INSTITUT DES HAUTES ETUDES SCIENTIFIQUES

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SEMINAR ON COMBINATORIAL TOPOLOGY

by

Eric-Christopher ZEEMAN

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FASCICULE 1

(Exposés I à V inclus)

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Seminar on Combinatorial Topology

by E.C. ZEEMAN

INTRODUCTION

The purpose of these seminars is to provide an introduction to combinatorial topology. The topics to be covered are :

1. The combinatorial category and subdivision theorems,
2. The polyhedral category,
3. Regular neighbourhoods,
4. Unknotting of spheres,
5. General Position,
6. Engulfing lemmas,
7. Embedding and isotopy theorems.

At first sight the unattractive feature of combinatorial theory as applied to manifolds is the kinkiness and unhomogeneity of a complex as compared with the roundness and homogeneity of a manifold. However this is due to a confusion between the techniques and subject matter. We resolve this confusion by separating into two different categories the tools and objects of study. The tools in the combinatorial category we keep as special as possible, namely finite simplicial complexes embedded in Euclidean space.
These possess two crucial properties :

i) finiteness, and the use of induction

ii) tameness, and niceness of intersection.

Meanwhile objects of study we make as general as possible. Our definition of polyhedral category contains not only

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i) polyhedra;

ii) manifolds (bounded or not, compact or not),

but also the following spaces, which have not been given a combinatorial structure before :

iii) non-paracompact manifolds, for example the Long Line;

iv) infinite dimensional manifolds, for example the orthogonal group,

v) joins of non-compact spaces; for example the suspension of an open interval.

vi) function spaces; for example the space of all piecewise linear embeddings of compact manifold in another manifold.

As the examples show, a polyhedral space need not be triangulable, and if it is, it does not have a specific triangulation, but is a set with a structure. The structure is, roughly speaking, a maximal family of subpolyhedra, and the structure determines the topology.

Our theory is directed towards the study of manifolds, and in particular of embeddings and isotopies. Recently it has become apparent that combinatorial results differ substantially from differential results; a striking case is S^3 in S^6, which knots differentially, and unknots combinatorially. In fact combinatorial theory seems to behave well in, and to have techniques to handle, most situations with codimension $\geqslant 3$. Just as differential theory behaves well and can handle most situations in the stable range.

We shall therefore concentrate on geometry in codimension $\geqslant 3$. This means we shall neglect a number of interesting and allied topics that depend more on algebra, for example

i) codimension 2

ii) immersion theory

iii) relations with differential theory.

Chapter I : THE COMBINATORIAL CATEGORY

Simplexes

Let E^p denote Euclidean p-space. An n-simplex $(n \geqslant 0)$ A in E^p is the convex hull of $n+1$ linearly independent points.
We call the points *vertices*, and say that A *spans* them. A is closed and compact; $\dot{A}$ denotes the boundary, A the interior. A simplex B spanned by a subset of the vertices is called a *face* of A, written $B < A$. Simplexes A,B are *joinable* if their vertices are linearly independent. If A, B are joinable we define the *join* AB to be the simplex spanned by the vertices of both; otherwise the join is undefined.

Complexes

A *finite simplicial complex*, or *complex*, K in E^p is a finite collection of simplexes such that

(i) if $A \in K$, then all the faces of A are in K,

(ii) if $A, B \in K$, then $A \cap B$ is empty or a common face.

The *star* and *link* of a simplex A K are defined :

$$st(A,K) = \{B;\ A < B\}, \qquad lk(A,K) = \{B;\ AB \in K\}.$$

Two complexes K,L in E^p are *joinable* provided :

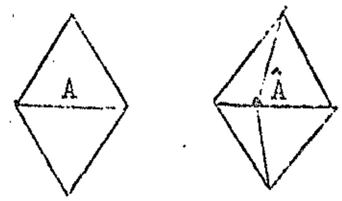
(i) if $A \in K$, $B \in L$ then A,B joinable

(ii) if $A,A' \in K$ and $B,B' \in L$, then $AB \cap A'B'$ is empty or a common face.

If K,L are joinable, we define the *join* $KL = K \cup L \cup [AB; A \in K, B \in L]$; otherwise the join is undefined.

The underlying point set $|K|$ of K is called a *euclidean polyhedron* :

L is called a <u>subdivision</u> of K if $|L| = |K|$, and every simplex of L is contained in some simplex of K.



<u>Examples</u>, 1) Choose a point $\hat{A} \in \mathring{A}$. Let

$$L = (K - st(A,K)) \cup \hat{A}\dot{A}\ lk(A,K).$$

Then L is a subdivision of K, and we say L is obtained from K by starring A (at $\hat{A}$).

2) <u>A first derived</u> $K^{(1)}$ of K is obtained by starring all the simplexes of K in some order such that if $A > B$ then A preceedes B (for example in order of decreasing dimension).
Another way of defining $K^{(1)}$ is to define the subdivision of each simplex, inductively in order of increasing dimension, by the rule $A' = \hat{A}\dot{A}'$. Therefore a typical simplex of $K^{(1)}$ is $\hat{A}_0 \hat{A}_1 \ldots \hat{A}_p$, where $A_0 < A_1 < \cdots < A_p$ in K. An <u>r^{th} derived</u> $K^{(r)}$ is defined inductively as the first derived of an $(r-1)^{th}$ derived. <u>The barycentric first derived</u> is obtained by starring at the barycentres.

<u>Convex linear cells</u>

A <u>convex linear cell</u>, or <u>cell</u>, A in E^p is a non-empty compact subset given by
$$\begin{cases} \text{linear equations} & f_1 = 0, \ldots, f_r = 0 \quad \text{and} \\ \text{linear inequalities} & g_1 \geq 0, \ldots, g_s \geq 0 . \end{cases}$$

A <u>face</u> B of A is a cell (i.e. non-empty) obtained by replacing some of the inequaties $g_i \geq 0$ by equations $g_i = 0$.

The 0-dimensional faces are called vertices. It is easy to deduce the following elementary properties :

1) A is the convex hull of its vertices
2) A is a closed compact topological n-cell, where $n + 1$ is the maximum number of linearly independent vertices,

3) A simplex is a cell.
4) The intersection or product of two cells is another.
5) Let x be a vertex of the cell A, and let B be the union of faces of A not containing x.
   Then $A =$ the cone xB.

A <u>convex linear cell complex</u>, or <u>cell complex</u>, K is a finite collection of cells such that

(i) if $A \in K$, then all the faces of A are in K,
(ii) if $A, B \in K$, then $A \cap B$ is empty or a common face.

<u>Lemma 1</u> : <u>A convex linear cell complex can be subdivided into a simplicial complex without introducing any more vertices.</u>

<u>Proof</u> : Order the vertices of the cell complex K. Write each cell A as a cone $A = xB$, where x is the first vertex. Subdivide the cells inductively, in order of increasing dimension. The induction begins trivially with the vertices. For the inductive step, we have already defined the subdivision A' of A, and so define A' to be the cone $A' = xB'$. The definition is compatible with subdivision C' of any face C of A containing x, because since x is the first vertex of A, it is also the first vertex of C. Therefore each cell, and hence K, is subdivided into a simplicial complex.

<u>Corollary 1</u> : <u>The underlying set of a cell complex is a euclidean polyhedron.</u>

<u>Corollary 2</u> : <u>The intersection or product of two euclidean polyhedra is another.</u>

For the intersection or product of simplicial complexes is a cell complex.

<u>Maps.</u>

Suppose K in E^p, L in E^q.

A <u>map</u> $f : K \to L$ is a continuous map $|K| \to |L|$.

Call f <u>simplicial</u> if it maps vertices to vertices and simplexes linearly to simplexes. Call f an <u>isomorphism</u>, written $f : K \cong L$, if it is a simplicial homeomorphism. The graph Γf of f is defined as usual

$$\Gamma f = \left\{ (x,fx) \; ; \; x \in |K| \right\} \subset |K| \times |L| \subset E^{p+q} .$$

Call f <u>piecewise linear</u> if either of the two definitions hold :

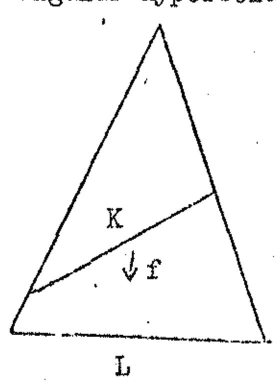
(1) The graph Γf of f is a euclidean polyhedron

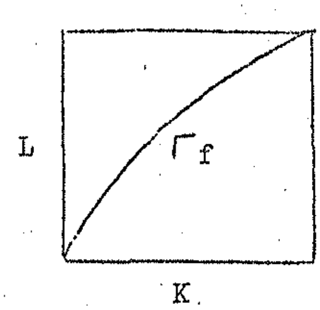
(2) There exist subdivision K', L', of K,L with respect to which f is simplicial .

Notice that condition (2) clearly implies condition (1), because the graph of a linear map from a simplex to a simplex is a simplex, and so the graph of a simplicial map $K \to L$ is a complex isomorphic to K. We shall prove the converse, and therefore the equivalence of the two definitions, in Lemma 7 . Definition (1) is the aesthetically simpler, while definition (2) is the one which is used continually in practice.

The reader is warned against the <u>standard mistake</u> of confusing projective maps with piecewise linear maps. For example the projection onto the base of a triangle from the opposite vertex of a line not parallel to the base is not piecewise linear .

Infact the graph Γf in the square $|K| \times |L|$ is part of a rectangular hyperbola .





<u>Lemma 2</u> : <u>The composition of two piecewise linear maps is another</u> .

<u>Proof</u> : We use definition (1), Given $K \xrightarrow{f} L \xrightarrow{g} M$, let

$$\Gamma = (\Gamma f \times |M|) \cap (|K| \times \Gamma g) \subset E^p \times E^q \times E^r .$$

Then Γ consists of all points (x, fx, gfx) , $x \in |K|$
Therefore the projection $\pi : E^p \times E^q \times E^r \longrightarrow E^p \times E^r$
maps Γ homeomorphically onto $\Gamma(g f)$.
Since f, g are piecewise linear, Γ is a euclidean polyhedron by Lemma 1 Corollary 2 . The image under the linear projection π of any complex triangulating Γ gives an isomorphic complex triangulating $\Gamma(gf)$. Hence $\Gamma(gf)$ is a euclidean polyhedron, and gf is piecewise linear .

<u>Definition</u> : Lemma 2 enables us to define the <u>combinatorial category</u> $\mathcal{C}$ with

$$\begin{cases} \text{objects : finite simplicial complexes} \\ \text{maps : piecewise linear maps.} \end{cases}$$

We shall also need the <u>subcategory of embeddings</u> $\mathcal{E}$ with

$$\begin{cases} \text{the same objects} \\ \text{maps : injective piecewise linear maps.} \end{cases}$$

We proceed to prove some useful subdivision theorems .

<u>Lemma 3</u> : <u>If $K \supset L$, then (i) any subdivision K' of K induces a subdivision L' of L , and (ii) any subdivision L' of L can be extended to a subdivision K' of K</u> .

<u>Proof</u> : (i) is obvious
(ii) subdivide, inductively in order of increasing dimension, those simplexes of $K-L$ that meet L , by the rule $A' = \hat{A}\dot{A}'$ where $\hat{A}$ is an interior point .

<u>Corollary</u> : <u>Given a simplicial embedding $f : K \to L$, and a subdivision K' of K, there exists a subdivision L' of L such that $f : K' \to L'$ is simplicial</u>.

<u>Lemma 4</u> : <u>If $|K| \supset |L|$, then there exists an r^{th} derived $K^{(r)}$ of K and a subdivision L' of L such that L' is a subcomplex of $K^{(r)}$</u>.

<u>Proof</u> : By induction on the number of simplexes of L. The induction starts trivially when $L = \emptyset$. If A is a principal simplex of L (principal means not the face of another), then by induction choose $K^{(r-1)}$ to contain a subdivision of $L-A$.

Choose a derived $K^{(r)}$, by starring each simplex $B \in K^{(r-1)}$ at a point in $\mathring{A} \cap \mathring{B}$ if $\mathring{A}$ meets $\mathring{B}$, and arbitrarily otherwise. Then $K^{(r)}$ contains subdivision of $L-A$, A and hence of L.

<u>Corollary 1</u>. <u>If $|K| = |L|$, then K, L have a common subdivision</u>.

<u>Corollary 2</u>. <u>If $|K| \supset |L_i|$, $i = 1,\ldots,r$, then there exist subdivision K', L'_i such that all the L'_i are subcomplexes of K'</u>.

<u>Corollary 3</u>. <u>The union of two euclidean polyhedra is another</u>.

For subdivide a large simplex containing them both, so that each appears as a subcomplex. The union is also a subcomplex.

<u>Lemma 5</u> : <u>Given a simplicial map $f : K \to L$, and a subdivision L' of L, then there exists a subdivision K' of K such that $f : K' \to L'$ is simplicial</u>.

<u>Proof</u> : Let $K_1 = f^{-1}L'$, which is a cell complex subdividing K. By Lemma 1 we can choose a simplicial complex K' subdividing K_1, introducing no new vertices. Then each simplex of K' is

mapped linearly to a simplex of L', and so $f : K' \to L'$ is simplicial.

<u>Définition</u> : A map $f : K \to E^q$ is <u>linear</u> if each simplex is mapped linearly.

<u>Lemma 6</u> : <u>Let e be the inclusion $L \subset E^q$. Given a map $f : K \to L$, such that $ef : K \to E^q$ is linear, then there exist subdivisions K', L' of K, L with respect to which f is simplicial</u>.

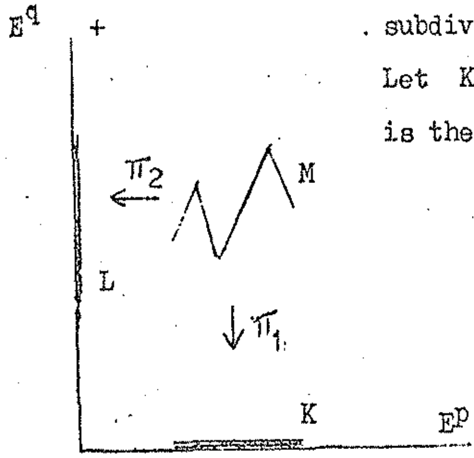
<u>Proof</u> : If $A_i \in K$, let $B_i = f A_i$.

By linearity B_i is a cell, possibly of lower dimension than A_i, and $|B_i| \subset |L|$. By Lemma 4 Corollary 2, choose simplicial subdivisions L', B_i' of L, B_i such that each B_i' is a subcomplex of L'. Then for each i, $f^{-1}B_i'$ is a cell complex subdividing A_i, and the union $f^{-1}L'$ is a cell complex subdividing K. By Lemma 1 choose a simplicial subdivision K' of $f^{-1}L'$, introducing no new vertices. Then $f : K' \to L'$ is simplicial.

<u>Lemma 7</u> : <u>The two definitions of piecewise linearity are equivalent</u>.

<u>Proof</u> : We have observed $(2) \Longrightarrow (1)$ trivially. Therefore we shall prove $(1) \Longrightarrow (2)$. Suppose K in E^p, L in E^q and let $f : K \to L$ be a map whose graph Γf is a euclidean polyhedron. In other words, there exist a complex M in E^{p+q} such that $|M| = \Gamma f$.

The projection $E^p \times E^q \xrightarrow{\pi_1} E^p$ maps M homeomorphically onto K, and linearly into E^p; therefore by Lemma 6, there exist subdivisions M', K' with respect to which π_1 is simplicial. Hence $\pi_1 : M' \to K'$ is an isomorphism. Similarly



$\pi_2 : E^p \times E^q \to E^q$ maps M into L (not necessarily homeomorphically), and linearly into E^q; therefore there exist subdivisions M'', L' with respect to which π_2 is simplicial. Let K'' be the subdivision of K' isomorphic to M''. Then f is the composition of the simplicial maps

$$K'' \xrightarrow{\pi_1^{-1}} M'' \xrightarrow{\pi_2} L' .$$

Hence $f : K \to L$ is piecewise linear by definition (2)

Let T be a finite subset of $\mathcal{C}$, such that if a map is in T so is its range and domain. The <u>diagram</u> of T is the 1-complex obtained by replacing each complex by a vertex and each map by an edge. Call T a <u>tree in $\mathcal{C}$</u> if its diagram is simply-connected. Call T <u>one-way</u> if each complex is the domain of at most one map. Therefore in a one-way tree there is exactly one complex that is the domain of no map, and every other complex is the domain of exactly one map. Call T <u>simplicial</u> if every map of T is simplicial. Call T' a <u>subdivision</u> of T if it has the same diagram, and each complex of T' is a subdivision of the corresponding complex of T, and each map of T' (qua map between the underlying polyhedra) is the same as the corresponding map of T.

<u>Theorem 1</u> : <u>If T is a one-way tree in $\mathcal{C}$, or a tree in $\mathcal{S}$, then T has a simplicial subdivision</u>.

<u>Proof</u> by induction on the number of maps in T. Let T be a one-way tree in $\mathcal{C}$.

The induction begins trivially with no maps.

Suppose T has at least one map. Then there exist complex K and a map $f : K \dashrightarrow L$ in T, such that K is not the range or domain of any other map in T.

By Lemma 7, there exist subdivisions K', L' of K, L with respect to which f is simplicial. Let T_* be the one-way tree obtained from T by omitting K and f, and replacing L by L'. By induction there is a simplicial subdivision T_*' of T_*. In particular T_*' contains a subdivision L'' of L'. By Lemma 5 there exists a subdivision K'' of K' such that $f : K'' \to L''$ is simplicial. Let $T' = T_*'$ together with K'' ans f. Then T' is a simplicial subdivision of T.

Now suppose T is a tree in $\mathcal{C}$, not necessarily one-way. There is a complex K which is the range or domain of exactly one map. If K is the domain, proceed as before. If K is the range, let the map be $f : L \to K$. Proceed as before, except that we can use the Collorary to Lemma 3 instead of Lemma 5 to form K'', since f is an embedding. The proof of Theorem 1 is complete.

The following two examples show that the hypotheses of Theorem 1 are necessary as well as sufficient.

<u>Example 1</u>. It is necessary that a tree in $\mathcal{C}$ be one-way, otherwise it contains a subtree

$$\begin{tikzcd} & L \\ K \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & M \end{tikzcd}$$

We can choose f, g so that there exists no simplicial subdivision as follows :

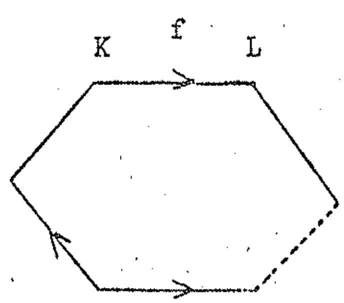
Let $K = L = M = I$, the unit interval, and let

$$f \text{ map } \begin{cases} 0 \longrightarrow 0 \\ 1/3 \longrightarrow 1 \\ 1 \longrightarrow 0 \\ [0,1/3] , [1/3,1] \qquad \text{linearly ,} \end{cases}$$

$$g \text{ map } \begin{cases} 0 \longrightarrow 0 \\ 2/3 \longrightarrow 1 \\ 1 \longrightarrow 0 \\ [0,2/3] , [2/3,1] \qquad \text{linearly .} \end{cases}$$

Suppose there is a simplicial subdivision, containing K' .
Let p,q,r be the numbers of vertices of K' between, respectively, 0 and 1/3, 1/3 and 2/3, 2/3 and 1 .
Since f is simplicial on K' , we have $p = q + 1 + r$.
From g similarly, $p + 1 + q = r$. Hence $q = -1$ a contradiction .
Therefore there is no simplicial subdivision .

