GAUGE THEORY AND THE DIVISION ALGEBRAS

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Abstract. We present a novel formulation of the instanton equations in 8-dimensional Yang–Mills theory. This formulation reveals these equations as the last member of a series of gauge-theoretical equations associated with the real division algebras, including flatness in dimension 2 and (anti-)self-duality in 4. Using this formulation we prove that (in flat space) these equations can be understood in terms of moment maps on the space of connections and the moduli space of solutions is obtained via a generalised symplectic quotient: a Kähler quotient in dimension 2, a hyperkähler quotient in dimension 4 and an octonionic Kähler quotient in dimension 8. One can extend these equations to curved space: whereas the 2-dimensional equations make sense on any surface, and the 4-dimensional equations make sense on an arbitrary oriented manifold, the 8-dimensional equations only make sense for manifolds whose holonomy is contained in Spin(7). The interpretation of the equations in terms of moment maps further constraints the manifolds: the surface must be oriented, the 4-manifold must be hyperkähler and the 8-manifold must be flat.

1. Introduction

Gauge theory in higher than four dimensions is rapidly coming of age. Recent developments in superstring theory, particularly related to the Matrix Conjecture of [5], point to the existence of supersymmetric quantum gauge theories in dimensions where traditionally we would have expected none to exist: five and six dimensions so far, but possibly higher. More recent work [20] also suggests that higher-dimensional instantons [10, 27] dominate certain regimes in the moduli space of M-theory. In addition, these higher-dimensional instantons are intimately linked to supersymmetry [7, 2, 6, 8, 1, 13, 14] and to the geometry of riemannian manifolds of special holonomy: Calabi–Yau and hyperkähler geometries, and especially the exceptional geometries in seven and eight dimensions [24, 11, 26]. At the same time, very little is known about these generalised instantons: very few solutions are known explicitly [12, 15, 21], and almost nothing is known about the moduli spaces, although the deformation complexes are elliptic and formulae for the virtual dimensions can be obtained [23, 24]. This result notwithstanding, the equality between the virtual dimension and the dimension of the moduli space (at least at irreducible points) hinges on the vanishing of the higher cohomology of the deformation complex—a question which has yet to be addressed.

Judging by the 4-dimensional case, instanton moduli space has a rich geometry worthy of study on its own right. It is likely that a similarly rich geometry will emerge out of the study of the moduli spaces of higher-dimensional instantons. This note is a first step in this direction. We focus on the generalised instantons in eight dimensions, proving that they fit inside a family of gauge-theoretical solitons associated with the division algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, and including the flat connections in dimension 2 ($\mathbb{C}$) and the (anti-)self-dual connections in dimension 4 ($\mathbb{H}$). From this fact, and by analogy with well-known results in the lower dimensions, we establish some facts concerning the moduli space of octonionic instantons. Among other things, we exhibit the moduli space of octonionic instantons on a flat 8-dimensional manifold, as an infinite-dimensional octonionic Kähler quotient. The notion of an octonionic Kähler structure is defined and some of its properties are explored in the appendix; although a more detailed discussion will be postponed to a separate publication.

This note is organised as follows. In Section 2 we discuss the family of instanton equations in $\mathbb{R}^N$ associated to the division algebras $\mathbb{C}$ (for N=2), $\mathbb{H}$ (for N=4) and $\mathbb{O}$ (for N=8). To the best of our knowledge, this formulation of the octonionic instanton equations is novel and has the advantage of exhibiting these equations as the last member of a well-established sequence. Using this reformulation, we show in Section 3 that the instanton equations can be obtained as the zero loci of generalised moment maps and that the moduli spaces of instantons can be understood as a generalised symplectic quotient. This is of course well known in the complex and quaternionic case. In Section 4 we investigate the extension of these results to more general riemannian manifolds. This will single out 8-manifolds of Spin(7) holonomy as those admitting the 8-dimensional instanton equations, and flat 8-dimensional manifolds as those for which the instanton moduli space can be interpreted as an octonionic Kähler quotient. Section 5 contains some conclusions and the paper ends with an appendix on octonionic geometry and a possible extension of the notion (introduced earlier in paper) of an octonionic Kähler structure.

2. Instanton equations in $\mathbb{R}^N$

In this section we introduce the (generalised) instanton equations on $\mathbb{R}^N$ where $N = 2, 4, 8$. These equations consist in setting to zero the imaginary part of the Yang–Mills curvature in a way that we will make precise. In dimension 2 this equation makes the connection flat, in dimension 4 (anti-)self-dual, and in dimension 8 it becomes the octonionic instanton equation introduced in [10].

