_A\rangle\,\mathrm{dvol}\,,$$

whence the Yang–Mills condition becomes the following differential equation:

$$d_A^* F_A = 0\,.$$

Done? ❑ **Exercise 3.3.** Show that $\star d_A^* F_A = d \star F_A$.

We therefore conclude that the Yang–Mills condition is equivalent to the equation

$$\boxed{d_A \star F_A = 0\,,} \tag{16}$$

which together with the Bianchi identity $d_A F_A = 0$ constitutes of a nonlinear version of the conditions for a 2-form to be harmonic.

Notice that because the Yang–Mills action is gauge-invariant, if A solves the Yang–Mills equations, so will any gauge transformed A^Φ. In other words, the gauge group acts on the space $\mathscr{A}_{YM}$ of Yang–Mills connections. The quotient $\mathscr{A}_{YM}/\mathscr{G}$ is the space of classical solutions. In general it is infinite-dimensional, but we will see that it has interesting finite-dimensional subspaces.

3.3 Coupling to matter

Gauge fields are responsible for the "forces" in Nature. Matter fields, on the other hand, are modelled as sections of certain bundles over M. For bosonic matter fields, these are simply associated fibre bundles to P: typically associated vector bundles, but more generally associated fibre bundles in the case of nonlinear realisations (σ-models,...). Fermionic matter fields are sections of a tensor product of a spinor bundle on M (assumed spin) and an associated vector bundle to P.

For simplicity, let us consider a bosonic matter field φ which is a section of an associated vector bundle $P \times_G V$ over M with representation $\varrho : G \to GL(V)$, preserving an inner product $\langle -, - \rangle$ on V. Let $d_A : \Omega^0(M; P \times_G V) \to \Omega^1(M; P \times_G V)$ denote the covariant derivative and let $|d_A\varphi|^2 \in C^\infty(M)$ denote the (squared) norm of $d_A\varphi$ using both the inner product on forms and the one on V. The coupling of this matter to the gauge fields is described by the action functional

$$S_{\text{matter}} = \tfrac{1}{2} \int_M |d_A\varphi|^2 \, \text{dvol} \; .$$

Done? ❑ **Exercise 3.4.** Show that the field equation for φ obtained by extremising the above action is given by

$$d_A \star d_A \varphi = 0 \; ,$$

which is a nonlinear version of Laplace's equation.

Of course, the inclusion of matter fields also changes the Yang–Mills equations. It's easy enough to work out the new equations by demanding that A be a critical point of the action $S_{\text{YM}} + S_{\text{matter}} : \mathscr{A} \to \mathbb{R}$, for fixed φ.

Done? ❑ **Exercise 3.5.** Show that in the presence of the matter field φ the Yang–Mills equations are modified by a quadratic term in φ:

$$d_A^* F_A + T(A, \varphi) = 0 \; ,$$

where $T = T(A, \varphi) \in \Omega^1(M; \operatorname{ad} P)$ is defined by

$$\langle T, \tau \rangle = \langle d_A\varphi, \varrho(\tau)\varphi \rangle$$

for every $\tau \in \Omega^1(M; \operatorname{ad} P)$.