<u>Example 2</u> . It is necessary that the diagram in $\mathcal{E}$ be a tree, otherwise it contains a circular subdiagram

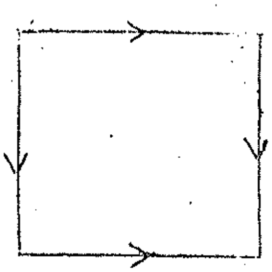


We can choose the maps so that there is no simplicial subdivision as follows : Let all the complexes be I , and all the maps be the identity except f , and let

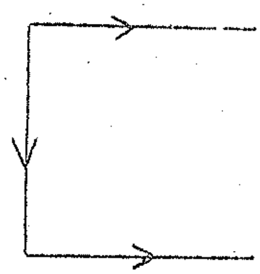
$$f \text{ map } \begin{cases} 0 \longrightarrow 0 \\ 1/3 \longrightarrow 2/3 \\ 1 \longrightarrow 1 \\ [0,1/3] \quad [1/3\ ,\ 1] \quad \text{linearly .} \end{cases}$$

Suppose there is a simplicial subdivision containing $f : K' \to L'$. Going round all the other maps we have the identity map simplicial, and so $K' = L'$. Using the same notation as in Example 1 , since f is simplicial, we deduce $p = p + 1 + q$. Hence $q = -1$, again a contradiction. Therefore T has no simplicial subdivision .

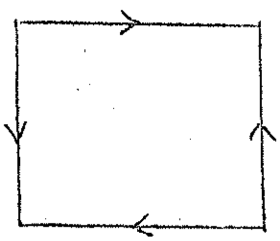
Remark . A "commutative diagram" in $\mathcal{E}$ has simplicial subdivision if the maps are determined by a maximal tree. For example :



is determined by



but



is not determined by a tree.

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Seminar on Combinatorial Topology

by E.C. ZEEMAN

Chapter 2 : THE POLYHEDRAL CATEGORY

In this chapter we give mainly definitions and examples to describe the category. We omit the proofs to most statements to make the reading easier, and because later chapters do not depend on them.

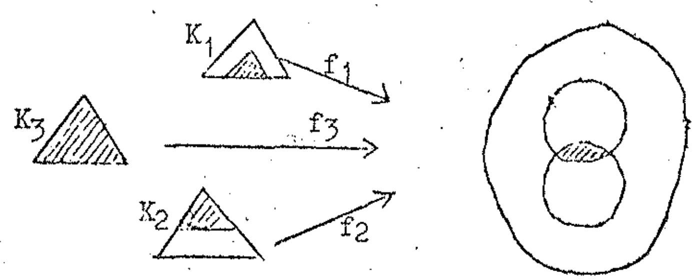
Let X be a set (without as yet any topology). A polyhedron in X is an injective function $f : K \to X$ where K is a finite simplicial complex. By a function we mean, as usual, a function from the set of points of the underlying euclidean polyhedron $|K|$ to the set X. We write

$$\operatorname{dom} f = K \quad , \qquad \operatorname{im} f = fK \; .$$

Two polyhedron $f_1 : K_1 \to X$ and $f_2 : K_2 \to X$ are related if there is a third $f_3 : K_3 \to X$, such that

$$\text{i)} \quad \operatorname{im} f_3 = \operatorname{im} f_1 \cap \operatorname{im} f_2$$

$$\text{ii)} \quad f_1^{-1} f_3 \, , \; f_2^{-1} f_3 \in \mathcal{E} \; .$$



A *family* $\mathcal{F}$ of polyhedra in X is a set in which any two are related. Write $\operatorname{im} \mathcal{F} = \{ \operatorname{im} f \,;\, f \in \mathcal{F} \}$.

A *polystructure*, (or more briefly a *structure*), $\mathcal{F}$ on X is a family such that

i) $\operatorname{im} \mathcal{F}$ covers X

ii) $\operatorname{im} \mathcal{F}$ is a lattice of subsets of X

iii) $\mathcal{F}\mathcal{E} = \mathcal{F}$

The last condition means that given $K \xrightarrow{g} L \xrightarrow{f} X$ with $f \in \mathcal{F}$ and g a piecewise linear embedding, then $fg \in \mathcal{F}$.

A *polyspace* $X = (X, \mathcal{F})$ is a set X together with a polystructure $\mathcal{F}$ on X.

Topology

The *topology* $T(\mathcal{F})$ of the structure $\mathcal{F}$ is the identification topology

$$X = \operatorname{dom} \mathcal{F} / \mathcal{F}.$$

Here $\operatorname{dom} \mathcal{F}$ means the disjoint union of the euclidean polyhedra $\{ \operatorname{dom} f \,;\, f \in \mathcal{F} \}$, and the identification is given by $\mathcal{F} : \operatorname{dom} \mathcal{F} \to X$.

We can deduce (non-trivially) :

i) Each $f : X \to X$ is a homeomorphism into

ii) A set $U \subset X$ is open (or closed) if and only if $U \cap f K$ is open (or closed) in $f K$, each $f \in \mathcal{F}$.

If X is a topological space, then a polystructure on X is one with the same topology.

Example 1. The *discrete* structure on a set X is given by maps of points into X. This gives the discrete topology.

<u>Example 2</u> . The natural structure $\mathcal{F}(E^n)$ on Euclidean space E^n is the set of all piecewise linear embeddings $K \rightarrow E^n$. This gives the natural topology .

<u>Example 3</u> . The natural structure $\mathcal{F}(K)$ on a complex K is the set of all piecewise linear embeddings $L \rightarrow K$. The natural structure on the euclidean polyhedron $|K|$ is the same .

<u>Example 4</u> . Suppose $f : |K| \rightarrow X$ is a homeomorphism from a euclidean polyhedron onto a topological space X . Then $f\mathcal{F}(K)$ gives a polyhedral structure $\mathcal{F}(X)$ on X .

We call X , with this structure, a <u>polyhedron</u> .

Notice that $\mathcal{F}(X)$ contains the triangulation f , and all related triangulations . Conversely the structure is uniquely determined by any triangulation in it .

<u>Remark 1</u> . We have used the word polyhedron in three ways

i) euclidean-polyhedron

ii) polyhedron-in-a-set

iii) polyhedron .

The usage is coherent, because (i) with its natural structure is an example of (iii) , and the image of (ii) with its induced (see below) structure is an example of (iii) .

<u>Remark 2</u> . It is possible to have many structures on a set ; more examples are given below . However it can be shown (non-trivially) that

1) Any structure is maximal with respect to its topology : the topology of a strictly smaller structure is strictly finer (more open sets) .

2) The natural structures of E^n and of polyhedra are maximal .

Subspaces

Let $\mathfrak{X} = (X, \mathfrak{F})$ be a polyspace. If $Y \subset X$, we define the <u>induced structure</u> on Y to be

$$\mathfrak{F} | Y = \{ f \in \mathfrak{F} \ ; \ \text{im} f \subset Y \} .$$

It is easy to verify that $\mathfrak{F} | Y$ is a polystructure on Y.

We call $Y = (Y, \mathfrak{F} | Y)$ a <u>polysubspace</u> if it has the induced topology :

$$T(\mathfrak{F} | Y) = T(\mathfrak{F}) | Y .$$

In general $T(\mathfrak{F} | Y)$ has a finer topology.

<u>Example 1</u>. E^n is a polysubspace of S^n.

This is a particularly satisfactory example, because combinatorially it is always a little embarrassing to regard the infinite triangulation of E^n as a satisfactory "substructure" of the finite triangulation of S^n.

We state elementary properties of polysubspaces, leaving the proofs to the reader :

i) Any open set of X is a polysubspace.

ii) Any polyhedron in X is a polysubspace.

iii) A polysubspace of a polysubspace is a polysubspace.

iv) The intersection of two polysubspaces is a polysubspace.

Therefore the notion of polysubspace substantially enlarges the concept of "tame" set to include both polyhedra and open sets.

<u>Example 2</u>. The union of two polysubspaces is not necessarily poly. For example let A = open disk in E^2

B = a boundary point.

Then $A \cup B$, with structure $\mathcal{F} \mid A \cup B$ is locally compact; a compact neighbourhood of B in $A \cup B$ is a closed disk D, having B on its boundary, and with $D - B \subset A$. But with the induced topology $A \cup B$ is <u>not</u> locally-compact, because B has no compact neighbourhood.

<u>Example 3</u>. A circle in E^2 is not a polysubspace, because the induced structure is discrete.

<u>Example 4</u>. A closed disk in E^2 is not a polysubspace. With the induced structure it is non-compact; any subset of the boundary being closed. It is like the Prüfer manifold with each attached disk shrunk to a point.

<u>Maps</u>

A function $f : X \to Y$ between two polyspaces is called a <u>polymap</u> if $f\mathcal{F}(X) \subset \mathcal{F}(Y)\mathcal{C}$.

In other words, given $g \in \mathcal{F}(X)$, then fg can be factored through the structure of Y, $fg = g'f'$ for some $g' \in \mathcal{F}(Y)$, where f' is piecewise linear,

$$\begin{array}{ccc} K & \xrightarrow{f'} & L \\ \downarrow g & & \downarrow g' \\ X & \xrightarrow{f} & Y \end{array}$$

It is easy to deduce

1) A polymap is continuous with respect to the structure topologies, and is therefore a map between the underlying topological spaces.

2) $f : K \to L$ is a polymap if and only if it is piecewise linear.

3) Identities and compositions of polymaps are polymaps.

Therefore we can define the polyhedral category $\mathcal{P}$ to consist of polyspaces and polymaps .

Call a polymap a polyhomeomorphism written $f : X \cong Y$, if $f\,\mathcal{F}(X) = \mathcal{F}(Y)$.

We deduce 1) it is a homeomorphism, and

2) f^{-1} is also a polyhomeomorphism .

Call a polymap a polyembedding , written $f : X \subset Y$, if it is an embedding , (i.e. a polyhomeomorphism only a polysubspace of Y) .

We deduce 3) $f : X \to Y$ is a polymap if and only if its graph

$1 \times f : X \to X \times Y$ is a polyembedding .

Remark

It would be natural to call a polymap $f : X \to Y$ injective if $f\,\mathcal{F}(X) = \mathcal{F}(Y) \mid fX$. This definition is weaker than polyembedding , because it does not require the image fX to be a polysubspace of Y . But it is of interest for the following reason . Consider the categories :

(1) space and embeddings

(2) polyspaces and injective polymaps

(3) polyspaces and polyembeddings.

Then (1) $\cap$ (2) = (3) . Now some constructions such as join and mapping cylinder are functorial in (2) but not in (1) , and therefore not in (3) . For these constructions the polystructure is more natural than its accompanying topology .

Bases.

A base $\mathcal{B}$ for a polystructure on a set X is a family of polyhedra such that im $\mathcal{B}$ covers X (i.e. only the first structure axiom). As with structures, the topology $T(\mathcal{B})$ is the identification topology $X = \operatorname{dom} \mathcal{B}/\mathcal{B}$. We say $\mathcal{B}$ is a base for $\mathcal{F}$ if

i) $\mathcal{B} \subset \mathcal{F}$

ii) every set of im $\mathcal{F}$ is contained in a finite union of sets of im $\mathcal{B}$.

We can deduce

1) Every structure has a base (trivially).

2) Every base is the base for a unique structure; and the base and structure have the same topology.

Example 1. Any polyspace has a base of simplexes, $\mathcal{B} = \{ f \in \mathcal{F} ; \operatorname{dom} f = \text{simplex} \}$.

Example 2. E^n has a base of all n-simplexes.

Example 3. A polyhedron X has a base of one element, namely a triangulation $f : K \to X$.

Example 4. The Woven Square. Let X be the square I^2. Let $\mathcal{B}$ be the base consisting of all horizontal and vertical intervals, or, more precisely, all horizontal and vertical linear embeddings of I. The resulting structure is smaller than the natural structure, because it contains no 2-dimensional polyhedra. The resulting topology is finer than the natural topology, and is therefore Hausdorff, but is not locally compact, nor simply-connected. A typical open neighbourhood of a point looks like a maltese cross. Any subset of the diagonal is a closed set.

Example 5. (The pathological Woven Square). We enlarge the structure of the Woven Square by weaving in one more thread so badly, that it produces a non-Hausdorff topology . Let $d: I \to I^2$ be the diagonal map ; and let $e: I \to I$ be the function that is the identity on the irrationals, but reflects tho rationals about the mid-point . We add to the base of the structure of the Woven Square one more element , the polyhedron $f = de : I \to I^2$. The topology of the Woven Square is thereby coarsened , so that the ends of the diagonal cannot be separated by disjoint open sets . (The proof uses measure theory, and depends upon the non-countability of the base) .

<u>Definition</u>, We call a base $\mathfrak{B}$ <u>topological</u> if it is also a base for the topology $T(\mathfrak{B})$ in the following sense : given $x \in X$, there exists $f \in \mathfrak{B}$, such that $\operatorname{im} f$ is a (closed) neighbourhood of x in X in the topology $T(\mathfrak{B})$. For instance , in Example 2 above , the set of all n-simplexes in E^n is a topological base , But in Example 4 , the base for the woven square is not topological . The structures for infinite manifolds and function spaces that we give below will not be topological .

<u>Triangulable Spaces</u>

The pathological examples 4 and 5 above indicate some of the consequences of the definitionsof polyspace . However since our interest lies towards manifolds, we do not stress the pathology , but rather use it to obtain insight into the structure of important polyspaces such as function spaces . One of the advantages of polyspace is that it is more general than the triangulable space , even if we use infinite triangulations . In fact we avoid infinite triangulations , because we regard them as alien to the subject , being too diffuse a tool , and defining too restrictive a space . The algebraic elegance of infinite complexes should not be confused with their geometric limitations . However it is worth mentioning the relationship between polyspaces and triangulable spaces .

Given a polyspace X , then there are six possibilities :

i) X is a polyhedron , i.e. its structure contains finite triangulations .

ii) X is not a polyhedron , but we can enlarge the structure of X to be a polyhedron - for example the woven square .

iii) There is a locally-finite infinite triangulation $f : K \to X$, whose restriction to any finite subcomplex is in the structure - as for example in E^n .

If X is connected , then a necessary and sufficient condition for this is that the structure have a countable topological base . A consequence is that the topology is paracompact , Hausdorff , and locally compact .

iv) The structure can be enlarged to give (iii) .

v) The structure is maximal , but (i) and (iii) are not true ; for example the Long Line (see below) .

vi) The structure is not maximal , but (ii) and (iv) are not true ; for example the pathological woven square, or ∞ -dimensional manifolds, or function spaces (see below) .

C o m p a c t n e s s

<u>Question 1</u> . Is a compact polyspace a polyhedron ? The answer is yes if it has a countable base , or if it has a topological base , but is unsolved otherwise .

<u>Question 2</u> . Does the lattice of compact subsets of a polyspace refine the lattice of polyhedra ?

The question is important for studying the homotopy structure of function spaces .

M a n i f o l d s

An <u>n-polyball</u> is a polyhedron triangulated by an n-simplex .
An <u>n-polysphere</u> is a polyhedron triangulated by the boundary of an (n+1)-simplex.

<u>Definition</u> : an <u>n-polymanifold</u> M is a polyspace , each point of which has an n-polyball neighbourhood .

More precisely , each point has a closed neighbourhood (with respect to the structure topology) which is a polysubspace , and which, with the induced structure , is an n-polyball . The <u>boundary</u> $\dot{M}$ is the closed polysubspace of those points which lie on the boundary of their neighbourhoods , and is an (n-1)-polymanifold . The <u>interior</u> $\mathring{M} = M - \dot{M}$ is the complementary open polysubspace .

We call M <u>closed</u> if compact and $\dot{M} = \emptyset$.
<u>bounded</u> if compact and $\dot{M} \neq \emptyset$.
<u>open</u> if non-compact and $\dot{M} = \emptyset$.

If M compact then any triangulation in the structure is a combinatorial manifold (i.e. the link of every vertex is an (n-1)-sphere or ball according as to whether the vertex is in the interior or boundary ; the proof is by verifying that the property is invariant under subdivision, and true in an n-simplex) .

<u>Example 1</u> . The <u>Long Line</u> is obtained by filling in (with unit intervals) all the ordinals up to the first non-countable , and is given the order topology . Then it can be shown (non trivially) that the Long Line has a 1 - polymanifold structure, although it is non-paracompact, and therefore non-triangulable .

<u>Example 2</u> . The Prüfer-manifolds are non-triangulable n-manifolds , $n \geq 2$.

<u>Direct Limits</u>

Suppose X_n , $n = 0, 1, 2, \ldots$, is a sequence of polyspaces , such that, for each n , X_n is a polysubspace of X_{n+1} . Define the <u>limit structure</u>

on $X = UX_n$ to be

$$\mathcal{J}(X) = U\mathcal{J}(X_n) .$$

The topology of $\mathcal{J}$ is the same as the limit topology .

Example 1

Let $X_n = E^n$. Assume $E^n \subset E^{n+1}$ linearly .
Then $E^{\infty} = U E^n$ is Euclidean ∞ -space . This is not to be confused with , nor homeomorphic (in either topology) to , R^{∞} , Hilbert space , which is the product of countable copies of the reals .

Example 2

Let B^o = point ; $B^n = S B^{n-1}$, the suspension . Then B^n is an n-polyball , and $B^{\infty} = U B^n$ the ∞ -polyball . This is not to be confused with , nor is homeomorphic to , I^{∞} , the Hilbert cube .

Example 3

Let S^o = two points ; $S^n = SS^{n-1}$, the suspension . Then S^n is an n-polysphere, and $S^{\infty} = US^n$ the ∞ -polysphere . It is true that B^{∞} has S^{∞} as a closed subpolyspace , with complementary open subspace $B^{\infty} - S^{\infty} \cong E^{\infty}$. Nevertheless we do not call these boundary and interior because it is fairly easy to show polyhomeomorphisms

$$E^{\infty} \cong B^{\infty} \cong S^{\infty} .$$

Therefore B^{∞} is homogeneous without boundary , because S^{∞} is .

Example 4

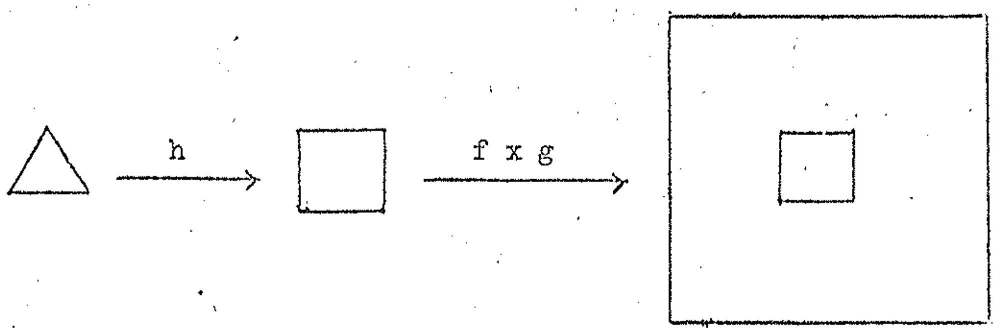
Let $O = U O_n$ be the infinite orthogonal group . Any triangulation of O_n can be extended to a triangulation of O_{n+1} . The resulting structures define a polystructure on O .