2.1. $\mathbb{C}$-instantons on $\mathbb{R}^2$. A gauge field on $\mathbb{R}^2$ has components $A_\mu(x)$ for $\mu = 1, 2$. It is convenient to consider complex-valued gauge fields $A(x) = A_1(x)i + A_2(x)$. Multiplication by i defines a 2×2 real matrix I as follows:

$$iA(x) = I_{1\mu} A_\mu(x) i + I_{2\mu} A_\mu(x) \ .$$

The matrix I is given by the standard symplectic structure

$$I = \tau_2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

where τ_2 given above is a Pauli matrix. We say that $A_\mu(x)$ is a $\mathbb{C}$-*instanton* if its curvature $F_{\mu\nu}$ satisfies

$$I \cdot F(x) \equiv I_{\mu\nu} F_{\mu\nu}(x) = 0 \ . \tag{1}$$

From the explicit form of I we see that $\mathbb{C}$-instantons are nothing but flat connections: $F_{\mu\nu}(x) = 0$.

2.2. $\mathbb{H}$-instantons on $\mathbb{R}^4$. Gauge fields $A_\mu(x)$ in $\mathbb{R}^4$ can be thought of as quaternion-valued:

$$A(x) = A_\mu(x) q_\mu = A_1(x)i + A_2(x)j + A_3(x)k + A_4(x) \ ,$$

where we have introduced a basis $q_\mu = \{i, j, k, 1\}$ for the quaternion units. Left multiplication by the imaginary units defines real 4×4 matrices I, J and K as before:

$$\begin{aligned} iA(x) &= I_{\mu\nu} A_\nu(x) q_\mu \ , \\ jA(x) &= J_{\mu\nu} A_\nu(x) q_\mu \ , \\ kA(x) &= K_{\mu\nu} A_\nu(x) q_\mu \ . \end{aligned}$$

Explicitly, we have

$$I = \begin{pmatrix} 0 & \tau_2 \\ \tau_2 & 0 \end{pmatrix} , \quad J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} , \quad \text{and} \quad K = \begin{pmatrix} -\tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix} .$$

The matrices I, J and K obey the quaternion algebra $I^2 = J^2 = K^2 = -\mathbf{1}$ and $IJ = K$, etc. They also obey the anti-self-duality equation

$$I_{\mu\nu} = -\tfrac{1}{2} \varepsilon_{\mu\nu\rho\sigma} I_{\rho\sigma} \ ,$$

and similarly for J and K. We say that $A_\mu(x)$ defines an $\mathbb{H}$-*instanton* if the following equations are satisfied:

$$I \cdot F(x) = J \cdot F(x) = K \cdot F(x) = 0 \ . \tag{2}$$

This means that $F_{\mu\nu}(x)$ is self-dual:

$$F_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \ . \tag{3}$$

In other words, $A_\mu(x)$ is an instanton in the ordinary sense.

The anti-instanton equations are recovered by considering right multiplication by the conjugate imaginary units on the quaternionic gauge field $A(x)$. This gives rise to matrices $\tilde{I}$, $\tilde{J}$ and $\tilde{K}$ which are now self-dual. The matrices are different because $\mathbb{H}$ is not commutative. The analogous equations to (2) but with the tilded matrices, now say that $F_{\mu\nu}(x)$ is anti-self-dual—in other words, $A_\mu(x)$ is an anti-instanton.

2.3. **$\mathbb{O}$-instantons on $\mathbb{R}^8$.** Let us consider a gauge field $A_\mu(x)$ in $\mathbb{R}^8$ and turn it into an octonion-valued field

$$A(x) = A_\mu(x)o_\mu = A_i(x)o_i + A_8(x) \ ,$$

where we have introduced a basis o_μ for $\mu = 1, \ldots, 8$ for the octonions such that o_i for $i = 1, \ldots, 7$ are the imaginary units and o_8 is the identity. Left multiplication by the imaginary units o_i gives rise to real 8×8 matrices I^i as follows:

$$o_i A(x) = I^i_{\mu\nu} A_\nu(x) o_\mu \ .$$

The matrices I^i cannot satisfy the octonion algebra, because unlike octonion multiplication, matrix multiplication is associative. Nevertheless they satisfy the 7-dimensional euclidean Clifford algebra $\mathrm{C}\ell(7)$:

$$I^i I^j + I^j I^i = -2\delta_{ij}\mathbf{1} \ . \tag{4}$$

We define an $\mathbb{O}$-*instanton* as a gauge field $A_\mu(x)$ subject to the seven equations

$$I^i \cdot F(x) = 0 \ . \tag{5}$$

Explicitly, for a particular choice of basis, these seven equations are given by

$$\begin{aligned} F_{12} - F_{34} - F_{58} + F_{67} &= 0 \\ F_{13} + F_{24} - F_{57} + F_{68} &= 0 \\ F_{14} - F_{23} + F_{56} - F_{78} &= 0 \\ F_{15} + F_{28} + F_{37} - F_{46} &= 0 \\ F_{16} - F_{27} + F_{38} + F_{45} &= 0 \\ F_{17} + F_{26} - F_{35} + F_{48} &= 0 \\ F_{18} - F_{25} - F_{36} - F_{47} &= 0 \ . \end{aligned} \tag{6}$$

They can be written in a way analogous to the self-duality equation (3):

$$F_{\mu\nu} = \tfrac{1}{2}\Omega_{\mu\nu\rho\sigma}F_{\rho\sigma} \ , \tag{7}$$

where $\Omega_{\mu\nu\rho\sigma}$ are the components of a 4-form in $\mathbb{R}^8$ given by

$$\Omega = -\tfrac{1}{6} I^i \wedge I^i \ . \tag{8}$$

In fact, equation (7) is the way in which the octonionic equations are usually presented (see, for example, [10]). Let us remark that in analogy with the classical four-dimensional instantons, a gauge field $A_\mu(x)$ obeying equation (5), automatically satisfies the Yang–Mills equations of motion: $D^\mu F_{\mu\nu} = 0$, as a consequence of the Bianchi identity: $D_{[\mu}F_{\nu\rho]} = 0$.