Infinite manifolds . The above definition is good for $n = \infty$. The above examples are all infinite manifolds. Similarly other classical groups , and the infinite Grassman and Stiefel manifolds . We observe that an ∞ -manifold has no boundary because an ∞ -ball has no boundary .

Products.

Let K, L be complexes in E^p , E^q . Then $K \times L$ is a cell complex in E^{p+q} , and so the natural structure $\mathcal{F}(K \times L)$ is uniquely defined . Given now two polyspaces X, Y , define the product structure .

$$\mathcal{F}(X \times Y) = \{ (f \times g)\, h \;;\; f \in \mathcal{F}(X) \,,\, g \in \mathcal{F}(Y),\, h \in \mathcal{F}(\mathrm{dom} f \times \mathrm{dom} g) \} .$$



We can deduce :

1) The product is functorial on $\mathcal{P}$.

2) A function $f : X \to Y$ is a polymap if and only if the graph $1 \times f : X \to X \times Y$ is a polyembedding .

Joins

The topological join $X * Y$ of two spaces X , Y is obtained from $X \cup (X \times I \times Y) \cup Y$ by identifying $x = (x, 0, y)$, $y = (x, 1, y)$ all $x \in X$, $y \in Y$, and giving the identification topology .

If X, Y are polyspaces we define the <u>join structure</u> $\mathcal{F}(X * Y)$ as follows.

Given K in E^p, L in E^q, we identify E^p, E^q with $E^p \times 0 \times 0$, $0 \times E^q \times 1$ in $E^p \times E^q \times I, \subset E^{p+q+1}$.

The images of K, L are joinable in E^{p+q+1}, and we define $K * L$ to be their join. The complex $K * L$ has a natural structure $\mathcal{F}(K * L)$. Define

$$\mathcal{F}(X * Y) = \{(f * g)h \; ; \; f \in \mathcal{F}(X), \; g \in \mathcal{F}(Y), \; h \in \mathcal{F}(\mathrm{dom} f * \mathrm{dom} g).$$

We can deduce that the topology of $\mathcal{F}(X * Y)$ is the same as the topology of the join $X * Y$ above.

<u>Remark</u>

The join $*$ is functorial on the category of maps, but not on the subcategory of embeddings. Give $X' \subset X$, $Y' \subset Y$ then $X' * Y'$ does not always have the topology induced from the inclusion $X' * Y' \subset X * Y$. For example let $X = I$, $X' = \mathring{I}$, and $Y = Y' =$ a point. Then the cone on $\mathring{I}$ is not a subspace of the cone on I; the cone on $\mathring{I}$ has a finer topology than the induced topology, and is locally compact at the vertex, whereas the induced topology is not (cf. the polysubspace Example 2).

On the other hand the join <u>is</u> functorial in the category of polysubspaces and injective polymaps; the naturality in the category dictates the topology to be chosen on the join, which is then not functorial in the subcategory of polyembeddings. The explanation is that the concept join is essentially a combinatorial idea, and so as we should expect, in this context the polystructure is more basic than the topology.

<u>Function Spaces</u>

Let X be a polyhedron, and Y a polyspace. Let Y^X be the set of polymaps $X \to Y$. We define the <u>function space structure</u> $\mathcal{F}(Y^X)$ on Y^X

as follows . If $f : K \to Y^X$ is an injective function , let $f' : X \times K \to Y$ be the associated function given by $f'(x , k) = (fk)\, x$. Define

$$\mathcal{F}(Y^X) = \{ f \,;\, f' \text{ is a polymap} \} .$$

<u>Lemma</u> (Hudson) <u>Any two such f'_s are related</u> .

Therefore $\mathcal{F}(Y^X)$ is a family of polyhedra in Y^X and the three axioms for a polystructure are easy to verify. We can deduce the following properties

1) The structure of Y^X is functorial on X, Y in $\mathcal{P}$. In other words if $f: X_1 \to X$ and $g : Y \to Y_1$ are polymaps, then the induced function

$$g^f : Y^X \longrightarrow Y_1^{X_1}$$

is also a polymap .

2) If X, Y are polyhedra and Z a polyspace then there is a natural polyhomeomorphism

$$(Z^Y)^X \cong Z^{Y \times X} .$$

<u>Remark 1</u> . If X not a polyhedron (not compact) then the above definition does not give a polystructure . For example if $X = E'$, $Y = E^2$ then Hudson's theorem fails; there exist two f's that are not related .

<u>Remark 2</u> . The topology of the structure is strictly finer than the compact open topology , and is therefore Hausdorff . If Y is a polyhedron or a manifold then both topologies give the same homotopy structure on Y^X . (Question : is this true for general Y ?)

<u>I s o t o p y</u>

Let $(X \subset Y)$ denote the polyspace of polyembeddings of X in Y, with strunture induced from Y^X . (Question : is it a polysubspace of Y^X ?)

One of the main reasons for the way we have developed the theory is that the following four definitions of isotopy are now trivially equivalent .

A <u>polyisotopy</u> of X in Y is

i) a point of $(X \subset Y)^I$

ii) a polymap $I \to (X \subset Y)$

iii) a polymap $X \times I \to Y$, which is a polyembedding at each level ,

iv) a level-preserving polyembedding $X \times I \to Y \times I$.

If $f, g : X \to Y$ are the beginning and end points of the isotopy , we say the isotopy <u>moves fX onto gX</u> , and that f , g are <u>isotopic</u> .

Let $H(Y)$ denote the polyspace of polyhomeomorphisms of Y onto itself, with structure inducec from Y^Y . An <u>ambient polyisotopy</u> of Y is a polyarc in $H(Y)$ starting at the identity, and finishing at e , say . If X is a polysubspace of Y we say the ambient isotopy <u>moves X onto eX</u> . If $f: X \to Y$ is a polyembedding (or polymap) we say f, ef are <u>ambient isotopic</u> . Later we shall prove a theorem of Hudson , which says that the notions of isotopy and ambient isotopy coincide for manifolds of codimension $\geqslant 3$. In codimension 2 they are essentially different, because ordinary knots in E^3 can be untied by isotopy , but not by ambient isotopy .

<u>Remark</u>

If $Y = E^n$, there is another definition of isotopy favoured by some writers , which we call <u>linear isotopy</u>, and it is worthwhile analysing the difference . A <u>linear homotopy</u> of X in E^n is constructed as follows : choose a fixed triangulation K of X , and for each vertex $v \in K$, a polymap $f_v : I \to E^n$. For each t , let $g_t : K \to E^n$ be the linear map determined by the vertex map $v \to f_v(t)$. Then $\{g_t\}$ or $g : K \times I \to E^n$ is the linear homotopy . If g is an embedding at each level we call g a linear isotopy . We make the following observations :

i) Not every linear isotopy is poly, because in general the track $g(K \times I)$ left by the linear isotopy is a curvilinear ruled surface rather than a euclidean polyhedron .

ii) Not every polyisotopy is linear, as shown by the example below .

iii) If two polymaps are linearly isotopic then they are polyisotopic . The converse is also true (non-trivially) if X is a manifold of codimension $\geqslant 3$.

iv) Linear isotopy is not functorial. We justify this last statement by defining a polystructure on $(X \subset E^n)$ that exactly captures linear isotopy ; more precisely we shall construct a polystructure, $\mathcal{F}_L$ say , on $(E^n)^X$ such that linear homotopies are the polymaps $I \longrightarrow (E^n)^X$ with respect to $\mathcal{F}_L$, and linear isotopies are the polymaps $I \to (X \subset E^n)$ with respect to the induced structure .

Define $\mathcal{F}_L$ as follows: if K is a triangulation of X with k vertices , then the set M_K of linear maps $K \longrightarrow E^n$ can be given a polystructure $M_K \cong E^{kn}$. If K' is a subdivision of K , then $M_K \subset M_{K'}$ is a polyembedding .
Therefore $(E^n)^X = \bigcup_K M_K$, the union taken over all triangulations in the structure of X , and $\mathcal{F}_L$ is defined by the limit polystructure . We shall show that if $E^n \to E^n$ is a polyhomeomorphism, then the induced function $(E^n)^X \longrightarrow (E^n)^X$, which is a polyhomeomorphism with respect to the function space structure, is not even continuous with respect to $\mathcal{F}_L$.

<u>Example 1</u> . Let $X = I$ and $Y = E^2$, and consider the isotopies $I \to (I \subset E^2)$ performed by a caterpillar crawling firstly along a straight twig , and secondly along a bent twig . The first isotopy , f say , is linear , and therefore also poly . The second isotopy , g say , is poly but <u>not</u> linear , because we cannot describe it in terms of a fixed triangulation of $X = L$. This shows $\mathcal{F} \not\subset \mathcal{F}_L$. Now suppose the caterpillar

performs g by starting with his nose, and finishing with his tail, at the bend in the twig. Then $g\mathring{I}$ is a closed set in the topology of $\mathcal{F}_L$, whereas $f\mathring{I}$ is not. There is an obvious polyhomeomorphism η of E^2 bending a straight twig into a bent twig, and the induced map of $(I \subset E^2)$ into itself maps $f\mathring{I}$ into $g\mathring{I}$. Therefore it cannot be continuous with respect to the topology of $\mathcal{F}_L$.

The explanation is that $\mathcal{F}$ is functorial on $X, Y \in \mathcal{P}$, whereas $\mathcal{F}_L$ is functorial only on $X \in \mathcal{P}$ and Y in the subcategory of euclidean spaces and linear maps. Since our theory is directed towards isotopies of manifolds in manifolds, we favour $\mathcal{F}$ and reject $\mathcal{F}_L$.

Example 2. Let H^n denote the set of polyhomeomorphisms of E^n onto itself having compact support. The hypothesis of compact support enables us to define on H^n, as above, both a function space polystructure $\mathcal{F}_1$, and a linear polystructure $\mathcal{F}_2$. Let H_1^n, H_2^n be the resulting topological spaces, both having H^n as underlying set. Then it appears that H_1^n, H_2^n have different homotopy structures. By Alexander's Lemma on isotopy, it is easy to show that H_1^n is contractible. However Kuiper has used the queer differential structures on S^7 to show that either $\pi_0(H_2^6) \neq 0$ or $\pi_1(H_2^5) \neq 0$. This is essentially a phenomenon of codimension zero.

Degeneracy

Let $f: X \to Y$ be a polymap. Define the non-degenerate structure $\eta(f)$ of f by

$$\eta(f) = \{ g \in \mathcal{F}(X) ; fg \in \mathcal{F}(Y) \} .$$

Note that in general $fg \notin \mathcal{F}(Y)$ because it is not injective. Call f non-degenerate if $\eta(f)$ is a base for $\mathcal{F}(X)$. Otherwise f is degenerate.

Example 1. A polyembedding is non-degenerate.

Example 2 . A polyimmersion (local embedding) is non-degenerate .

Example 3 . A simplicial map is non-degenerate if and only if it maps each simplex non-degenerately .

Example 4 . We shall show that any map of a polyhedron of dimension $\leq n$ to an n-manifold can be put into "general position" where it is non-degenerate .

The mapping cylinder problem

The problem is to define a natural structure on the mapping cylinder $\mathcal{C}$ of a map $f : X \to Y$. We explain why this problem is, in a sense, insoluble .

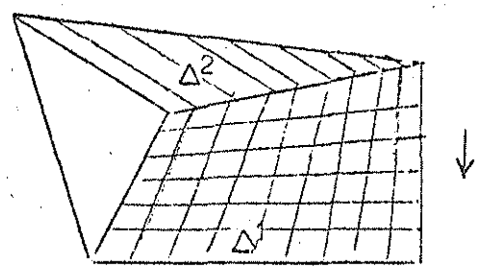
1. Topological . The topological mapping cylinder $\mathcal{C}$ is obtained from $X \times I \cup Y$ by identifying $(x,1) = fx$, all $x \in X$, and is given the identification topology . Then $\mathcal{C}$ is functorial on the category of maps .

2. Combinatorial . Suppose $f : K \to L$ is a simplicial map . Whitehead gave a rule for defining the simplicial mapping cylinder , G say , of f , which is a triangulation $g : G \to C$ of the topological mapping cylinder . This rule is functorial on the category of simplicial maps , but not on the category of piecewise linear maps . For suppose K', L' are subdivisions of K , L , giving rise to the simplicial cylinder $g' : G' \to C$. Then , although G, G' are piecewise linearly homeomorphic , g , g' are not in general related . Therefore the identity maps $K' \to K$, $L' \to K$ induce the identity $C \to C$, but only a piecewise projective map $G' \to G$.

3. Polyhedral . The inclusion $C \subset X * Y$ of the mapping cylinder in the join induces a natural polystructure $\mathcal{F}(C)$ on C that is functorial on the category of polymaps . However $\mathcal{F}(C)$ gives the wrong topology (too fine a one) .

Example 1 . The identity on I has mapping cylinder a square , and polystructure the Woven Square (of example 4 above) .

Example 2 . The mapping cylinder of a simplicial map of a 2-simplex onto a 1-simplex epitomises the problem , because when embedded in E^3 it looks like the prow of a ship .



The structure $\mathcal{F}(C)$ has a base consisting of all horizontal sections , and all vertical sections going athwartships , but no 3-dimensional stuff .

Example 3 . If f is simplicial , then the simplicial mapping cylinder $G \to C$ is related to $\mathcal{F}(C)$. In other words , $\mathcal{F}(C)$ can be enlarged (non-naturally) to contain any simplicial cylinder, and is the intersection of all the structures determined by the simplicial cylinders .

Example 4 . On the subcategory of non-degenerate polymaps the natural structure $\mathcal{F}(C)$ can be enlarged to a structure $\mathcal{F}_1(C)$ that (i) is functorial on this subcategory (ii) contains all simplicial cylinders , and (iii) gives the correct topology . A base for $\mathcal{F}_1(C)$ is

$$\mathcal{B} = \mathcal{F}(Y) \quad \cup \left\{ (g \times 1)\, h \,;\ \ g \in \eta(f) \,,\ h \in \mathcal{F}(\mathrm{dom}\, g \times I) \right\} .$$

Concluding Remarks

We can enlarge or change $\mathcal{P}$ by enlarging or changing the tool $\mathcal{C}$.

Example 1 . Enlarge $\mathcal{C}$ to contain piecewise projective maps .

Example 2 . Further enlarge $\mathcal{C}$ to contain piecewise algebraic complexes and piecewise algebraic maps. Then algebraic varieties in E^n

would become polysubspaces .

Example 3 . Replace $\mathcal{C}$ by the category of open subsets of E^n and differential maps . Then $\mathcal{P}$ would be the category of differential manifolds and differential maps.

Gabrielle has pointed out that a polyspace is equivalent to a contravariant fuctor from $\mathcal{C}$ to the category of sets and functions , obeying two axioms of intersection and union; a polymap is a natural transformation between two such functors .

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INSTITUT DES HAUTES ETUDES SCIENTIFIQUES

1 9 6 3

Seminar on Combinatorial Topology

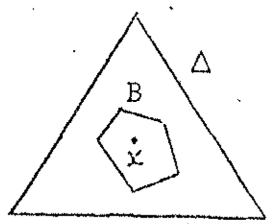
by E.C. ZEEMAN

Chapter 3 : REGULAR NEIGHBOURHOODS

From now on we shall omit the prefix "poly", and whenever we say space, map, manifold, etc., we mean polyspace, polymap, polymanifold, etc...

Lemma 8. A convex linear cell is a ball

Proof : Given a convex linear cell B we have to exhibit a specific piecewise linear homeomorphism from a simplex Δ onto B. Since B is in some Euclidean space, we can choose $\Delta \supset B$. Let x be a point in $\mathring{B}$. Then radial projection from x gives a homeomorphism $\dot{\Delta} \longrightarrow \dot{B}$, but this is not piecewise linear by the Standard Mistake. We get round this difficulty by defining a pseudo radial projection as follows. Let $\dot{\Delta}'$ be the cell subdivision of $\dot{\Delta}$ consisting of all cells



$\Delta_1 \cap x B_1$, $\Delta_1 \in \dot{\Delta}$, $B_1 \in \dot{B}$. Let $\dot{\Delta}''$ be a simplicial subdivision of $\dot{\Delta}'$. Radial projection of $\dot{\Delta}''$ determines an isomorphic subdivision $\dot{B}''$ of $\dot{B}$, and radial projection of the vertices determines the simplicial isomorphism, which is of course piecewise linear. Joining to x gives the required homeomorphism $A \rightarrow B$.

Corollary. Joins of spheres and balls obey the rules :

i) $B^p B^q \simeq B^{p+q+1}$

......

ii) $B^p S^q \cong B^{p+q+1}$

iii) $S^p S^q \cong S^{p+q+1}$.

Proof. Since the structure of a join is functorial, it suffices to prove one example .

i) The join of two simplexes is a simplex

ii) In E^{p+q+1} choose B^p, B^{q+1} to be simplexes crossing at their barycentres. Then $B^p B^{q+1}$ is a convex linear cell.

iii) Take the boundary of ii) .

We call a complex J a <u>combinatorial n-manifold</u> if the link of each vertex is an $(n-1)$-sphere or an $(n-1)$-ball.

<u>Lemma 9 . Suppose $|J| = M$. Then J is a combinatorial manifold if and only if M is a manifold</u> .

<u>Proof</u> . One way is trivial; for if J is a combinatorial manifold , then the closed vertex stars of J give a covering of M by balls, such that each point of M has some ball as a neighbourhood .

Conversely suppose M is an n-manifold , and let x be a vertex of J in $\mathring{M}$. By the definition of manifold (polymanifold), there is a piecewise linear embedding $f : \Delta \longrightarrow J$ covering a neighbourhood of x , where Δ is an n-simplex , such that $f^{-1}x \in \mathring{\Delta}$. Subdivide so that $f : \Delta' \longrightarrow J'$ is simplicial; we have piecewise linear homeomorphisms

$$\dot{\Delta} \longrightarrow \mathrm{lk}(f x , \Delta') \longrightarrow \mathrm{lk}(x, J') \text{ —— } \mathrm{lk}(x, J)$$

Where the middle arrow is an isomorphism and the other two arrows are pseudo radial projections . Hence $\mathrm{lk}(x,J)$ is an $(n-1)$-sphere .

If x is a vertex of J in $\dot{M}$, there is a similar situation except that $f^{-1}_{x} \in \dot{\Delta}$, and so it follows that $\mathrm{lk}(x,J)$ is a ball.

Corollary 1 . Let $|J| = M$ be an n-manifold . If A is a p-simplex of J , then

$$\text{either } \operatorname{lk}(A,J) = (n-p-1)\text{-sphere and } \mathring{A} \subset \mathring{M}$$

$$\text{or } \operatorname{lk}(A,J) = (n-p-1)\text{-ball and } A \subset \dot{M} .$$

Proof . We show the link is a sphere or ball by induction on p , the induction starting at $p = 0$ by the Lemma . If $p > 0$, write $A = xB$, and then $\operatorname{lk}(A,J) = \operatorname{lk}(x, \operatorname{lk}(B,J))$, which is the link of a vertex in an $(n-p)$-sphere or ball, by induction, and is therefore an $(n-p-1)$-sphere or ball by the Lemma .

Any point of $\mathring{A}$ has $A \operatorname{lk}(A;J)$ as a closed neighbourhood , and so lies in $\mathring{M}$ or $\dot{M}$ according as to whether it lies in the interior or boundary of this neighbourhood, i.e. according as to whether $\operatorname{lk}(A,J)$ is a sphere or ball. Therefore if the link is a sphere then $\mathring{A} \subset \mathring{M}$, and if the link is a ball then $A \subset \dot{M}$, since $\dot{M}$ is closed .

Corollary 1 justifies the following definition : if J is a combinatorial manifold, define the boundary $\dot{J}$ to be the subcomplex

$$\dot{J} = \{ A \in J \; ; \; \operatorname{lk}(A,J) = \text{ball} \}$$

and the interior to be the open subcomplex $\mathring{J} = J - \dot{J}$.

We deduce at once :

Corollary 2. If $|J| = M =$ manifold, then $|\dot{J}| = \dot{M}$.

Definition . If B^{n-1} is an $(n-1)$-ball contained in the boundary $\dot{M}^n$ of an n-manifold M^n , we call B^{n-1} a face of M^n and write $B^{n-1} < M^n$. We are particularly interested when $M^n = B^n$ a ball also. Let Δ^n denote an n-simplex .

Theorem 2 . If $B^{n-1} < B^n$ and $\Delta^{n-1} < \Delta^n$ then any homeomorphism $B^{n-1} \rightarrow \Delta^{n-1}$ can be extended to a homeomorphism

$B^n \longrightarrow \Delta^n$.

Corollary . If two balls meet in a common face , then their union is a ball . (For by Theorem 2 the union is homeomorphic to the suspension of a simplex) .

Theorem 3 . If $B^n \subset S^n$ then $\overline{S^n - B^n}$ is a ball

Remark 1 .

The original proofs of Theorem 2 and 3 were given by Newman and Alexander in the 1920's and 30's and used "stellar theory" instead of combinatorial theory . The essential notion of the proof is to replace the finite simplicial structure of a ball by some ordered finite structure, and then use induction on the number of steps in the ordering (the induction starting trivially with a simplex) . Newman and Alexander used an ordering by stellar subdivisions; we give a new proof here, based on ordering by collapsing . The collapsing technique was invented by Whithehead in 1939, and is more powerful than stellar theory because it includes the theories of regular neighbourhoods and simple homotopy type . Notice that some concept of an ordered structure seems vital, because without it we cannot prove :

Schönflies Conjecture : If $S^{n-1} \subset S^n$ then the closures of each component of the complement is a ball .

The conjecture is true for $n \leqslant 3$, but unsolved for $n > 3$. It is known by Morton Brown's result that they are triangulated topological balls, but not known whether they are polyballs . Our ignorance of whether they are polyballs when $n = 4$ implies our ignorance of whether they are even polymanifolds when $n = 5$ (the links of boundary vertices may go haywire) .

Remark 2 .

The proof of Theorem 2 and 3 is done together by induction on n . The induction starts trivially with $n = 0$. We shall show first that

Theorem 2_n is equivalent to Theorem 3_{n-1} . The inductive step is achieved by showing that

$$\left.\begin{array}{l} \text{Theorem } 2_r \text{ , } r \leq n \\ \\ \text{Theorem } 3_r \text{ , } r < n \end{array}\right\} \Longrightarrow \text{Theorem } 3_n$$

The inductive step is long, involving Lemmas 10 - 17 and Theorems 4 - 8 , during which we shall often have to make inductive use of Theorems 2 and 3 . However we can avoid going round in a circle by

i) assuming everything to be of dimension $\leq n$

ii) avoiding the use of Theorem 3_n

until Theorem 3_n is proved . To emphasise which statements are involved in the induction, and at the same time avoid repetition, we put a star against all those lemmas or theorems which depend upon Theorem 2_r and its Corollary , $r \leq n$, and Theorem 3_r , $r < n$.

<u>Lemma 10</u> . <u>Any homeomorphism between the boundaries of two balls can be extended to the interiors</u> .

<u>Proof</u> . We are given $f : \dot{B}_1 \rightarrow \dot{B}_2$.

Choose triangulations $g_i : \Delta \longrightarrow B_i$. Define $h : \dot{\Delta} \longrightarrow \dot{\Delta}$ by the commutative diagram

$$\begin{array}{ccc} \dot{\Delta} & \xrightarrow{h} & \dot{\Delta} \\ \downarrow{g_1} & & \downarrow{g_2} \\ \dot{B}_1 & \xrightarrow{f} & \dot{B}_2 \end{array}$$

Extend h conewise to a homeomorphism $h' : \Delta \longrightarrow \Delta$.

Then the required homeomorphism $f' : B_1 \rightarrow B_2$ is given by the commutative diagram

$$\begin{tikzcd} \Delta \arrow[r, "h'"] \arrow[d, "g_1"] & \Delta \arrow[d, "g_2"] \\ B_1 \arrow[r, "f'"] & B_2 \end{tikzcd}$$

<u>Theorem 2_n is equivalent to Theorem 3_{n-1}</u> .

<u>Proof</u> . Assume Theorem 2_n . Given $B^{n-1} \subset S^{n-1}$, then joining to a point x , we have $B^{n-1} < xS^{n-1}$. Let Δ^n be a simplex with face Δ^{n-1} and opposite vertex y . Choose a homeomorphism $B^{n-1} \longrightarrow \Delta^{n-1}$, and extend it to $xS^{n-1} \longrightarrow \Delta^n$. Therefore $\overline{S^{n-1} \quad B^{n-1}}$ is homeomorphic to the ball $y\,\dot{\Delta}^{n-1}$.

Conversely assume Theorem 3_{n-1} . Given $B^{n-1} < B^n$, then we know $\overline{\dot{B}^n - B^{n-1}}$ is a ball . Therefore given a homeomorphism $B^{n-1} \longrightarrow \Delta^{n-1}$, we can extend $\dot{B}^{n-1} \to \dot{\Delta}^{n-1}$ to a homeomorphism $\dot{B}^n - B^{n-1} \longrightarrow y\,\dot{\Delta}^{n-1}$, by Lemma 10 . Therefore we have defined $\dot{B}^n \to \dot{\Delta}^n$, and can extend to $B^n \longrightarrow \Delta^n$, again by Lemma 10 .

<u>Stellar subdivision</u>

Recall from Chapter 1 that an <u>elementary stellar subdivision</u> of K is given by

$$K' = (K - st(A,K)) \cup a\,\dot{A}\,lk(A,K) \qquad \text{where} \quad a \in \overset{o}{A}, \quad A \in K .$$

A <u>stellar subdivision</u> of K , written σK , is the result of a finite number of elementary ones .

<u>Examples</u> i) An r^{th} derived is a stellar .

ii) If $K \supset L$, then any stellar subdivision of K determines a unique stellar subdivision of L , and conversely .

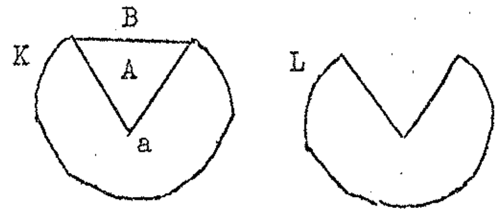
iii)  is <u>not a</u> stellar subdivision of a triangle .

C o l l a p s i n g

If $K \supset L$, we say there is an <u>elementary simplicial collapse</u> from K to L if $K - L$ consists of a principal simplex A of K together with a free face . Therefore if $A = aB$, then

$$K = L \cup A$$

$$a\dot{B} = L \cap A$$

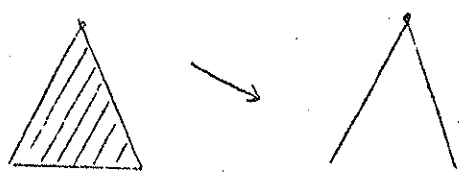


We describe the elementary simplicial collapse by saying <u>collapse A onto $a\dot{B}$</u> , or <u>collapse A from B</u> .

We say K <u>simplicially collapses</u> to L , written $K \overset{s}{\searrow} L$, if there is a sequence of elementary simplicial collapses going from K to L . If L is a point we call K <u>simplicially collapsible</u>, and write $K \overset{s}{\searrow} 0$.

<u>Examples</u> .

i) A cone simplicially collapses onto any subcone . For just collapse all the other simplexes in towards the vertex .



More precisely let aK be the cone on K , and aL be the subcone on L , where $L \subset K$. Then order the simplexes $B_1, \ldots, B_r$ of $K - L$ in order of decreasing dimension , and collapse aB_i from B_i , $i = 1, \ldots, r$.

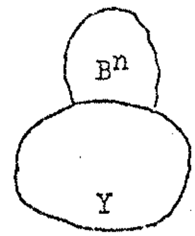
ii) A cone is simplicially collapsible .

iii) A simplex is simplicially collapsible . Both these are special cases of i) .

We now repeat the definition for polyhedra . If $X \supset Y$ are polyhedra , we say there is an <u>elementary collapse</u> from X to Y if there exists $B^n \succ B^{n-1}$ such that

$$X = Y \cup B^n$$

$$B^{n-1} = Y \cap B^n$$



We describe the elementary collapse by saying <u>collapse B^n onto B^{n-1}</u> , or <u>collapse B^n from $\overline{\dot{B}^n - B^{n-1}}$</u> .

We say <u>X collapsesto Y</u> , written $X \searrow Y$, if there is a sequence of elementary collapses going from X to Y . If Y is a point we call X <u>collapsible</u> , and write $X \searrow 0$. For example a ball is collapsible .

We now investigate the relationship between simplicial collapsing and collapsing . We write $K \searrow L$ if $|K| \searrow |L|$. The significance of this last definition is that the balls across which the collapse takes place may not be subcomplexes of K . It is trivially true that

$$K \overset{s}{\searrow} L \Longrightarrow K \searrow L ,$$

but the converse is unknown . What we can prove is :

<u>* Theorem 4</u> . <u>If $K \searrow L$, then there exists a subdivision K', L' of K , L such that $K' \overset{s}{\searrow} L'$</u> .

<u>* Corollary 1</u> . <u>If $X \searrow Y$ then there exists a triangulation such that $K \overset{s}{\searrow} L$</u> .

<u>* Corollary 2</u> . <u>If X is collapsible, then there exists a triangulation that is simplicially collapsible</u> .

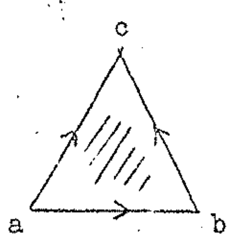
Before proving Theorem 4 we digress a little to indicate the consequences of the definition of collapsing .

Simple homotopy type

The relation $X \searrow Y$ between X and Y is ordered . If we forget the ordering, then we generate an equivalence relation between polyhedra called <u>simple homotopy type</u> . Since a collapse is a homotopy equivalence, this is a finer equivalence relation than homotopy type . It is strictly finer, because, for example, the lens spaces $L(7,1$, $L(7,2)$ are of the same homotopy type , but not of the same simple homotopy type . But for simply-connected spaces homotopy type = simply homotopy type , and there are simply-connected non-homeomorphic manifolds of the same homotopy type .

The Dunce Hat.

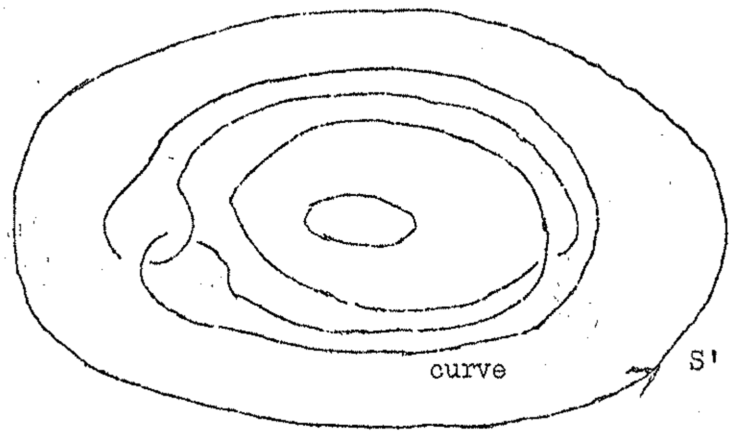
If we preserve the order $X \searrow Y$ then the relation between X , Y is much sharper . Trivially if X is collapsible then X is contractible (homotopywise) . But the converse is not true. For example consider the Dunce Hat D which is defined to be a triangle with its sides identified $ab = ac = bc$. Then D is contractible (although the contraction is hard to visualise) , and so D is the same simple homotopy type as a point; but D is <u>not</u> collapsible because there is nowhere to start. Although $D \not\searrow 0$, it can be shown that $D \times I \searrow 0$.



<u>Conjecture</u> . <u>If K^2 is a contractible 2-complex then $K \times I \searrow 0$</u>

This conjecture is interesting because it implies the 3-dimensional Poincaré Conjecture, as follows . Let M^3 be a compact contractible 3-manifold; it is sufficient to show that M^3 is a ball . Call X is a <u>spine</u> of M if $M \searrow X$. Now M^3 has a contractible spine K^2 . By the conjecture $M^3 \times I \searrow K^2 \times I \searrow 0$, and we shall show in Theorem 8 Corollary 1 that this implies $M^3 \times I = B^4$. Hence $M^3 \subset \dot{B}^4 = S^3$, and by the Schönflies Theorem $M^3 = B^3$, a ball.

In particular the conjecture is true for the Dunce Hat , and so any M^n , $n = 3$, having D as a spine is a ball . This is also true for $n \geq 5$ because D unknots in $\geqslant 5$ dimensions . However it is not true for $n = 4$ because there is an $M^4 \neq B^4$ (in fact $\pi_1(\dot{M}^4) \neq 0$) having D as a spine . The construction of M^4 is due to Mazur, and defined by attaching a 2-handle to $S^1 \times B^3$, by a curve in the boundary that is homotopic , but not isotopic , to the first factor :



Lemma 11 . If $K \overset{s}{\searrow} L$, then we can reorder the elementary collapses so that they are in order of decreasing dimension .

Proof . Suppose $K_1 \searrow K_2 \searrow K_3$ are consecutive elementary collapses, the first being across A^p from B^{p-1} , and the second across C^q from D^{q-1}. We shall show that if $p < q$ then we can interchange the order of the collapses (which is not true if $p \geqslant q$) . The lemma follows by performing a finite number of such interchanges .

Since $p < q$, C^q is not a face of A^p or B^{p-1} . Therefore C^q , which is principal in K_2 , remains principal in K_1 . Also $D^{q-1} \neq A$ or B , because A , B do not lie in K_2 , and so D^{q-1} cannot be a face of A or

B (again since $p \leqslant q-1$) . Therefore D remains a free face of C in K_2 . Therefore , if $K_2^* = K_1 - (C \cup D)$, then there is an elementary collapse $K_1 \searrow K_2^*$ across C from D . Meanwhile A remains principal in K_2^* , and B remains a free face . Therefore there is an elementary collapse $K_2^* \searrow K_1$ across A from B . The lemma is proved .

<u>Remark</u> . Although Lemma 11 indicates a certain freedom to rearrange the order of collapses, we cannot rearrange arbitrarily . For example if B^3 is a simplicially collapsible 3-ball , if we start collapsing B^3 carelessly we may get stuck before reaching a point - for instance the dunce hat is a spine of B^3 , so that by mistake we might get stuck at the dunce hat . This problem is the reason why the methods which classified 2-manifolds failed to classify 3-manifolds .

Again , if $K \overset{s}{\searrow} L$ and K' is an arbitrary subdivision of K , then trivially $K' \searrow L'$ but we do not know if $K' \overset{s}{\searrow} L'$. However we can prove a more limited result :

<u>Lemma 12</u> . <u>If $K \overset{s}{\searrow} L$ then $\sigma K \overset{s}{\searrow} \sigma L$ for any stellar subdivision σK of K .</u>

<u>Proof</u> . By induction we may assume both the simplicial collapse and stellar subdivision to be elementary . Suppose

$$K = L \cup A$$

$$a\dot{B} = L \cap A \text{ , and suppose}$$

σK is obtained by starring C at c . There are three cases

(i) If $C \not< A$, then the lemma is trivial

(ii) If $C \not< B$, then the cone $a(\sigma B)$ collapses to the subcone $a(\sigma \dot{B})$

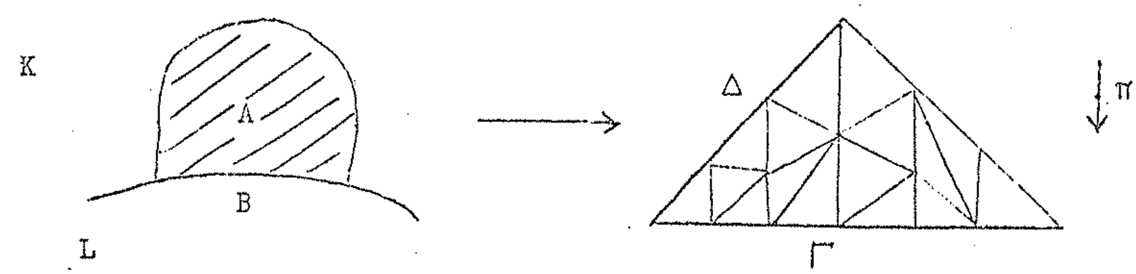
(iii) If $C < A$, but $C \not< B$, let $C = aB_1$, $B = B_1B_2$.

$$\text{Then } \sigma K \searrow \sigma L \cup \text{ cone } a(c\dot{B}_1B_2)$$
$$\searrow \sigma L \cup \text{ subcone } a(c\dot{B}_1B_2)^{\cdot} = \sigma L .$$

*<u>Lemma 13</u> . <u>If $K \searrow L$ is an elementary collapse , then there exists a subdivision such that $K' \overset{s}{\searrow} L'$ and L' is stellar (but K' may not be)</u> .

<u>Proof</u> . Let $A = \overline{K - L}$, and $B = A \cap L$. Then A is an n-ball and B a face . Let Δ , Γ be an n-simplex and an (n-1)-face . By Theorem 2_n choose a homeomorphism

$$h : A,B \longrightarrow \Delta , \Gamma .$$



Choose subdivisions so that h is a simplicial isomorphism $h : A',B' \xrightarrow{\cong} \Delta', \Gamma'$ Let $\pi : \Delta \longrightarrow \Gamma$ be the linear projection , mapping the vertex opposite Γ to the barycentre of Γ . Choose subdivisions Δ'' , Γ'' of Δ' , Γ' so that

$$\pi : \quad \Delta'' \longrightarrow \Gamma''$$

is simplicial . Call such a subdivision of Δ <u>cylindrical</u> . Let A'' , B'' be the isomorphic subdivisions of A' , B' . Let B''' be an r^{th} derived of B , subdividing B'' , and let Γ''' the corresponding subdivision of Γ'' . By Lemma 5 , choose a subdivision Δ''' of Δ'' such that $\pi : \Delta''' \longrightarrow \Gamma'''$ simplicial , and let A''' be the corresponding subdivision of A'' . Then B''' is a stellar subdivision of B , and induces a stellar subdivision L' of L . Define $K' = A''' \cup L'$. Since Δ''' is cylindrical , $\Delta''' \overset{s}{\searrow} \Gamma'''$ cylinderwise , in decreasing order of dimension . Hence $A''' \overset{s}{\searrow} B'''$, and so $K' \overset{s}{\searrow} L'$.