The 4-form Ω is self-dual, as can be seen by the following construction. Octonion multiplication defines a 3-form φ in $\mathbb{R}^7$ by:

$$o_i\, o_j = -\delta_{ij}\, o_8 + \varphi_{ijk}\, o_k \ .$$

Our choice of basis is such that

$$\varphi = o_{125} + o_{136} + o_{147} - o_{237} + o_{246} - o_{345} + o_{567} \ ,$$

where we have used the shorthand $o_{ijk} = o_i \wedge o_j \wedge o_k$. We now consider the 7-dimensional Hodge dual of φ:

$$\tilde{\varphi} \equiv \star_7 \varphi = o_{1234} - o_{1267} + o_{1357} - o_{1456} + o_{2356} + o_{2457} + o_{3467} \ .$$

Thinking of $\tilde{\varphi}$ as a 4-form in $\mathbb{R}^8$, its 8-dimensional Hodge dual is given by $\varphi \wedge o_8$, whence we can define a self-dual 4-form Ω in $\mathbb{R}^8$ as follows:

$$\begin{aligned} \Omega &= \tilde{\varphi} + \varphi \wedge o_8 \\ &= o_{1234} + o_{1258} - o_{1267} + o_{1357} + o_{1368} - o_{1456} + o_{1478} \\ &\qquad + o_{2356} - o_{2378} + o_{2457} + o_{2468} - o_{3458} + o_{3467} + o_{5678} \ . \end{aligned}$$

This is precisely the 4-form defined in (8).

Interpreting $\mathbb{R}^8$ as the vector representation of SO(8), the 4-form Ω is left invariant by a Spin(7) subgroup of SO(8), one under which the vector representation remains irreducible. There are three conjugacy classes of Spin(7) subgroups in Spin(8), which are related by triality. Each of these subgroups are maximal and they can be distinguished by which one of the three 8-dimensional irreducible representations of Spin(8) they split. Two of these subgroups, call them $\mathrm{Spin}(7)^{\pm}$, leave the vector representation irreducible, but split one of the two spinor representations. Let $\mathrm{Spin}(7)^{+}$ be the one leaving invariant the 4-form Ω in (8). There is a similar set of equations to the $\mathbb{O}$-instanton equations but using instead the 4-form $\tilde{\Omega}$ which is invariant by $\mathrm{Spin}(7)^{-}$. These equations are obtained analogously to (5) but using the matrices $\tilde{I}^i$ obtained by right multiplication by the conjugate imaginary units. Indeed the 4-form $\tilde{\Omega}$ is given by equation (8) but using the tilded matrices instead. A gauge field obeying these equations will be referred to an *octonionic anti-instanton* or $\mathbb{O}$-*anti-instanton.*

2.4. **Another reformulation.** The instanton equations can be reformulated in yet another way: as the reality of a laplacian-type operator defined on vector bundles of type $\mathbb{A}$ associated to the principal gauge bundle. We turn to this now.

Let $\{e_\mu\}$ denote generically a set of units for the division algebra $\mathbb{A}$, being one of $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, and let $\{\bar{e}_\mu\}$ denote their $\mathbb{A}$-conjugates. Let $N = \dim \mathbb{A}$ stand for the real dimension of $\mathbb{A}$. We will choose our set of units such that $e_N = \mathbf{1}$ and $\{e_i\}_{i=1}^{N-1}$ are imaginary. If $o \in \mathbb{A}$ we will let $\mathrm{Re}\, o$ denote its real part; that is, $\mathrm{Re}\, o_\mu e_\mu = o_N$. Let D_μ denote the covariant derivative, and let $D = D_\mu e_\mu$. We can think of D as acting on $\mathbb{A}$-valued fields ψ (with values in some unitary representation of the gauge group). Given two such $\mathbb{A}$-valued fields ψ, ϕ we define their inner product as

$$(\psi, \phi) = \int_{\mathbb{R}^N} \mathrm{dvol}\ \mathrm{Tr}\, \mathrm{Re}\, \psi^\dagger \phi\ ,$$

where N is the real dimension of $\mathbb{A}$, Tr means the gauge invariant inner product, and † involves conjugation in $\mathbb{A}$ as well as in the representation of the gauge group. Let $D^\dagger$ denote the formal adjoint of D relative to this inner product:

$$D^\dagger = -D_\mu \bar{e}_\mu\ .$$

It follows immediately from the first of the two identities

$$e_\mu \bar{e}_\nu = \delta_{\mu\nu}\mathbf{1} + I^k_{\mu\nu} e_k \qquad \text{and} \qquad \bar{e}_\mu e_\nu = \delta_{\mu\nu}\mathbf{1} + \tilde{I}^k_{\mu\nu} e_k \tag{9}$$

that

$$D^\dagger D = -D^2\mathbf{1} + e_k I^k \cdot F(A)\ ,$$

where $D^2 = D_\mu D_\mu$. Therefore we see that the $\mathbb{A}$-instanton equations (1), (2) and (5) are equivalent to

$$\operatorname{Im}\left(D^\dagger D\right) = 0 \quad \text{or equivalently} \quad D^\dagger D = -D^2 \mathbf{1} \ .$$

Similarly from the second identity in (9), it follows that

$$DD^\dagger = -D^2\mathbf{1} + e_k \tilde{I}^k \cdot F(A) \ ,$$

whence the $\mathbb{A}$-anti-instantons are the solutions to the opposite equation:

$$\operatorname{Im}\left(DD^\dagger\right) = 0 \quad \text{or equivalently} \quad DD^\dagger = -D^2 \mathbf{1} \ .$$

3. Instantons and moment maps

In this section we show that the instanton equations (1), (2) and (5) can be understood as the zeroes of moment maps associated to the gauge transformations on the space of connections. This will prove that the moduli space of instantons can be seen in each case as a generalised symplectic quotient: a Kähler quotient in the complex case [4], a hyperkähler quotient [19] in the quaternionic case [3], and an octonionic Kähler quotient in the octonionic case. To the best of our knowledge, this latter quotient construction is new.

As before we let $\mathbb{A}$ be any one of the division algebras $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, and let $N = \dim \mathbb{A}$ be its real dimension. Let us denote by $\mathcal{A}_\mathbb{A}$ the space of $\mathbb{A}$-valued gauge fields on $\mathbb{R}^N$. The space $\mathcal{A}_\mathbb{A}$ is an infinite-dimensional affine space modelled on the space of Lie-algebra valued 1-forms on $\mathbb{R}^N$, and it inherits some geometric structure: it is Kähler for $\mathbb{A} = \mathbb{C}$, hyperkähler for $\mathbb{A} = \mathbb{H}$ and octonionic Kähler (see below) for $\mathbb{A} = \mathbb{O}$. The group of gauge transformations leaves these structures invariant and will give rise to moment maps whose components are nothing but the $\mathbb{A}$-instanton equations. As a result the moduli space $\mathcal{M}_\mathbb{A}$ of $\mathbb{A}$-instantons can be understood as a generalised symplectic quotient of $\mathcal{A}_\mathbb{A}$. This is of course well-known for $\mathbb{A} = \mathbb{C}$ and $\mathbb{A} = \mathbb{H}$. In what follows we will treat all three cases simultaneously.

3.1. **$\mathcal{A}_\mathbb{A}$ as an infinite-dimensional $\mathbb{A}$-Kähler space.** We will use the following notation: $A(x)$ is a Lie algebra- and $\mathbb{A}$-valued gauge field on $\mathbb{R}^N$. We will let $\overline{A(x)}$ denote its $\mathbb{A}$-conjugate. We will let Tr denote the invariant metric on the Lie algebra and Re denote the real part of an element of $\mathbb{A}$. The tangent space to the space $\mathcal{A}_\mathbb{A}$ of connections is the space of Lie algebra- and $\mathbb{A}$-valued 1-forms. Let $\delta_1 A(x)$ and $\delta_2 A(x)$ be two such 1-forms. As above we will let e_μ denote a basis for the $\mathbb{A}$-units, with e_N being the identity and e_i for $i = 1, \ldots, N-1$ being imaginary. Then if $z = z_\mu e_\mu \in \mathbb{A}$ with $z_\mu \in \mathbb{R}$, we can define the following bilinear form:

$$\langle\!\langle \delta_1 A, \delta_2 A\rangle\!\rangle_z = \int_{\mathbb{R}^N} \mathrm{dvol}\ \mathrm{Tr}\,\mathrm{Re}\, z\, \delta_1 A\, \overline{\delta_2 A}\ .$$

Expanding this out, we have

$$\langle\!\langle \delta_1 A, \delta_2 A\rangle\!\rangle_z = z_i \omega^i(\delta_1 A, \delta_2 A) + z_N g(\delta_1 A, \delta_2 A)\ ,$$

where the metric g is defined by

$$g(\delta_1 A, \delta_2 A) = \int_{\mathbb{R}^N} \mathrm{dvol}\ \mathrm{Tr}\,\mathrm{Re}\, \delta_1 A\, \overline{\delta_2 A}\ , \tag{10}$$

and the $N-1$ 2-forms ω^i by

$$\omega^i(\delta_1 A, \delta_2 A) = \int_{\mathbb{R}^N} \mathrm{dvol}\ \mathrm{Tr}\,\mathrm{Re}\, e_i\, \delta_1 A\, \overline{\delta_2 A}\ . \tag{11}$$