Proof of Theorem 4

We are given a collapse $X \searrow Y$; that is to say a sequence of elementary collapses

$$|K| = X_r \searrow X_{r-1} \searrow \dots \searrow X_o = |L| .$$

By Theorem 1 we can find a subdivision K_r of K, such that, for each i, there is a subcomplex K_i covering X_i. Therefore we may write the elementary collapses

$$K_r \searrow K_{r-1} \searrow \dots \searrow K_o .$$

If $r = 1$ the result follows by Lemma 13. If $r > 1$ we show the result by induction. Assume we have found a subdivision K'_{r-1} of K_{r-1} such that $K'_{r-1} \overset{s}{\searrow} K'_o$. By Lemma 3 extend K'_{r-1} to a subdivision K'_r of K_r. Apply Lemma 13 to the elementary collapse $K'_r \searrow K'_{r-1}$, to obtain a simplicial collapse $K''_r \overset{s}{\searrow} K''_{r-1}$, where K''_{r-1} is a stellar subdivision of K'_{r-1}. The latter fact enables us to appeal to Lemma 12 to deduce $K''_{r-1} \overset{s}{\searrow} K''_o$, and so $K''_r \overset{s}{\searrow} K''_o$.

Full subcomplexes

If $K \subset J$ are complexes, we say K is full in J if no simplex of $J - K$ has all its vertices in K. We can deduce the elementary properties of fullness :

(i) If $K \subset J$, and J' a first derived complex of J then K' is full in J'.

(ii) If K full in J, and J^* any subdivision of J, then K^* full in J^*.

(iii) If K full in J, and A a simplex of J, then $A \cap K$ is empty or a face of A.

(iv) If K full in J, then there is a unique simplicial map $f : J \to I$ (the unit interval) such that $f^{-1}0 = K$.

Neighbourhoods

Let J be a complex and let $X \subset |J|$. The <u>simplicial neighbourhood</u> $N(X,J)$ is the smallest subcomplex of J containing a topological neighbourhood of X. It consists of all (closed) simplexes of J meeting X, together with their faces.

Now suppose X is a polyhedron in an n-manifold M. We construct derived neighbourhoods of X in M as follows. If M is compact choose a triangulation J, K of N, X. If M is <u>not</u> compact choose a triangulation J, K of M_o, X where M_o is a subpolyhedron containing a topological neighbourhood of X in M. Now in general M_o will not be a manifold round the edges, but it will be a manifold near X, which is all that matters. More precisely, if $A \in N(X, J)$ then $\operatorname{lk}(A,J) = \begin{cases} S^{n-1}, & A \subset \mathring{M} \\ B^{n-1}, & A \subset \dot{M} \end{cases}$

For simplicity of exposition we identify $M_o = |J|$, $X = |K|$.
Choose now an r^{th} derived complex $J^{(r)}$ of J.
Call $N = N(X,J^{(r)})$ an <u>r^{th} derived neighbourhood</u> of X in M.
If $r = 1$ and K full in J we call N a <u>derived neighbourhood</u> of X in M.
A fortiori if $r \geq 2$, then any r^{th} derived neighbourhood is a derived neighbourhood, because $K^{(r-1)}$ full in $J^{(r-1)}$. If J', J'' denote first and second deriveds, it is easy to show that

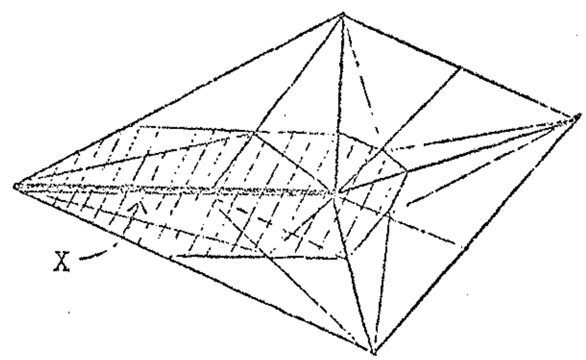
(i) $N(X, J') = \bigcup \overline{\operatorname{st}(x,J')}$, the union taken over all vertices $x \in K$,

(ii) $N(X, J'') = \bigcup \overline{\operatorname{st}(\hat{A}, J'')}$, the union taken over all simplexes $A \in K$, where $\hat{A}$ denotes the point at which A is starred in J'

<u>Lemma 14 . Any two derived neighbourhoods of X in M are homeomorphic , keeping X fixed</u> .

<u>Proof</u> . Let $N_1 = N(X, J'_1)$, $N_2 = N(X, J'_2)$ be the two given neighbourhoods . If M is compact , let J_o be a common subdivision of J_1, J_2 . If M is not compact choose subdivisions of J_1, J_2 that intersect in a common subcomplex , and let J_o be this subcomplex . Choose a first derived J'_o of J_o and let $N_o = N(X, J'_o)$.

Let $f : J_1 \longrightarrow I$ be the unique simplicial map such that $f^{-1}0 = X$, which exists by the hypothesis of fullness . Choose $\varepsilon > o$ and such that $\varepsilon < f x$, for all vertices $x \in J_o$, $x \notin X$. Let J_i^{ε} $(i = o, 1)$ denote a first derived of J_i obtained by starring $A \in J_i$ on $f^{-1}\varepsilon$ if $fA = I$ and arbitrarily otherwise . Then $|N(X, J_i^{\varepsilon})| = f^{-1}[0, \varepsilon]$, $i = 0, 1$.



Therefore $N_1 \cong N(X, J_1^{\varepsilon})$, isomorphic

$\simeq N(X, J_o^{\varepsilon})$, homeomorphic by identity map

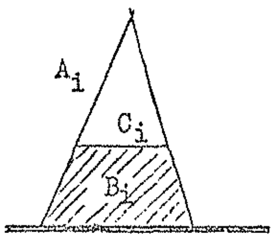
$\cong N_o$, isomorphic

$\cong N_2$, similarly .

<u>Remark</u> . Lemma 14 fails for first derived neighbourhoods without the fullness condition, which indicates the reason for having to pass to the second derived in general to obtain a derived neighbourhood. For example suppose X is the boundary of a 1-simplex in J . Then the first derived neighbourhood is connected , but the second derived is not .

<u>Corollary</u> . <u>Any derived neighbourhood of X in M collapses to X</u> .

<u>Proof</u> . By Lemma 14 it suffices to prove for one particular derived neighbourhood. Therefore choose a triangulation J , K of M , X such that K is full in J , and let $N = N\,(X\,,\,J^{\varepsilon})$ where J^{ε} is defined as in the proof of Lemma 14 .



Order the simplexes $A_1, \ldots, A_r$ of $J - K$ that meet K in order of decreasing dimension . Each A_i meets N in a convex cell B_i , with a face $C_i = A_i \cap f_{\varepsilon}^{-1}$. There is an elementary collapse of B_i from C_i , and the sequence of collapses $i = 1\,,\ldots,\,r$ determines the collapses $N \searrow X$.

<u>Lemma 15</u> . <u>Let $h : K \to K$ be a homeomorphism of a complex that maps each simplex onto itself, and keeps a subcomplex L fixed . Then h is ambient isotopic to the identity keeping L fixed</u> .

<u>Proof</u> . The obvious isotopy moving along straight paths is not piecewise linear by the standard mistake . However it is easy to construct a piecewise linear isotopy $H : K \times I \to K \times I$ inductively on the prisms $A \times I$, $A \in K$, in order of increasing dimension . For each prism , $H | \dot{A} \times I$ is given by induction, $H \mid A \times 0$ is the identity , and $H\,(a,1) = ha$. Therefore $H | (A \times I)^{\circ}$ is already defined, so map the centre of the prism to itself and join linearly . By construction H keeps L fixed .

<u>Corollary 1</u> . <u>The isomorphism between any two first derived complexes is ambient isotopic to the identity</u> .

<u>Corollary 2</u> . <u>Any two derived neighbourhoods of X in M are ambient isotopic , keeping X fixed . If $X \subset \mathring{M}$, the isotopy can be chosen to keep $\dot{M}$ fixed</u> .

For in the proof of Lemma 14 the homeomorphism was achieved by two isomorphisms between first deriveds, both keeping X fixed. The first deriveds can be chosen to agree outside the neighbourhoods, and so the isotopy keeps $\dot{M}$ fixed if $X \subset \overset{o}{M}$.

*<u>Theorem 5</u>. <u>A derived neighbourhood of a collapsible polyhedron is an n-manifold is an n-ball</u>.

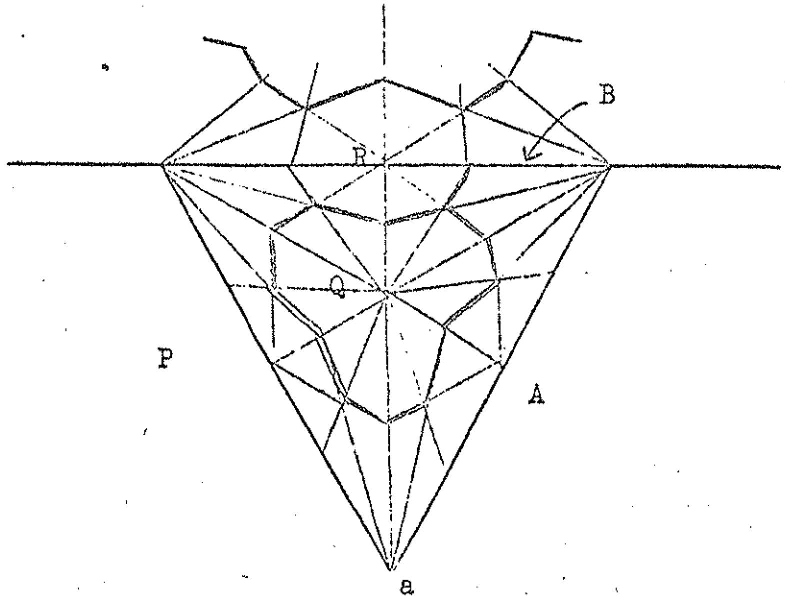
<u>Proof</u>. By induction on n, starting trivially with $n = 0$. By Lemma 14 it suffices to prove the theorem for one particular derived neighbourhood, and so we choose a second derived neighbourhood $N = N(X, J'')$, where $X = |K|$, $K \subset J$, and J'' is the second barycentric derived complex of J. Since X is collapsible, we can choose K such that $K \overset{s}{\searrow} 0$ by Theorem 4.

Let r be the number of elementary simplicial collapses involved in $K \overset{s}{\searrow} 0$. We show N is a ball by induction on r. The induction starts trivially with $r = 0$, for then K is a point, and N its closed star, which is a ball by Lemma 9. For the inductive step, let $K \searrow L$ be the first elementary simplicial collapse, collapsing a simplex A from B, say, where $A = aB$. Let $\hat{A}$, $\hat{B}$ denote the barycentres of $\hat{A}$, $\hat{B}$. Now

$$N = N(K, J'') = P \cup Q \cup R ,$$

where $P = N(L, J'')$, $Q = N(\hat{A}, J'')$, $R = N(\hat{B}, J'')$.

Now P is a ball by induction, and Q, R are balls since they are closed stars of vertices. If we can show that Q is glued onto P by a common face, then $P \cup Q$ is a ball by the Corollary to Theorem 2_n; similarly if R is glued onto $P \cup Q$ by a common face then N is a ball. Therefore the proof is reduced to showing the $P \cap Q$, and $(P \cup Q) \cap R$ are $(n-1)$-balls because if they are balls then they must be common faces, since the interiors of P, Q, R are disjoint.



Now $P \cap Q \subset \dot{Q} = \mathrm{lk}(\hat{A}, J'')$. Let

$$J_* = \mathrm{lk}(A', J') = \dot{A}' (\mathrm{lk}(A,J))',$$

where the prime thought this proof always denotes the barycentric first derived complex. There is an isomorphism

$$\dot{Q} \xrightarrow{\cong} J'_*$$

determined by the vertex map $\widehat{AC} \longrightarrow \hat{C}$, for all $C \in J_*$. Under this isomorphism

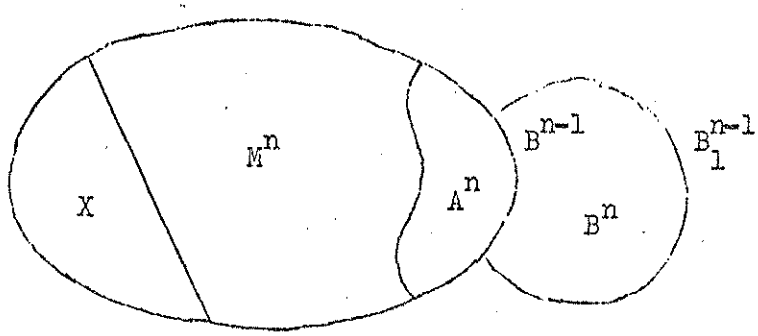
$$P \cap Q \xrightarrow{\cong} N(a\dot{B}, J'_*).$$

Now $a\dot{B}$ is collapsible, being a cone, and $(a\dot{B})'$ is full in J_*, which is an $(n-1)$-sphere or ball, by Lemma 9. Therefore $N(a\dot{B}, J'_*)$ is a derived neighbourhood of a collapsible polyhedron, and is an $(n-1)$-ball by induction on n. Hence $P \cap Q$ is an $(n-1)$-ball.

Similarly $(P \cup Q) \cap R \subset \dot{R}$, and if we now choose $J_* = \mathrm{lk}(\hat{B}, J')$, then there is an isomorphism $\dot{R} \xrightarrow{\cong} J'_{*}$, throwing $(P \cup Q) \cap R$ onto $N(\hat{A}\,\dot{B}\,,\,J'_{*})$. For the same reason as before we deduce $(P \cup Q) \cap R$ is an $(n-1)$-ball. This completes the proof of Theorem 5.

*<u>Theorem 6</u>. <u>Suppose the manifold M^n and the ball B^n meet in a common face. Let X be a closed subset of M^n not meeting B^n. Then there is a homeomorphism $M^n \to M^n \cup B^n$ keeping X fixed</u>.

<u>Proof</u>. Since X is closed, $M^n - X$ is a manifold. Let B^{n-1} be the common face, and let A^n be a derived neighbourhood of B^{n-1} in $M^n - X$, which is a ball by Theorem 5. Since $\overset{o}{A}{}^n \subset \overset{o}{M}{}^n$, B^{n-1} does not meet $\overset{o}{A}{}^n$,



and so B^{n-1} is a face of A^n. Since A^n, B^n meet in the common face B^{n-1}, their union is a ball by the Corollary to Theorem 2n. Let $B_1^{n-1} = \dot{B}^n - \overset{o}{B}{}^{n-1}$, which is a ball by Theorem 3_{n-1}.

We now construct the homeomorphism h. Define h to be the identity on $(M^n - A^n) \cup (\dot{A}^n - \overset{o}{B}{}^{n-1})$. In particular h is the identity on X. Extend $h = 1 : \dot{B}^{n-1} \to \dot{B}_1^{n-1}$ to a homeomorphism $B^{n-1} \to B_1^{n-1}$ by Lemma 10. Similarly extend $h : \dot{A}^n \to (A^n \cup B^n)^{\cdot}$ to the interiors. Then h has the desired properties.

<u>Lemma 16</u>. <u>Any homeomorphism of a ball onto itself keeping the boundary fixed is isotopic to the identity keeping the boundary fixed</u>.

Proof . It suffices to prove for a simplex . Given $h : \Delta \longrightarrow \Delta$, we construct the isotopy $f : \Delta \times I \quad \Delta \times I$ as follows . Let

$$f(x, t) = \begin{cases} hx, & t = 0 \\ x, & t = 1 \text{ or } x \in \dot{\Delta} . \end{cases}$$

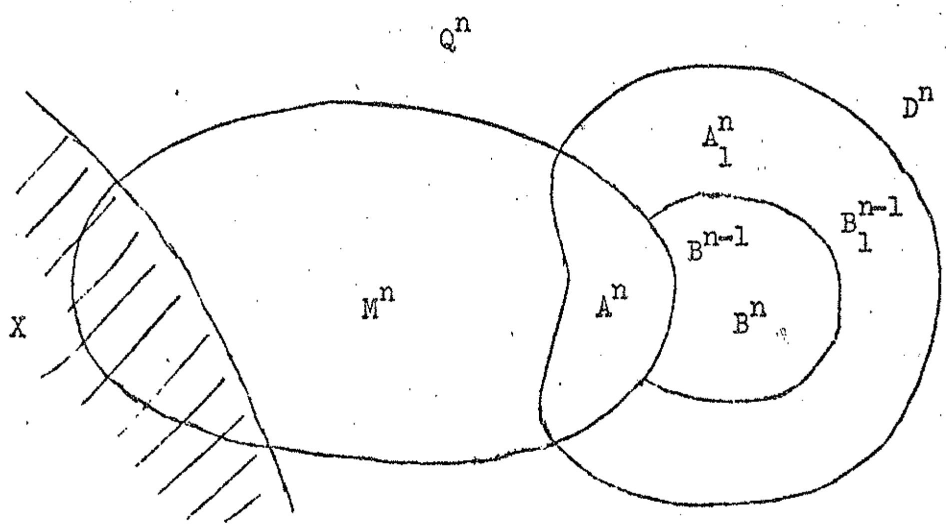
This gives f level preserving on $(\Delta \times I)^{\cdot}$. Define f level preserving on $\Delta \times I$ by mapping the centre of the prism to itself, and joining to the boundary linearly . Then f is the desired isotopy .

*Lemma 17 . Suppose $M^n \subset Q^n$ are manifolds , and that M^n is a closed subset of $\mathring{Q}^n$. Then $\overline{Q^n - M^n}$ is a manifold .

Proof . Let $M_1^n = \overline{Q^n - M^n}$. We have to show that every point $x \in M_1^n$ has a ball neighbourhood in M_1^n . If $x \in Q^n - M^n$, then x has a ball neighbourhood in Q^n that is contained in M_1^n, because M^n is closed in Q^n . If , on the other hand , $x \in M^n \cap M_1^n$, then $x \in \mathring{Q}^n$ by hypothesis , and so x lies in the interior of a ball in Q^n . Triangulate this ball so that x is a vertex , and so that its meets M^n in a subcomplex . If S^{n-1} is the link of x , then $S^{n-1} \cap M^n$ is a ball by Lemma 9 . Therefore the closure of the complement , $S^{n-1} \cap M_1^n$, is a ball by Theorem 3_{n-1} . Hence x has a ball neighbourhood in M_1^n .

*Theorem 7 . Suppose $M^n \subset Q^n$ are manifolds , and that M^n is a closed subset of $\mathring{Q}^n$. Let B^n be an n-ball in $\mathring{Q}^n$ meeting M^n in a common face . Let X be a closed subset of Q^n not meeting B^n . Then there is an ambient isotopy of Q^n moving M^n onto $M^n \cup B^n$, and keeping $X \cup \dot{Q}^n$ fixed .

<u>Proof</u> .



Let B^{n-1} be the common face, and let $B_1^{n-1} = \dot{B}^n - \overset{\circ}{B}{}_1^{n-1}$, which is a ball by Theorem 3_{n-1}. Let $M_1^n = \overline{Q^n - (M^n \cup B^n)}$, which is a manifold by Lemma 17, since $M^n \cup B^n$ is a manifold by Theorem 6. Let D^n be a derived neighbourhood of B^n in the manifold $Q^n - \dot{Q}^n - X$. Then D^n is a ball by Theorem 5. Let $A^n = D^n \cap M^n$, $A_1^n = D^n \cap M_1^n$. If when constructing D^n we choose a triangulation that meets M^n, B^n in subcomplexes, this ensures that A^n, A_1^n are respectively derived neighbourhoods of B^{n-1}, B_1^{n-1} in M^n, M_1^n and therefore are balls. A^n meets B^n in the common face B^{n-1}, and A_1^n, meets B^n in the common face B_1^{n-1}. Therefore $A^n \cup B^n$, $A_1^n \cup B^n$ are balls by Corollary to Theorem 2_n.

Next we construct a homeomorphism h of D^n onto itself as follows. Define $h = 1$ on $\dot{D}^n \cup (\dot{A}^n - \overset{\circ}{B}{}^{n-1})$. Extend $h : \dot{B}^{n-1} \to \dot{B}_1^{n-1}$ to the interiors by Lemma 10. Similarly extend $\dot{A}^n \to (A^n \cup B^n)^{\cdot}$ and $(B^n \cup A_1^n)^{\cdot} \to \dot{A}_1^n$ to the interiors. By Lemma 16 the identity is isotopic to h, keeping $\dot{D}^n$ fixed. Extend this to an ambient isotopy of Q^n keeping fixed $Q^n - \overset{\circ}{D}{}^n$ (in particular $X \cup \dot{Q}^n$). By construction this isotopy moves M^n onto $M^n \cup B^n$.

Regular neighbourhoods .

The definition of regular neighbourhood is more powerful than that of derived neighbourhood because it is intrinsic , and leads at once to an existence and uniqueness theorem .

Let X be a polyhedron in a manifold M . A regular neighbourhood N of X in M is a polyhedron such that

i) N is a neighbourhood of X in M .

ii) N is an n-manifold ($n = \dim M$)

iii) $N \searrow X$.

*Theorem 8

(1) Any derived neighbourhood of X in M is regular .

(2) Any two regular neighbourhoods of X in M are homeomorphic , keeping X fixed ,

(3) If $X \subset \overset{o}{M}$, then any two regular neighbourhoods of X in $\overset{o}{M}$ are ambient isotopic keeping $X \cup \dot{M}$ fixed .

Remark .

Clearly (3) is stronger than (2) . However it is valuable to have (2) in cases where (3) does not apply . For example suppose X is a spine of M in the interior of M ; then by (2) M is homeomorphic to any regular neighbourhood N of X in M . But obviously M and N are not ambient isotopic .

Proof of Theorem 8 .

Part (1) . Let $N = N(X , J')$ be a derived neighbourhood of X in M . We have to verify the three conditions for regularity . Condition (i) follows from the definition, and (iii) from the Corollary to Lemma 14 . To verify (ii) we check the link of each vertex $x \in N$. Let $L = \mathrm{lk}\,(x, J')$. If $x \in X$, then $\mathrm{lk}\,(x, N) = L$, which is a sphere or ball. If $x \notin X$, then $x \in \overset{\circ}{A}$, where A is a unique simplex in $J - K$, K being the subcomplex of J convering X. By the fullness of K in J, $A \cap K = B$, a face of A.

Now $L = \dot{A}'S$, where S is isomorphic to $(\mathrm{lk}\,(A,J))'$, and so is a ball or sphere. Since S lies in the interior of st (A,K) it does not meet X, and therefore $L \cap X = \dot{A}' \cap X = B'$. Therefore

$$\begin{aligned} \mathrm{lk}\,(x,N) &= N\,(B',L) \\ &= N\,(B',\dot{A}'S) \\ &= N\,(B',\dot{A}')S \end{aligned}$$

which is a ball, because $N(B',\dot{A}')$ is a ball by Theorem 5, being a derived neighbourhood of B in $\dot{A}$. The proof of part (1) is complete.

For part (2) it suffices by Lemma 14 to show that any regular neighbourhood is homeomorphic to a derived neighbourhood, keeping X fixed. If N is the regular neighbourhood, use Theorem 4 to choose a triangulation J, K of N, X such that J collapses simplicially to K

$$J = K_r \searrow K_{r-1} \searrow \dots \searrow K_o = K$$

Let J'' be the barycentric second derived of J, and let $N_i = N\,(K_i\,, J'')$. Then N_o is a derived neighbourhood of X in M, and $N_r = N$. As in the proof of Theorem 5, N_i is obtained from N_{i-1} by glueing on two balls. Neither of these balls meets X, because N_{i-1} is a neighbourhood of X, and so by Theorem 6 there is a homeomorphism $N_{i-1} \to N_i$ keeping X fixed. Composing these, we have the desired homeomorphism $N_o \to N$.

For part (3) we make the same construction as for part (2), and instead of Lemma 14 and Theorem 6 we use Corollary 2 to Lemma 15 and Theorem 7 to show that the two neighbourhoods are ambient isotopic keeping $X \cup \dot{M}$ fixed. The proof of Theorem 8 is complete.

Proof of Theorem 3_n

At last we come to the end of our mammoth induction. We recall that in the proofs of Theorem 4 - 8 we have used Theorem 2_r, $r \leq n$ and Theorem 3_r, $r < n$, but not Theorem 3_n. We now use Theorem 8 to prove Theorem 3_n. This will make Theorem 2 - 8 and the accompanying lemmas valid for all n.

Given $B^n \subset S^n$ we have to show that $\overline{S^n - B^n}$ is a ball. Choose a homeomorphism $f : \dot{\Delta}^{n+1} \to S^n$ throwing a vertex x of Δ^{n+1} onto a point $y \in \overset{\circ}{B}{}^n$. Let $A^n = f\,(\mathrm{st}(x, \dot{\Delta}^{n+1}))$. Then the balls A^n, B^n are both regular neighbourhoods of y in S^n, and so by Theorem 8 Part 3 are ambient isotopic. Therefore the closures of their complements are homeomorphic. But $\overline{S^n - A^n} = f\ \Delta^n$, where Δ^n is the face of Δ^{n+1} opposite x. Hence $\overline{S^n - B^n}$ is a ball.

We conclude the chapter with some useful corollaries to Theorem 8.

Corollary 1. A manifold is collapsible if and only if it is a ball. For if it is collapsible, then it is a regular neighbourhood (in itself) of any point, and therefore a ball by Theorem 5.

Corollary 2. If $X \subset \overset{\circ}{M}$, and N, N_1 are regular neighbourhoods of X in M, such that $N_1 \subset \overset{\circ}{N}$, then $N - \overset{\circ}{N}_1 \cong \dot{N} \times I$.

Proof. Construct two derived neighbourhoods as in the proof of Lemma 14.

$$N^* = f^{-1}[0, \varepsilon], \quad N_1^* = f^{-1}[0, \delta]$$

where $0 < \delta < \varepsilon < 1$. Then

$$N^* - \overset{\circ}{N}{}_1^* = f^{-1}[\delta, \varepsilon] \cong f^{-1}\varepsilon \times I = \dot{N}^* \times I.$$

Therefore the result is true for N^*, N_1^*. By Theorem 8 (2) choose a homeomorphism $h : N^* \to N$ keeping X fixed. Now hN_1^*, N_1 are both regular neighbourhoods of X in $\overset{\circ}{N}$, and so by Theorem 8 (3) we can ambient isotope hN_1^* onto N_1 keeping $\dot{N}$ fixed. Therefore

$$N - \overset{\circ}{N}_1 \cong N^* - \overset{\circ}{N}{}_1^* = \dot{N}^* \times I = \dot{N} \times I .$$

<u>Corollary 3. The combinatorial annulus theorem. If A, B are two n-balls such that $\overset{\circ}{A} \supset B$, then $A - \overset{\circ}{B} \cong S^{n-1} \times I$</u>. Proof by Corollary 2.

<u>Corollary 4. Suppose $X, Y \subset \overset{\circ}{M}$, and suppose X is a spine of M (i.e. $M \searrow X$). If $X \searrow Y$ or $Y \searrow X$ then Y is also a spine of M</u>.

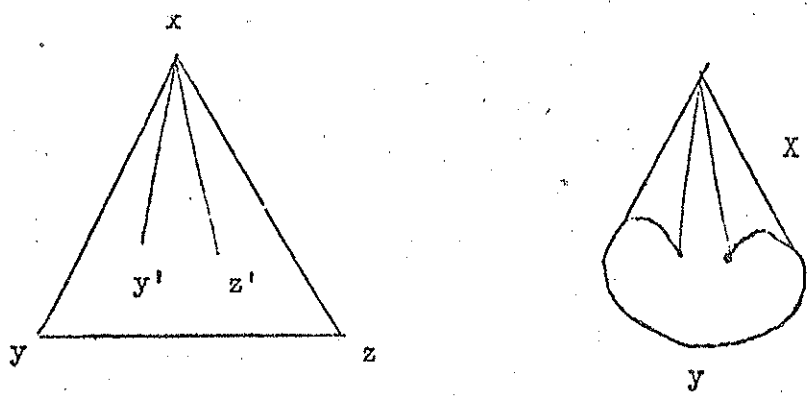
<u>Proof</u>. If $X \searrow Y$ the result is trivial, because then $M \searrow X \searrow Y$.

If $Y \searrow X$, let N be a regular neighbourhood of Y in $\overset{\circ}{M}$. Then $N \searrow Y \searrow X$, and so N is also a regular neighbourhood of X. By Corollary 2, $M - \overset{\circ}{N} \cong \dot{M} \times I$, and so $M \searrow N$. Therefore $M \searrow N \searrow Y$, and so Y is a spine of M.

<u>Remark 1</u>. Corollary 4 is a form of factorization of the collapsing process. However such factorization is only true for manifolds, and not true for polyhedra in general. For instance

$$\left.\begin{array}{c} X \searrow 0 \\ Y \searrow 0 \\ X \supset Y \end{array}\right\} \not\Longrightarrow X \searrow Y$$

Consider the following example. Let $x\,y\,z$ be a triangle, and let y', z' be two interior points not concurrent with x. Let X be the space obtained by identifying the intervals $xy = xy'$, $xz = xz'$, and let Y be the image of yz in X.



Then $X \searrow 0$ conewise, and $Y \searrow 0$ because Y is an arc. But $X \not\searrow Y$ because any initial elementary simplicial collapse of any triangulation of X must have its free face in Y, and so must remove part of B. Similarly can build examples to show that

$$\left.\begin{array}{r} X \searrow 0 \\ Y \searrow 0 \\ X \cap Y \searrow 0 \end{array}\right\} \not\Rightarrow X \cup Y \searrow 0$$

<u>Remark 2</u>. Corollary 4 is useful for simplifying spines. For example the spine of a bounded 3-manifold can be <u>normalised</u> in the following sense : we can find a spine, which is a 2-dimensional cell complex in which every edge bounds exactly 3 faces, and every vertex bounds exactly 4 edges and 6 faces. For choose a spine in the interior; expand each edge like a banana and collapse from one side; then expand each vertex like a pineapple and collapse from one face. By Corollary 4 any sequence of expansions and collapsesleaves us with a spine, and the process described makes it normal.

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INSTITUT DES HAUTES ETUDES SCIENTIFIQUES

1963

Seminar on Combinatorial Topology

by E.C. ZEEMAN

Chapter 4 : UNKNOTTING BALLS AND SPHERES

Suppose $M^m \subset M^q$ are manifolds; we say the embedding is proper if $\dot{M}^m \subset \dot{M}^q$ and $\overset{o}{M}{}^m \subset \overset{o}{M}{}^q$. A (q,m)-manifold pair $M^{q,m} = (M^q, M^m)$ is a pair such that $M^m \subset M^q$ properly. The codimension of the pair is $c = q - m$. The boundary $\dot{M}^{q,m} = (\dot{M}^q, \dot{M}^m)$ is a pair of the same codimension. We write $M^{p,n} \subset M^{q,m}$ if $M^p \subset M^q$ and $M^n = M^p \cap M^m$.

In this chapter we are interested in sphere pairs $S^{q,m}$ and ball pairs $B^{q,m}$. The boundary of a ball pair is a sphere pair. If $B^{q,m} \subset \dot{M}^{q+1,m+1}$ we call $B^{q,m}$ a face of $M^{q+1,m+1}$.
The standard (q,m)-ball pair is $\Delta^{q,m} = (\Sigma^{q-m} \Delta^m, \Delta^m)$ where Δ^m is the standard m-simplex, and Σ^{q-m} denotes (q-m)-fold suspension. The standard (q,m)-sphere pair is $\dot{\Delta}^{q+1,m+1}$. We say a sphere or ball pair is unknotted if it is homeomorphic to a standard pair. The cone on an unknotted (q,m) ball or sphere pair gives an unknotted (q+1,m+1) ball pair.

Theorem 9. *Any sphere or ball pair of codimension ≥ 3 is unknotted.*

Remark 1. In codimension 2 the theorem fails for both spheres and balls. The (3,1) sphere pairs give classical knot theory, and in higher dimensions knots can be tied for example by suspending and spinning (3,1) knots.

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Conjecture 1 . The sphere pair (S^q, S^{q-2}) is unknotted if $S^q - S^{q-2}$ is a homotopy S^1 . If $q = 3$ the result is true by a theorem of Papakyriakopoulos . If $q \geqslant 5$ an analogous topological theorem of Stallings says that if the sphere is topologically locally unknotted then it is topologically unknotted.

Conjecture 2 . Sphere and ball pairs unknot in codimension 1 . This is the Schönfiles conjecture which is true for $q \leqslant 3$, and unsolved for $q \geqslant 4$.

Conjecture 3 . If B is a ball pair contained in an unknotted sphere pair of the same dimension, then B is unknotted . This is true for codimension $\geqslant 3$ by Theorem 9 . It is true for codimension 2 when $q = 3$ by the unique factorization of classical knot theory (an unknotted curve is not the sum of two knots) . It is true for codimension 1 when $q \leqslant 3$ by the Schönflies Theorem . But otherwise in codimensions 1 and 2 is unsolved .

A modified result is that $B \subset S$ are both unknotted then the complementary ball pair $\overline{S - B}$ is also unknotted . This proved by Theorem 8 part 3 generalised to relative regular neighbourhoods .

Remark 2 . In differential theory Theorem 9 is no longer true because Haefliger has knotted S^{4k-1} differentially in S^{6k} . Above this critical dimension, in the stable range, he has unknotted all sphere pairs .

Plan of the proof of Theorem 9 .

Most of this chapter is devoted to proving Theorem 9 . The proof is by induction on m keeping the codimension $c = q - m$ fixed . We eventually show that

$$\text{Theorem } 9_{q-1,m-1} \Longrightarrow \text{Theorem } 9_{q,m}$$

The induction starts trivially with $m = 0$; for, given c , then $a(c,0)$ ball pair is a ball B^c with an interior point B^0 , which is homeomorphic to a standard pair .

Next we observe that :

<u>Unknotting of (q,m)-ball pairs implies unknotting of (q,m)-sphere pairs</u> .

<u>Proof</u> . Given $S^{q,m}=(S^q,S^m)$, triangulate the pair and choose a vertex $x \in S^m$. Let

$$B^{q,m} = (S^q - st(x,S^q) \ , \ S^m - st(x,S^m)) \ .$$

If $\Delta^{m+1} = y \ \Delta^m$ is the standard simplex, then

$$\Delta^{q+1,m+1} = \Delta^{q,m} \cup y \ \Delta^{q,m} \ .$$

By hypothesis choose an unknotting homeomorphism $B^{q,m} \longrightarrow \Delta^{q,m}$; then map x to y and extend linearly to an unknotting $S^{q,m} \longrightarrow \Delta^{q+1,m+1}$.

<u>Lemma 18</u> . <u>Let (B^q, B^m) and (C^q, C^m) be two unknotted ball pairs</u> . <u>Then any homeomorphisms $f : \dot{B}^q \longrightarrow \dot{C}^q$ and $g : B^m \longrightarrow C^m$ that agree on $\dot{B}^m$ can be extended to a homeomorphism $h : B^q \rightarrow C^q$</u> .

<u>Proof</u> . Extend f conewise to $\bar{f}: B^q \rightarrow C^q$ as in the proof of Lemma 10 . Let $e: C^m \rightarrow C^m$ be the composition

$$\begin{tikzcd} C^m & B^m \arrow[l, "\bar{f}"'] \arrow[r, "g"] & C^m \end{tikzcd}$$
$$C^m \xrightarrow{\quad e \quad} C^m \ .$$

Then e keeps $\dot{C}^m$ fixed , since f , g agree on $\dot{B}^m$. By the unknottedness we can suspend e to a homeomorphism $\bar{e}: C^q \rightarrow C^q$ fixed on $\dot{C}^q$. Then $h = \bar{e}\bar{f} : B^q \rightarrow C^q$ agrees with both f and g , and proves the Lemma .

<u>Corollary</u> . <u>Any homeomorphism between the boundaries of two unknotted ball pairs can be extended to the interiors</u> .

<u>Lemma 19</u> . <u>Assume Theorem $9_{q-1,m-1}$</u> . <u>Then if two unknotted (q,m) ball pairs meet in a common face their union is a unknotted ball pair</u> .

<u>Proof</u> . Let B_1, B_2 be the ball pairs meeting in the face F . Let $\Sigma\Delta$ be the suspension of the standard $(q-1,m-1)$ ball pair Δ , with suspension points x_1 , x_2 say . Choose an unknotting $F \longrightarrow \Delta$ by hypothesis. Extend $\dot{F} \longrightarrow \dot{\Delta}$ to unknottings $\dot{B}_i - \mathring{F} \longrightarrow x_i \dot{\Delta}$ by the above corollary . Similarly extend $\dot{B}_i \longrightarrow (x_i \Delta)^{\cdot}$ to the interiors . Then $B_1 \cup B_2$ is unknotted by the homeomorphism onto $\Sigma\Delta$.

<u>Lemma 20</u> . <u>If (B^q, B^m) is a ball pair of codimension $\geqslant 3$ then $B^q \searrow B^m$</u> .

<u>Remark</u> . Lemma 20 fails in codimension 2 ; for example a knotted arc properly embedded in B^3 is not a spine of B^3 . The proof of Lemma 20 involves some geometrical construction, and we postpone it until after Lemma 23, which is the crux of the matter . First let us show how Lemma 20 implies Theorem 9 .