In components, we have

$$g(\delta_1 A, \delta_2 A) = \int_{\mathbb{R}^N} \mathrm{dvol}\ \mathrm{Tr}\, \delta_1 A_\mu\, \delta_2 A_\mu\ ,$$

and

$$\omega^i(\delta_1 A, \delta_2 A) = \int_{\mathbb{R}^N} \mathrm{dvol}\ I^i_{\mu\nu}\, \mathrm{Tr}\, \delta_1 A_\nu\, \delta_2 A_\mu\ .$$

It is then easy to see that the metric is indeed symmetric and that the 2-forms are antisymmetric. Moreover both g and ω^i are constant (i.e., do not depend on the connection $A(x)$ on which they are defined) and hence covariantly constant relative to the Levi-Cività connection corresponding to g. We see that $\mathcal{A}_{\mathbb{A}}$ is therefore (formally) Kähler for $\mathbb{A} = \mathbb{C}$ and hyperkähler for $\mathbb{A} = \mathbb{H}$. For $\mathbb{A} = \mathbb{O}$, it not hard to see that this makes $\mathcal{A}_{\mathbb{A}}$ into what we call an *octonionic Kähler* space. We use this term in a rather narrow sense which we now explain.

We start with $\mathbb{R}^8$, which we think of as the octonions $\mathbb{O}$. The real matrices I^i defined by left (or right) multiplication by the imaginary unit octonions satisfy the Clifford algebra $\mathrm{C}\ell(7)$ in (4). In particular, each I^i is complex structure relative to which the standard euclidean metric is hermitian.

Let X be a riemannian manifold. For the purposes of this paper, we will say that X is *octonionic (almost) hermitian* if it admits orthogonal (almost) complex structures $\{I^i\}$ satisfying the algebra (4). In addition, we will say that an octonionic hermitian manifold X is *octonionic Kähler* (or *OK*) if the associated 2-forms ω^i are Kähler.

In the appendix it is shown that octonionic Kähler manifolds are severely constrained and it is therefore not clear that this is an interesting geometrical concept. We therefore discuss a potentially much more interesting extension of this notion akin to the notion of quaternionic Kähler.

If we do not need to specify $\mathbb{A}$, we will simply say that $\mathcal{A}_{\mathbb{A}}$ is $\mathbb{A}$-*Kähler*. Therefore $\mathbb{C}$-Kähler means Kähler, $\mathbb{H}$-Kähler does *not* mean quaternionic Kähler but hyperkähler, and $\mathbb{O}$-Kähler means OK.

3.2. **The $\mathbb{A}$-valued moment map.** The group of gauge transformations acts by conjugation on the tangent vectors δA and commute with the action of $\mathbb{A}$. Because Tr is pointwise invariant under conjugation, we see that both the metric and the Kähler forms are gauge invariant. Let us analyse more closely the invariance of the Kähler forms under infinitesimal gauge transformations; that is, under $\delta A = D\epsilon$, where $D = e_\mu D_\mu$ is the $\mathbb{A}$-valued covariant derivative and ϵ is a Lie algebra valued function on $\mathbb{R}^N$. Taking our cue from the finite-dimensional case, when the Lie derivative along a vector field v of a closed 2-form ω is zero, the contraction $\imath(v)\omega$ is locally exact, whence there exists (at least locally) a function $\Phi(v)$ so that $\imath(v)\omega = d\Phi(v)$. The functions $\Phi(v)$ are the components of the moment map.

In our case, we have that the contraction of the closed 2-form ω^i with the infinitesimal gauge transformation $D\epsilon$ is given by

$$\begin{aligned}\omega^i(D\epsilon, \delta A) &= \int_{\mathbb{R}^N} \mathrm{dvol}\; \operatorname{Tr}\operatorname{Re} e_i\, D\epsilon\, \overline{\delta A}\\ &= \int_{\mathbb{R}^N} \mathrm{dvol}\; I^i_{\mu\nu} \operatorname{Tr} D_\nu\epsilon\, \delta A_\mu\\ &= \int_{\mathbb{R}^N} \mathrm{dvol}\; I^i_{\mu\nu} \operatorname{Tr} \epsilon\, D_\mu \delta A_\nu\\ &= \int_{\mathbb{R}^N} \mathrm{dvol}\; I^i_{\mu\nu} \operatorname{Tr} \epsilon\, \delta F_{\mu\nu}\ .\end{aligned}$$

In other words, the components of the moment map are

$$\Phi^i(\epsilon) = \int_{\mathbb{R}^N} \mathrm{dvol}\; \operatorname{Tr} \epsilon I^i \cdot F\ .$$

The moment map itself is given by

$$\Phi^i = I^i \cdot F(A)\ ,$$

which can be thought of as a map from the space $\mathcal{A}_{\mathbb{A}}$ of connections to the dual of the Lie algebra $\mathrm{Lie}(\mathcal{G})$ of the group of gauge transformations. Acting on an element ϵ in $\mathrm{Lie}(\mathcal{G})$, we obtain $\Phi^i(\epsilon)$. The zero locus of the moment map Φ^i consists of those connections for which $I^i \cdot F = 0$.