<u>Proof of Theorem 9 assuming Lemma 20</u>

We assume Theorem $9_{q-1,m-1}$, where $q-m \geqslant 3$. By the observation that unknotting balls implies unknotting spheres, it suffices to show that a given ball pair $B = (B^q, B^m)$ is unknotted .

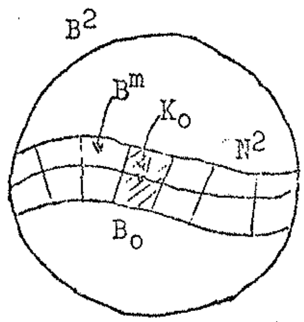
Choose a triangulation J , K of B^q, B^m such that K is simplicially collapsible

$$K = K_r \searrow K_{r-1} \searrow \dots \searrow K_0 = \text{point} .$$

Let J'' be the second barycentric derived of J, and let B_i be the ball pair

$$B_i = \left(N(K_i , J'') , \ N(K_i , K'')\right) .$$

We show inductively that B_i is unknotted.



The induction starts with $i = 0$, because B_0 is a cone on the ball or sphere pair $(\mathrm{lk}(K_0,J''),\mathrm{lk}(K_0,K''))$ which is unknotted by Theorem $9_{q-1,m-1}$. For the inductive step assume B_{i-1} unknotted. As in Theorem 5, we notice that B_i is obtained by glueing on two more small ball pairs, each by a common face, and each of which being unknotted like B_0. Hence B_i is unknotted by Lemma 19. At the end of the induction B_r is unknotted.

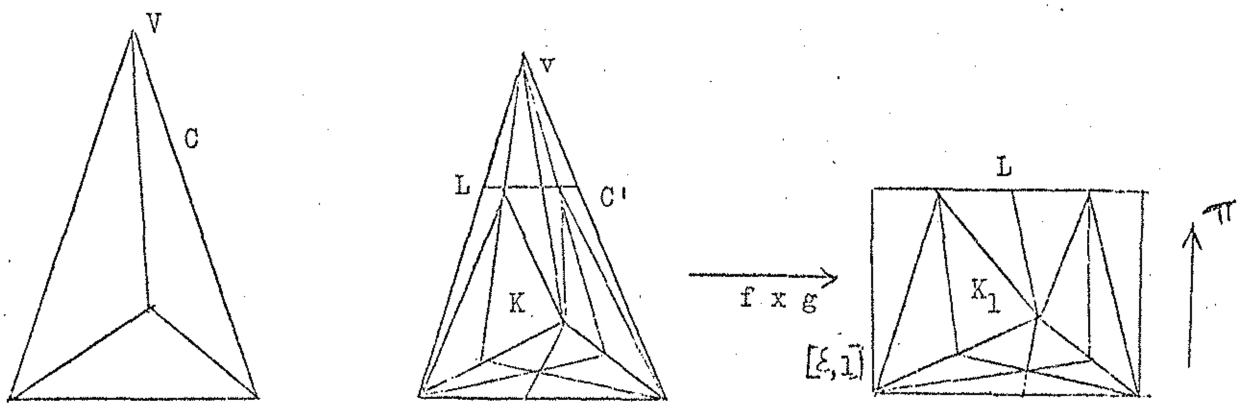
Now $B_r = (N^q,B^m)$, where N^q is a regular neighbourhood of B^m in B^q. But by Lemma 20, B^q itself is another regular neighbourhood. Therefore by Theorem 8 Part 2 there is a homeomorphism $B^q \to N^q$ keeping B^m fixed, or, in other words, a homeomorphism $B \to B_r$, showing B unknotted.

Conical subdivisions

We shall need a lemma about subdividing cones. Let $C = vX$ be a cone on a polyhedron X, with vertex v. If $Y \subset C$, the <u>subcone through Y</u> is the smallest subset of C containing Y of the form vZ, $Z \subset X$. For example a subcone through a point is a <u>generator</u> of the cone. A triangulation of C is called <u>conical</u> if the subcone through each simplex is a subcomplex.

<u>Lemma 21</u>. <u>Any triangulation of C has a conical subdivision</u>.

<u>Proof</u> . Let C also denote the given triangulation . Let $f: C \to I$ denote the piecewise linear map such that $f^{-1}(0) = v$, $f^{-1}(1) = x$, and such that f maps each generator linearly . Choose $\varepsilon > 0$, and such that $\varepsilon < fx$ for every vertex $x \in C$, $x \neq v$. Choose a first derived C' of C such that each simplex of C meeting $f^{-1}(\varepsilon)$ is starred on $f^{-1}(\varepsilon)$. Then $f^{-1}[\varepsilon, 1]$, $f^{-1}(\varepsilon)$ are subcomplexes , K , L say , of C' , and $C' = K \cup vL$. Let $g : K \to L$ be radial projection , which is a projective map and not piecewise linear . Then $f \times g : K \to [\varepsilon, 1] \times L$ is a projective homeomorphism ,



that maps K projectively onto an isomorphic complex , K_1 say , triangulating $[\varepsilon, 1] \times L$. The projection $\pi : K_1 \to L$ onto the second factor is piecewise linear, and so there are subdivisions such that $\pi : K_1' \to L'$ is simplicial . Let $K' = (f \times g)^{-1} K_1'$. Then K' is a subdivision of K , containing L' as a subcomplex , because $\pi(f \times g) : L \to L$ is the identity . Let $C'' = K' \cup vL'$. Then C'' is a subdivision of C , and is conical because K_1' is cylindrical .

<u>S h a d o w s</u>

Let I^q be the q-cube . We single out the last coordinate for special reference and write $I^q = I^{q-1} \times I$. Intuitively we regard I as vertical , and I^{q-1} as horizontal , and identify I^{q-1} with the base of the cube . Let X be a polyhedron in I^p . Imagine the sun vertically overhead , causing X to cast a shadow; a point of I^p lies in the shadow of X if it is vertically below some point of X .

Definition . Let X^* be the closure of the set of points of X that lie in the same vertical line as some other point of X (i.e. the set of points of X that either overshadow , or else are overshadowed by , some other point of X) . Then X^* is a subpolyhedron of X .

Lemma 22 . Given a ball pair (B^q, B^m) of codimension ≥ 3 , then there is a homeomorphism $B^q, B^m \longrightarrow I^q, X$ such that

i) X does not meet the base of the cube

ii) X meets each vertical line finitely

iii) $\dim X^* \leq m - 2$.

Proof . First choose the homeomorphism to satisfy i) , which is easy . Now triangulate I^q, X . Then shift all the vertices of this triangulation by arbitrary small moves into general position , in such a way that any vertex in the interior of I^p remains in the interior, and any vertex in a face of I^p remains inside that face . If the moves are sufficiently small, the new positions of vertices determine an isomorphic triangulation , and a homeomorphism of I^p onto itself . The general position ensures that conditions (ii) and (iii) are satisfied , because $m \leq q - 3$,

$$\text{and so} \quad \dim X^* \leq (m + 1) + m - q \leq m - 2 \quad .$$

Remark . The "general position" of the above proof may be analysed more rigorously as follows . Each vertex is in the interior of some face , and has coordinates in that face . The set of all such coordinates of all vertices determine a point $\mathcal{X}$ some high dimensional euclidean space , and the sufficient smallness means that x is permitted to vary in an open set, U say. To satisfy the conditions (ii) and (iii) we merely have to choose $x \in U$, so as to avoid a certain finite set of proper linear subspaces.

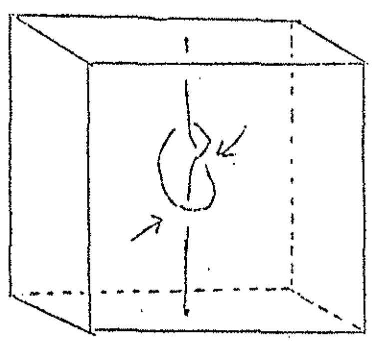
Sunny collapsing

Suppose we are given polyhedra $I^q \supset X \supset Y$, such that $X \searrow Y$ is an elementary collapse. We call this collapse <u>sunny</u> if no point of $X - Y$ lies in the shadow of X. We call a sequence of elementary sunny collapses a <u>sunny collapse</u>, and if Y is a point we call X <u>sunny collapsible</u>.

<u>Corollary to Theorem 4</u>. <u>If X is sunny collapsible then some triangulation is simplicially sunny collapsible</u>. For each elementary sunny collapse is factored in a sequence of elementary simplicial sunny collapses.

<u>Lemma 23</u>. <u>If (I^q, X) is (q,m)-ball pair of codimension ≥ 3 satisfying the conditions of Lemma 22 then X is sunny collapsible</u>.

<u>Remark</u>. Lemma 23 fails with codimension 2. The classical example of a knotted arc in I^3 gives a good intuitive feeling for the obstruction to a sunny collapse: locking down from above it is possible to start collapsing away until we hit underpasses, which are in shadow and so prevent any further progress.



<u>Definition</u>. A <u>principal k-complex</u> is a complex in which every principal simplex is k-dimensional.

Proof of Lemma 23 .

We shall construct inductively a decreasing sequence of subpolyhedra

$$X = X_o \supset X_1 \supset \ldots \supset X_m = \text{a point} ,$$

and , for each i , a homeomorphism

$$f_i : X_i \longrightarrow v\, K^{m-i-1}$$

onto a cone on a principal $(m-i-1)$-complex, satisfying the three conditions :

1) $f_i X_i^*$ does not contain the vertex of the cone , and meets each generator of the cone finitely .

2) $\dim X_i^* \leqslant m-i-2$.

3) There is a sunny collapse $X_{i-1} \searrow X_i$.

The induction starts with $X = X_o$.

Condition (2) is by hypothesis and (3) is vacuous . Choose a homeomorphism $f_o : X_o \longrightarrow \Delta$, where Δ is the standard m-simplex . Since $f_o X^*$ is a sub-polyhedron of dimension $\leqslant m-2$, we can choose a point $v \in \mathring{\Delta} - f_o X^*$, and in general position relative to $f_o X^*$. Starring Δ at v makes Δ into the cone $v\, \dot{\Delta}$ on $\dot{\Delta}$, which is principal . Condition (2) is satisfied by our choice of v .

The induction finishes with X_m = a point , and so we shall have a sunny collapse

$$X \searrow X_1 \searrow X_2 \searrow \ldots \searrow X_m$$

which will prove the lemma .

The hard part is the inductive step .

Suppose we are given $f_{i-1} : X_{i-1} \longrightarrow v\, K^{m-i}$, satisfying the three conditions, we have to construct f_i , X_i , K^{m-i-1} and prove the three conditions .

Let C , L triangulate $v K^{m-i}$, $f_{i-1} X^*_{i-1}$. By Lemma 21 we can choose C to be conical . In particular C contains a subdivision $(K^{m-i})'$ of K^{m-i} . Define

K^{m-i-1} = the $(m-i-1)$-skeleton of $(K^{m-i})'$, which is a principal complex since K^{m-i} was principal . Let C_o be the subcomplex of C triangulating the subcone $v K^{m-i-1}$. Let $e_o : C_o \to C$ be the inclusion map . We shall construct another embedding

$$e : C_o \to C$$

that differs slightly , but significantly , from e_o . Having chosen e , then there is a unique subpolyhedron X_i , and homeomorphism f_i , such that the diagram

$$\begin{array}{ccc} X_i & \xrightarrow{\subset} & X_{i-1} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} \\ C_o & \xrightarrow{e} & C \end{array}$$

is commutative .

It is no good choosing $e = e_o$, because then $X^*_i = X^*_{i-1}$ which would be of too high a dimension . In fact this is the crux of the matter : we must arrange some device for collapsing away the top-dimensional shadows of X^*_{i-1} . The first thing to observe is that the triangulation L of $f_{i-1} X^*_{i-1}$ is in no way related to the embedding of X^*_{i-1} in the cube I^q . The inverse images of simplexes of L may wrap around and overshadow each other hopelessly, so our next task is to take a subdivision that remedies this confusion . We have piecewise linear maps

$$L \xrightarrow{f_{i-1}^{-1}} X_{i-1}^{*} \xrightarrow{\pi} I^{q-1}$$

where the first is a homeomorphism , and π is vertical projection onto the base of the cube I^q . By Theorem 1 we can choose subdivisions L' of L , $(I^{q-1})'$ of I^{q-1} and a triangulation M of X_{i-1}^{*} such that the maps

$$L' \xleftarrow{f_{i-1}} M \xrightarrow{\pi} (I^{q-1})'$$

are simplicial .

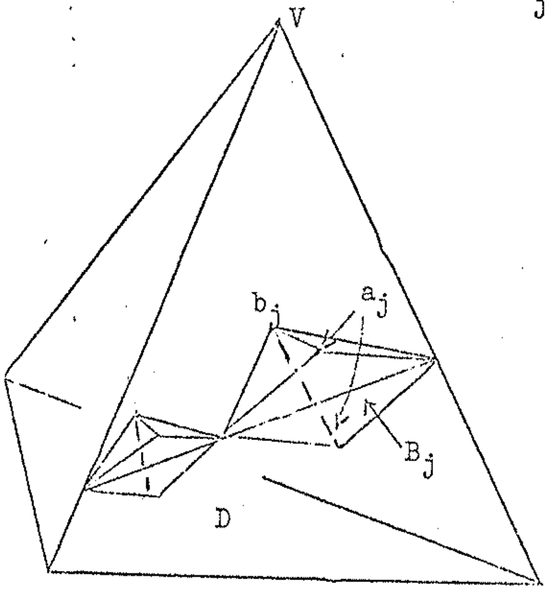
Recall that $\dim M \leqslant m-i-1$, by induction on i . Let A_1 $A_1, A_2, \ldots, A_r$ be the $(m-i-1)$-simplexes of M . Each A_j is projected non-degenerately by π , because of Lemma 22 (ii) . If $j \neq k$ there are two possibilities : either $\pi A_j \neq \pi A_k$ or $\pi A_j = \pi A_k$. In the first case π maps $\mathring{A}_j$, $\mathring{A}_k$ disjointly and so no point of $\mathring{A}_j \cup \mathring{A}_k$ overshadows any other . In the second case either $\mathring{A}_j$ overshadows $\mathring{A}_k$ or vice versa . Consequently overshadowing induces a partial ordering amongst the A'_s , and we choose the ordering $A_1, A_2, \ldots, A_r$ to be compatible with this partial ordering . We summarise the conclusion :

<u>Sublemma</u> . <u>All points of X that overshadow $\mathring{A}_k$ are contained in</u> $\bigcup_{j<k} \mathring{A}_j$

We now pass to L' . Let $B_j = f_{i-1} A_j \in L'$. The next step in the proof is to construct a little $(m-i+1)$-dimensional blister Z_j about each B_j in the cone C . The blisters are the device that enable us to make the sunny collapse, and the fact that there is just sufficient room to construct them is an indication of why codimension $\geqslant 3$ is a necessary and sufficient condition for unknotting .

Fix j. Let $\hat{B}_j$ be the barycentre of B_j. Since the base of the cone is principal, there is simplex $D^{m-i} \in (K^{m-i})'$ such that B_j is contained in the subcone $v\dot{D}$. There are two cases depending on whether or not B_j lies in $\dot{D}$, in the base of the cone. If $B_j \subset \dot{D}$, let b_j be a point in $\mathring{D}$ near $\hat{B}_j$, and let a_j be a point in the generator vB_j near B_j. If $B_j \not\subset \dot{D}$, let b_j be a point in $v\mathring{D}$ near $\hat{B}_j$, and let a_j be a pair of points on the generator through $\hat{B}_j$, near $\hat{B}_j$ and either side of $\hat{B}_j$. In either case define the blister

$$Z_j = a_j b_j B_j .$$



We choose the points sufficiently near to the barycentres so that no two blisters meet more than is necessary (i.e. $Z_j \cap Z_k = \dot{B}_j \cap \dot{B}_k$). The bottom of the blister is $a_j B_j$ and the top is $a_j b_j \dot{B}_j$. Let e_j ne the map

$$e_j : a_j B_j \longrightarrow a_j b_j \dot{B}_j$$

that raises the blister, and is given by mapping $\hat{B}_j \longrightarrow b_j$.

Now C_o meets each blister in its bottom. Therefore we can define the embedding $e : C_o \to C$ by choosing $e = e_j$ on the intersection with each blister, and $e = 1$ otherwise. In other words e is a map that raises all the blisters. Having defined e, we have completed the definition of X_i and $f_i : X_i \longrightarrow v\,K^{m-i-1}$.

There remains to verify the three conditions. Condition (2) holds because by our construction $X_i^* = X_{i-1} \cap X_i$

$$= \text{the } (m-i-2)\text{-skeleton of } M .$$

Condition (1) holds , because

$$f_i X_i^* = f_{i-1} X_i^*$$

$$\subset f_{i-1} \quad X_{i-1}^* \quad ,$$

for which the condition holds by induction .

Finally we come to condition (3) . Let $Z = \bigcup Z_j$. For each $(m-i)$-simplex $D \in (K^{m-i})'$, $\overline{v\, D - Z}$ is a ball because it is a simplex with a few blisters pushed in round the edge, and $\overline{D - Z}$ is a face . Therefore collapse each $v\,\overline{D - Z}$ from $\overline{D - Z}$. We have collapsed

$$C \searrow e\, C_o \cup Z \quad .$$

and the inverse image under f_{i-1} determines a sunny collapse

$$X_{i-1} \searrow X_i \cup f_{i-1}^{-1} Z \quad ,$$

sunny because we have not yet removed any point of X_{i-1}^* .

We now collapse the blisters as follows . Each blister meets $e\, C_o$ in its top , and so by collapsing each blister onto its top in turn, $j=1,\ldots,r$, we effect a collapse

$$e\, C_o \cup Z \searrow e\, C_o \quad .$$

If $Y_j = f_{i-1}^{-1} Z_j$, then the inverse image of this collapse determines a sequence of elementary collapses

$$X_i \cup \bigcup_1^r Y_j \searrow X_i \cup \bigcup_2^r Y_j \searrow \ldots \searrow X_i \quad .$$

Each of these elementary collapses is sunny by the Sublemma, because by the time we come to collapse Y_k , say , the only points that might have been in shadow are those in $\mathring{A}_k$, but these are sunny for we have already removed everything that overshadows them. We have demonstrated the sunny collapse $X_{i-1} \searrow X_i$, which completes the proof of Lemma 23.

Proof of Lemma 20.

We can now return to the proof of Lemma 20, which will conclude the proof of Theorem 9. Given a ball pair (B^q, B^m) of codimension $\geqslant 3$, we have to show $B^q \searrow B^m$. By Lemma 22 it suffices to show for a ball pair (I^q, X) satisfying the conditions of Lemma 22.

By Lemma 23 and the Corollary to Theorem 4 we can choose a triangulation K of X that is simplicially sunny collapsible.