Furthermore the moment map, as a function: $\Phi^i : \mathcal{A}_{\mathbb{A}} \to \mathrm{Lie}(\mathcal{G})^*$, is equivariant under the infinitesimal action of $\mathcal{G}$, acting on $\mathcal{A}_{\mathbb{A}}$ as infinitesimal gauge transformations and on $\mathrm{Lie}(\mathcal{G})^*$ as the coadjoint representation. To see this notice that, if $\epsilon, \eta \in \mathrm{Lie}(\mathcal{G})$, then

$$\begin{aligned}
\delta_\epsilon \Phi^i(\eta) &= \int_{\mathbb{R}^N} \mathrm{dvol}\, \mathrm{Tr}\, \eta\, I^i \cdot \delta_\epsilon F(A) \\
&= \int_{\mathbb{R}^N} \mathrm{dvol}\, \mathrm{Tr}\, \eta\, I^i \cdot [F, \epsilon] \\
&= \int_{\mathbb{R}^N} \mathrm{dvol}\, \mathrm{Tr} [\epsilon, \eta]\, I^i \cdot F \\
&= \Phi^i([\epsilon, \eta]) \ .
\end{aligned}$$

This means that the zero locus of the moment map Φ^i is preserved by $\mathcal{G}$ and we can consider the orbit space. Let $\mathcal{A}^0_{\mathbb{A}} \subset \mathcal{A}_{\mathbb{A}}$ denote the set of connections A for which $\Phi^i = 0$ for all i. This is nothing but the space of $\mathbb{A}$-instantons, whence the orbit space $\mathcal{A}^0_{\mathbb{A}}/\mathcal{G}$ is then the moduli space $\mathcal{M}_{\mathbb{A}}$. It is then possible to prove that the moduli space $\mathcal{M}_{\mathbb{A}}$ of $\mathbb{A}$-instantons on $\mathbb{R}^N$ is (formally) an infinite-dimensional $\mathbb{A}$-Kähler quotient of the space $\mathcal{A}_{\mathbb{A}}$ of connections. This is of course well-known for $\mathbb{A} = \mathbb{C}$ (respectively, $\mathbb{A} = \mathbb{H}$), where the moduli space inherits the structure of a Kähler (respectively, hyperkähler) manifold. It can be shown that this persists in the octonionic case. Details will appear elsewhere.

4. Instantons on riemannian manifolds

In this section we investigate whether the instanton equations (1), (2) and (5) make sense on manifolds other than $\mathbb{R}^2$, $\mathbb{R}^4$ and $\mathbb{R}^8$ respectively, and whether the interpretation in terms of moment maps persists. We will see that although the complex and quaternionic instantons make sense on any (oriented) riemannian manifold of the right dimension, the octonionic equations only make sense in a manifold whose holonomy is contained in Spin(7). Moreover the interpretation of the $\mathbb{H}$-instanton equations in terms of moment maps will force the manifold to be hyperkähler, whereas for the $\mathbb{O}$-instanton it will force it to be flat.

4.1. **The instanton equations on riemannian manifolds.**

In order for the $\mathbb{A}$-instanton equations to make sense on an arbitrary manifold, it is necessary that the structure group of the tangent bundle preserve the subbundle of 2-forms which define the equations. We will take all our manifolds to be riemannian, so that the group of the tangent bundle reduces to $\mathrm{O}(N)$. Any further reduction of the structure group can then be understood as a reduction of the holonomy of a metric connection with torsion.

In the 2-dimensional case, the bundle of 2-forms is a line bundle, hence under a change of coordinates the 2-form I will always go back to a multiple of itself. Therefore the $\mathbb{C}$-instanton equation makes sense on any 2-dimensional manifold.

In four dimensions, the 2-forms I, J, and K are a local basis for the anti-self-dual 2-forms. The maximal subgroup of $\mathrm{O}(4)$ which respects the split $\bigwedge^2 = \bigwedge^2_+ \oplus \bigwedge^2_-$ into self-dual and anti-self-dual 2-forms is $\mathrm{SO}(4)$, whence provided that the manifold is oriented, the $\mathbb{H}$-instanton equations make sense. This can also be understood from the alternate form (3) of the $\mathbb{H}$-instanton equations: we now need that the volume form $\varepsilon_{\mu\nu\rho\sigma}$ exist globally, which again means that the manifold is oriented.

In eight dimensions we obtain a stronger restriction on the manifold. The structure group must respect the split $\bigwedge^2 = \bigwedge^2_7 \oplus \bigwedge^2_{21}$, where $\bigwedge^2_7$ is the subbundle spanned by the I^i and $\bigwedge^2_{21}$ is its orthogonal complement. This latter subbundle is spanned by the antisymmetric products $I^iI^j - I^jI^i$ and hence corresponds to the Lie algebra $\mathfrak{so}(7)$. The above split is the eigenspace decomposition of the map $\bigwedge^2 \to \bigwedge^2$ defined by $\omega \mapsto \star(\Omega \wedge \omega)$ with Ω defined by (8). The maximal subgroup of $\mathrm{SO}(8)$ which preserves Ω, and hence the above split, is $\mathrm{Spin}(7)^+$. Therefore the manifold must admit a $\mathrm{Spin}(7)^+$ structure. This is not all, however. The Bianchi identity will imply that any instanton obeys the Yang–Mills equations of motion, provided that the 4-form Ω be closed. By a result of Bryant [9] this is equivalent to the holonomy group of the metric being contained in $\mathrm{Spin}(7)^+$. In other words, octonionic instantons are only defined on riemannian manifolds with holonomy contained in $\mathrm{Spin}(7)^+$. Similarly, octonionic anti-instantons are only defined on 8-manifolds admitting a metric with holonomy contained in $\mathrm{Spin}(7)^-$. Hence a generic manifold will not admit both $\mathbb{O}$-instantons and $\mathbb{O}$-anti-instantons. For this to be the case, the manifold must admit a metric whose holonomy is contained in $\mathrm{Spin}(7)^+ \cap \mathrm{Spin}(7)^- \cong G_2$, so that the manifold is locally reducible.

4.2. **Moment maps for instantons on riemannian manifolds.** Finally we investigate the persistence of the interpretation of the $\mathbb{A}$-instanton equations as the zero locus of a moment map in the space $\mathcal{A}_{\mathbb{A}}$ of connections, and hence of the moduli space as an infinite-dimensional $\mathbb{A}$-Kähler quotient.

For this to be the case, we have to endow $\mathcal{A}_{\mathbb{A}}$ with the structure of an infinite-dimensional $\mathbb{A}$-Kähler manifold. It is not hard to show that now it is no longer sufficient to preserve the subbundle of 2-forms spanned by the I^i but that each I^i must be invariant under the holonomy group. In two dimensions this constrains the surface to be Kähler, which is simply the condition that it be oriented. In four dimensions, the fact that I, J, and K are constant under the holonomy group, trivialises the bundle of anti-self-dual forms. The holonomy must then be contained in one of the Sp(1) factors in SO(4); in other words, the manifold must be hyperkähler. Finally, in eight dimensions the fact that the I^i are parallel, means that the manifold is octonionic Kähler, which as discussed in the Appendix, implies that it is flat. We summarise these results in the following table.

		($\dim \mathbb{A}$)-manifolds admitting	
$\mathbb{A}$	$\mathbb{A}$-instanton	instanton equation	quotient construction
$\mathbb{C}$	$F = 0$	arbitrary	oriented
$\mathbb{H}$	$F = \pm \star F$	oriented	hyperkähler
$\mathbb{O}$	$F \in \bigwedge^2_{21}$	Spin(7) holonomy	OK ($\implies$ flat)

Table 1. $\mathbb{A}$-instanton equations and their allowed manifolds.

5. Conclusion

In this paper we have reformulated the eight-dimensional instanton equation introduced in [10] in a way that exhibits it naturally as a member of a family of equations associated to the real division algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, and comprising flatness in dimension 2 and self-duality in dimension 4. The usual way in which the octonionic equations are presented, namely equation (7), has the advantage of suggesting generalisations to geometries in which one has a co-closed 4-form, but at the same time obscures the relative simplicity of the equations. Moreover, it does not distinguish the 8-dimensional case from the other ones, and it also treats both equation (7) and the dual equation [10],

$$F_{\mu\nu} = -\tfrac{1}{6}\Omega_{\mu\nu\rho\sigma}F_{\rho\sigma} \ , \tag{12}$$

on an equal basis. In fact, one often finds in the literature that equations (7) and (12) are referred to as the self-duality and anti-self-duality equations respectively. This nomenclature suggests a symmetry between these equations which is not present in the octonionic case since, for example, the spaces have different dimension. In our opinion, self-duality and anti-self-duality correspond to which way the division algebra $\mathbb{A}$ acts: if on the left or on the right, and are hence related by a change of orientation on the manifold. Although there has been some work in the literature concerning equation (12), we believe this equation not to be as fundamental as (7). This can already be seen not just in the results of the present paper but also, for example in [1], where it is shown that supersymmetry singles out equation (7). Another argument in favour of equation (7) is the following: at any given point in the manifold, equation (12) is a system of 21 equations for 8 unknowns and is hence over-determined; whereas on the other hand, equation (7) is a system of 7 equations (and the Bianchi identity) for 8 unknowns.

There is yet a third notion of eight-dimensional instantons associated to the octonions which has appeared in the literature. It is a classical observation credited to Trautman, that the natural connection on the Hopf bundle $S^3 \to S^7 \to S^4$ is a Yang–Mills instanton. Departing from this observation, several authors [17, 25, 22, 18] sought to endow the "last" Hopf map $S^7 \to S^{15} \to S^8$ with a similar gauge-theoretic interpretation, this time in eight dimensions. The 8-dimensional instanton obtained from the last Hopf map, however, is not a minima of the standard Yang–Mills action functional but of one which is quartic in the curvature. It is of course also the octonionic member in a sequence of gauge theoretic objects, namely the Hopf maps. As discussed above, the octonionic instanton equations (12) cannot be defined on S^8, in contrast with the ones coming from the Hopf map. Both equations are defined on $\mathbb{R}^8$ but as the (standard) quadratic Yang–Mills action is not conformally invariant in eight-dimensions, the equations do not extend to S^8. In contrast, the instanton equations associated to the Hopf map do. This can be seen in two ways. First of all, they minimise a conformally invariant action. Secondly, these equations imply the self-duality in eight dimensions of a 4-form constructed by squaring the Yang–Mills curvature, and it is well-known that the Hodge star operator acting on middle-dimensional forms is conformally invariant.

The original motivation for this paper was to examine the moduli space of octonionic instantons for Spin(7) holonomy 8-manifolds. Alas, we have found that unless the manifold is flat, the moduli space cannot be described as an octonionic Kähler quotient. Nevertheless the geometry of the manifold on which the instanton equations are defined does influence the geometry of the moduli space of instantons. For example, the moduli space of instantons on a Kähler 4-manifold is itself Kähler, even though it loses its interpretation as a Kähler quotient. Similarly it is possible to show that if the holonomy of the 8-manifold is further reduced, say to a subgroup of $SU(4) \subset \mathrm{Spin}(7)$, then the moduli space inherits a Kähler structure. In this case, the instanton equations are the celebrated Donaldson–Uhlenbeck–Yau equations.

In [14] we used supersymmetry to exhibit a relation between two very different spaces: the octonionic instanton moduli space on an 8-manifold $M \times K$, where M and K are hyperkähler 4-manifolds in the limit in which K shrinks to zero size; and the space of triholomorphic curves (or hyperinstantons) $M \to \mathcal{M}_{\mathbb{H}}(K)$. It seems plausible that the results in this paper can be used to understand the geometry of the space of hyperinstantons better.

In analogy to what happens in four dimensions, certain octonionic instantons can be understood as monopoles in seven dimensions. These equations, which generalise the Bogomol'nyi equation, can be defined on any riemannian 7-manifold M of G_2 holonomy. Very little is known about the moduli spaces of these monopoles, but it follows from the results in this paper that when M is flat, the moduli space is OK.

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Appendix A. Some octonionic geometry

In this appendix we summarise the basic notions of octonionic geometry *as used in this paper*. Octonionic geometries and their torsioned generalisations have been studied recently in [16] where they are exhibited as the geometries of the moduli space of some solitonic black holes. As these authors never define the term octonionic Kähler, the definition above is not in conflict with that paper. Nevertheless, our definition is rather narrow and in order to obtain interesting geometries one must relax it, either as was done in [16] or alternatively as we suggest below.

The existence of an octonionic Kähler structure on a riemannian manifold imposes strong constraints on the manifold. First of all we have that the dimension of a finite-dimensional octonionic (almost) hermitian manifold X is divisible by 8. This follows from the fact that each tangent space T_pX admits an action of $\mathrm{C}\ell(7)$, whose representations are always $8k$-dimensional. The geometry is also very constrained. For example, an 8-dimensional OK manifold X is necessarily flat. This can be proven as follows. The fact that X is OK means that it is Kähler with respect to each of the complex structures I^i, whence $\nabla I^i = 0$. Because the I^i generate $\mathrm{C}\ell(7)$, it follows that the holonomy group commutes with the action of $\mathrm{C}\ell(7)$. Since $\mathrm{C}\ell(7)$ acts irreducibly on the tangent space, the restricted holonomy group is trivial. It seems rather likely that the geometry of OK manifolds is similarly constrained in higher dimensions.

This prompts us to try to generalise OK geometry in such a way that it admits interesting examples. For example, as done in [16] one can relax the condition that ∇ be torsionless and also substitute $\nabla I^i = 0$ with a weaker condition (see (3.33) in [16]). This yields the so-called OKT geometries. Another approach, more in line with quaternionic Kähler geometry, would be to demand that the almost complex structures I^i, satisfying (4), only exist locally. Then one would impose that the 7-dimensional subbundle of the 2-forms spanned by the I^i, instead of being trivial, be preserved by the holonomy group. In eight dimensions, as we saw in Section 4.1, this singles out those riemannian manifolds whose holonomy group is contained in Spin(7). In $8k \geq 16$ dimensions any such manifold must be reducible, as can be gleaned from Berger's list of irreducible holonomy representations. Nevertheless it seems tempting to try and develop a theory of such manifolds and in particular to try to use them to construct 8-dimensional Spin(7) holonomy manifolds by a quotient construction.

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