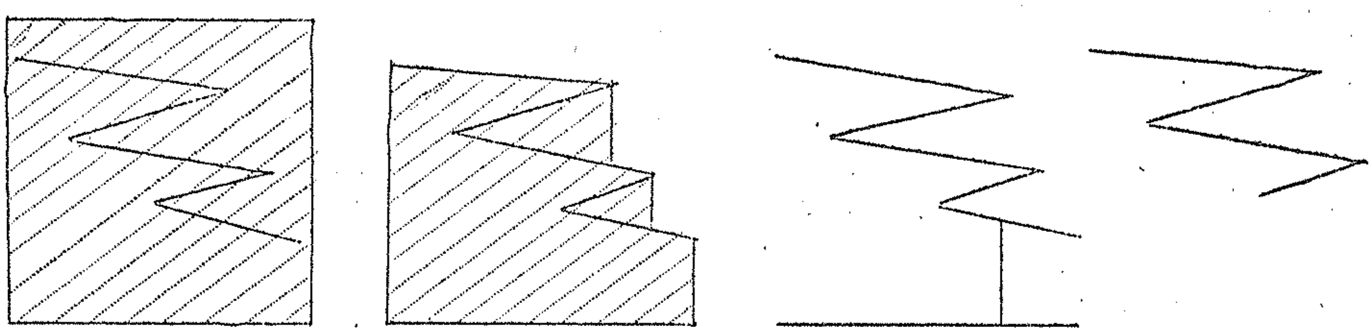
$$K = K_o \searrow K_1 \searrow \ldots \searrow K_r = \text{a point}.$$

Let L_i be the polyhedron consisting of $I^{q-1} \cup X$ together with all points in the shadow of K_i. We shall show that

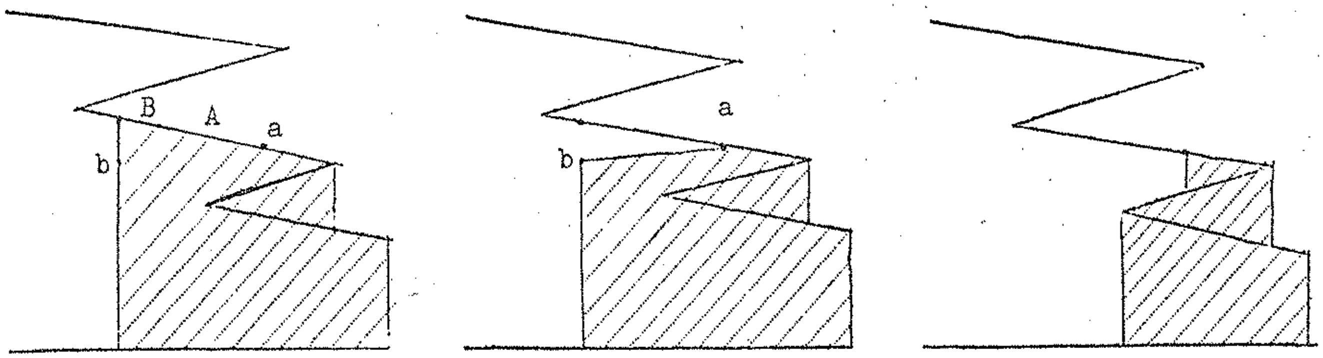
$$I^q \searrow L_o \searrow L_1 \searrow \ldots \searrow L_r \searrow X.$$

The first step is as follows. Choose a cylindrical subdivision $(I^q)'$ of I^q containing a subdivision L_o' of L. Then collapse $(I^q)' \searrow L_o'$ prismwise from the top, in order of decreasing dimension of the prisms.

The last step is easy, because L_r consists of $I^{q-1} \cup X$ joined by a single arc. Collapse I^{q-1} onto the bottom of this arc, and then collapse the arc. There remain the intermediate steps $L_{i-1} \searrow L_i$, $i = 1, \ldots, r$.



I^2 $\quad$ L_o $\quad$ L_r $\quad$ X



L_{i-1} $\qquad$ T $\qquad$ L_i

Fix i, and suppose the elementary simplicial sunny collapse $K_{i-1} \searrow K_i$ collapses A from B, when $A = aB$. Choose a point b below the barycentre $\hat{B}$ of B, and sufficiently close to $\hat{B}$ for $bA \cap X = A$ (this is possibly by Lemma 22 (ii)). Let $T = L_i$ together with all points in the shadow of $ab\dot{B}$. Since the collapse is sunny, no points of K_i overshadow $\mathring{A} \cup \mathring{B}$, and so $T \cap bA = a(bB)^{\bullet}$, $T \cup bA = L_{i-1}$. In other words collapsing bA from bB gives an elementary collapse

$$L_{i-1} \searrow T .$$

Finally collapse $T \searrow L_i$ prismwise downwards from $ab\dot{B}$, as in the first case. This completes the proof of Lemma 20 and Theorem 9.

Isotopies of balls and spheres

Recall that Lemma 16 proved that any homeomorphism of a ball onto itself keeping the boundary fixed is isotopic to the identity keeping the boundary fixed.

Corollary 1 to Theorem 9. If $q-m \geq 3$, then any two proper embeddings $B^m \subset B^q$ that agree on $\dot{B}^m$ are ambient isotopic keeping $\dot{B}^q$ fixed.

Proof. Let f, g be the embeddings. By Lemma 18 we can extend $1 : \dot{B}^q \to \dot{B}^q$ and $gf^{-1} : f\,B^m \to g\,B^m$, which agree on $\dot{B}^m$, to a homeomorphism between the ball pairs

$$h : (B^q, f\,B^m) \longrightarrow (B^q, g\,B^m) .$$

By construction $hf = g$, and, by Lemma 16, h is ambient isotopic to the identity keeping $\dot{B}^q$ fixed.

Theorem 10. Any orientation preserving homeomorphism of S^n is isotopic to the identity.

Proof: by induction on n, starting trivially with $n = 0$. Let f be the given homeomorphism. Choose a point $x \in S^n$, and ambient isotopic fx to x. This moves f to f_1, say, where $f_1 x = x$. Choose a ball B containing x in its interior. Then $B, f_1 B$ are regular neighbourhoods of x, and so by Theorem 8 ambient isotope $f_1 B$ onto B. This moves f_1 to f_2, say, where $f_2 B = B$. The restriction $f_2 | \dot{B}$ preserves orientation, and is therefore isotopic to the identity by induction. Extend the isotopy conewise to B and $\overline{S^n - B}$, making it into an ambient isotopy, that moves f_2 to f_3, say where $f_3 | \dot{B} = 1$. Apply Lemma 16 to each of B, $\overline{S^n - B}$ to ambient isotope f_3 into the identity.

Corollary 2 to Theorem 9. If $q-m \geq 3$, then any two embeddings $S^m \subset S^q$ are ambient isotopic.

Proof. If (S^q, S^m), $q > m$, is an unknotted sphere pair, then S^q is the $(q-m)$-fold suspension of S^m, and so there is

(1) an orientation reversing homeomorphism of S^q, throwing S^m

onto itself, and

(2) an orientation preserving homeomorphism of S^q, throwing S^m onto itself with reversed orientation.

Let $f, g : S^m \to S^q$ be the two given embeddings. Since $q - m \geqslant 3$, the unknotting gives a homeomorphism $(S^q, fS^m) \longrightarrow (S^q, gS^m)$, which we can choose to be orientation preserving on S^2 by (1), and which is therefore isotopic to the identity by Theorem 10. Therefore f is ambient isotopic to f_1, say, such that $f_1S^m = gS^m$. Let $h = gf_1^{-1} : f_1S^m \longrightarrow f_1S^m$. By (2) above and Theorem 10, we can choose f_1 so that h is orientation preserving.

Now apply Theorem 10 to the smaller sphere S^m, to obtain an isotopy from the identity to h; suspend this isotopy of S^m into an ambient isotopy of S^q moving f_1 into g.

<u>Remark</u>. The above two corollaries are also true for unknotted ball and sphere pairs of codimension 1 and 2. The aim of the next four chapters is to obtain similar results for arbitrary manifolds.

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INSTITUT DES HAUTES ETUDES SCIENTIFIQUES

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Seminar on Combinatorial Topology

by E.C. ZEEMAN

Chapter 5 : ISOTOPY

The natural way to classify embeddings of one manifold in another is by means of isotopy. But there are several definitions of isotopy, and the purpose of this chapter is to prove three of the definitions equivalent. The three that we consider (of which the first two were mentioned in Chapter 2) are :

(1) Isotopy, sliding the smaller manifold in the larger through a family of embeddings ;

(2) Ambient isotopy, rotating the larger manifold on itself, carrying the smaller with it ;

(3) Isotopy by moves, making a finite number of local moves, each inside a ball in the larger manifold, analogous to moving a complex in Euclidean space by shifting the vertices, like the moves of classical knot theory .

Since any homeomorphism of a ball keeping the boundary fixed is isotopic to the identity it follows at once that

$$\text{isotopy by moves} \Longrightarrow \text{ambient isotopy} \Longrightarrow \text{isotopy} .$$

In Theorems 11 and 12 we shall show that these arrows can be reversed . To reverse the second arrow, that is to cover an isotopy by an ambient isotopy, it is necessary to impose a local unknottedness condition on the isotopy .

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For otherwise the knots of classical knot theory give counterexamples of embeddings that are mutually isotopic but not ambient isotopic. However the results of Chapter 4 show that this phenomenon occurs only in codimension 2, and possibly codimension 1 .

Throughout this chapter we shall be considering embeddings of a <u>compact</u> m-manifold M in a q-manifold Q , which may or may not be compact . We restrict attention to <u>proper</u> embeddings $f : M \to Q$; recall that f is proper provided $f^{-1}\dot{Q} = \dot{M}$; in particular if M is closed , then any embedding of M in $\mathring{Q}$ is proper . By a <u>homeomorphism of</u> Q , we mean a homeomorphism of Q onto itself ; in particular a homeomorphism is a proper embedding .

<u>Definitions of Isotopy</u>

Recall definitions that have been given in previous chapters .

(1) A <u>homeomorphism</u> h of M is a homeomorphism of M onto itself . If $Y \subset M$ and $h \mid Y$ = the identity we say <u>h keeps Y fixed</u> .

(2) An <u>isotopy</u> of M in Q is a proper level preserving embedding

$$F : M \times I \to Q \times I \ .$$

Denote by F_t the proper embedding $M \to Q$ defined by $F(x,t) = (F_t x, t)$, all $x \in M$. The subspace $\bigcup_{t \in I} F_t M$ of Q is called the <u>track left by</u> the isotopy .

If $X \subset M$, we say F <u>keeps X fixed</u> if $F(x,t) = F(x,0)$, all $x \in X$ and $t \in I$.

(3) The embeddings f , $g : M \to Q$ are <u>isotopic</u> if there exists an isotopy F of M in Q with $F_0 = f$, $F_1 = g$.

(4) An <u>ambient isotopy</u> of Q is a level preserving homeomorphism $H : Q \times I \to Q \times I$ such that $H_0 =$ the identity, where as above H_t is defined by $H(x,t) = (H_t x, t)$, all $x \in Q$. We say that H <u>covers</u> the isotopy F if the diagram

$$\begin{tikzcd} Q \times I \arrow[dr, "H"] & \\ & Q \times I \\ M \times I \arrow[uu, "F_0 \times I"] \arrow[ur, "F"'] & \end{tikzcd}$$

is commutative; in other words $F_t = H_t F_0$, all $t \in I$.

(5) The embeddings $f, g : M \to Q$ are <u>ambient isotopic</u> if there is an ambient isotopy H of Q such that $H_1 f = g$.

<u>Remark</u>. If $M = Q$, then a proper embedding $M \times I \to Q \times I$ is the same as a homeomorphism $Q \times I \to Q \times I$. Therefore since we have restricted attention to proper embeddings, the only difference between an isotopy of Q in Q, and an ambient isotopy of Q, is that the latter has to start with the identity; consequently two homeomorphisms of Q are isotopic if and only if they are ambient isotopic.

(6) A homeomorphism or ambient isotopy of Q is said to be <u>supported</u> by X if it keeps $Q - X$ fixed. By continuity the frontier $\overline{X} \cap \overline{Q - X}$ of X in Q must also be kept fixed.

(7) An <u>interior move</u> of Q is a homeomorphism of Q supported by a ball keeping the boundary of the ball fixed. A <u>boundary move</u> of Q is a

homeomorphism of Q supported by a ball that meets $\dot{Q}$ in a face ; the complementary face is the frontier of the ball that is kept fixed by continuity.

(8) The embeddings f, g are isotopic by moves if there is a finite sequence $h_1, h_2, \ldots, h_n$ of moves of Q such that $h_1 h_2 \ldots h_n f = g$.

Locally unknotted embeddings

Let $f : M \to Q$ be a proper embedding . Let Q_0 be a regular neighbourhood of fM in Q . Let K, L be triangulations of M, Q_0 such that $f : K \to L$ is simplicial . We say that f is a locally unknotted embedding if, for each vertex $v \in K$, the pair

$$\left(\mathrm{lk}\ (fv, L), f(\mathrm{lk}(v, K) \right)$$

is unknotted . Notice that since the embedding is proper, the pair is either a sphere or ball pair according to whether $v \in \overset{\circ}{M}$ or $v \in \dot{M}$.

Corollary 3 to Lemma 9 . Any proper embedding of codimension ≥ 3 is locally unknotted . Therefore then we say "locally unknotted" in future we refer only to the cases of codimension 1 or 2 .

Remark 1 . The definition is independent of Q_0 , and the triangulations, because if all the links are unknotted, then the same is true for any subdivisions of K , L , and hence also true for any other triangulations .

Remark 2 . An equivalent condition is to say that the closed stars of vertices are unknotted ball pairs, but in codimensions 1 and 2 the equivalence, for a boundary vertex , depends upon a result that we have quoted, but not proved, that if an unknotted ball pair has an unknotted face then the complementary face is also unknotted .

Remark 3 . If $f : M \to Q$ is locally unknotted embedding, then so is the restriction to the boundaries $f : \dot{M} \to \dot{Q}$.

Remark 4 . We say a ball pair (B^q, B^m) is locally unknotted if the inclusion is so ; for example this always happens in codimension $\geqslant 3$ or in the classical case $(q,m) = (3,1)$. Suppose (B^q, B^m) is locally unknotted , and let N^q be a regular neighbourhood of B^m in B^q . Then although (B^q, B^m) may be (globally) knotted, it can be shown that (N^q, B^m) is unknotted , by adapting the proofs of Lemma 19 and Theorem 9 .

Locally unknotted isotopies

We say an isotopy $F : M \times I \to Q \times I$ is locally unknotted if

(i) each level $F_t : M \to Q$ is a locally unknotted embedding , and

(ii) for each subinterval $J \subset I$, the restriction $F : M \times J \to Q \times J$ is a locally unknotted embedding .

Remark 1 . If F is a locally unknotted isotopy, then so is the restriction to the boundaries $F : \dot{M} \times I \to \dot{Q} \times I$. The proof is non-trivial (as in Remark 2 above) and is omitted . As we need to use the fact in Corollary 1 to Theorem 12 below , we should either accept it without proof , or else add it as an additional condition in the definition of locally unknotted isotopy .

Remark 2 . Any isotopy of codimension $\geqslant 3$ is locally unknotted .

Remark 3 . The above definition is tailored to our needs . There is an alternative definition as follows ; we say an isotopy is locally

<u>trivial</u> if, for each $(x,t) \in M \times I$, there exists an m-ball neighbourhood A of x in M, and an interval neighbourhood J of t in I, and a commutative diagram

$$\begin{tikzcd} A \times J \arrow[r, "\subset"] \arrow[d, "\subset"] & \Sigma A \times J \arrow[d, "G"] \\ M \times I \arrow[r, "F"] & Q \times I \end{tikzcd}$$

where Σ denotes $(q-m)$-fold suspension, and G is a level preserving embedding onto a neighbourhood of $F(x,t)$. It is easy to verify that

F is a locally trivial isotopy

$\Downarrow$

F is a locally unknotted isotopy

$\Downarrow$

F is an isotopy and a locally unknotted embedding.

We shall prove in Corollary to Theorem 12 that the top arrow can be reversed. Therefore a locally trivial isotopy is the same as a locally unknotted isotopy. We conjecture the bottom arrow can also be reversed - it is a problem depending upon the Schönflies problem, and the unique factorisation of sphere knots.

We now state the theorems, and then prove them in the order stated.

<u>Theorem 11</u>. <u>Let H be an ambient isotopy of Q with compact support keeping Y fixed. Then H_1 can be expressed as the product of a finite number of moves keeping Y fixed</u>.

Addendum. Given any triangulation of a neighbourhood of X, then the moves can be chosen to be supported by the vertex stars. Therefore the moves can be made arbitrarily small.

Corollary. Let M be compact, let $f : M \to Q$ be a proper locally unknotted embedding, and let g be a homeomorphism of M that is isotopic to the identity keeping $\dot{M}$ fixed. Then g can be covered by a homeomorphism h of Q keeping $\dot{Q}$ fixed; in other words the diagram is commutative:

$$\begin{tikzcd} Q \arrow[r, "h"] & Q \\ M \arrow[u, "f"] \arrow[r, "g"] & M \arrow[u, "f"'] \end{tikzcd}$$

Remark.

In fact the corollary is improved by Theorem 12 below, to the extent of covering not only the homeomorphism but the whole isotopy. However we need to use the corollary in the proof of Theorem 14, in the course of proving Theorem 12.

Theorem 12. (Covering isotopy theorem).

Let $F : M \times I \to Q \times I$ be a locally unknotted isotopy keeping $\dot{M}$ fixed, and let N be a neighbourhood of the track left by the isotopy. Then F can be covered by an ambient isotopy supported by N keeping $\dot{Q}$ fixed.

Addendum. Let X be a compact subset of $\dot{Q}$ and N a neighbourhood of X in Q. Then an ambient isotopy of $\dot{Q}$ supported by X can be extended to an ambient isotopy of Q supported by N.

Corollary 1 . Theorem 2 remains true if we omit "keeping $\dot{M}$ fixed" from the hypothesis and "keeping $\dot{Q}$ fixed" from the thesis .

Corollary 2 . Let $f, g : M \to Q$ be two proper locally unknotted embeddings . Then the following four conditions are equivalent :

(1) f, g are isotopic by a locally unknotted isotopy

(2) f, g are ambient isotopic

(3) f, g are ambient isotopic by an ambient isotopy with compact support

(4) f, g are isotopic by moves .

Corollary 3 . An isotopy is locally trivial if and only if it is locally unknotted .

Proof of Theorem 11

We are given an ambient isotopy $H : Q \times I \to Q \times I$ with compact support, and have to show that H_1 is a composition of moves . We first prove the theorem for the case when Q is a combinatorial manifold , namely a simplicial complex in E^n, say . Then $Q \times I$ is a cell complex in $E^n \times I$. We regard E^n as horizontal and I as vertical.

Let K , L be subdivisions of $Q \times I$ such that $H : K \to L$ is simplicial (in fact a simplicial isomorphism) . Let A be a principal simplex of L , and B a vertical line element in A . Define $\theta(A)$ to be the angle between $H^{-1}(B)$ and the vertical . Since $H : K \to L$ is simplicial, this does not depend upon the choice of B . Since H is level preserving , $\theta(A) < \frac{\pi}{2}$. Define $\theta = \max \theta(A)$, the maximum taken over all principal simplexes of L . Then $\theta < \frac{\pi}{2}$.

Now let W denote the set of all linear maps $Q \longrightarrow I$ (i.e. maps that map each simplex of Q linearly into I) . Let

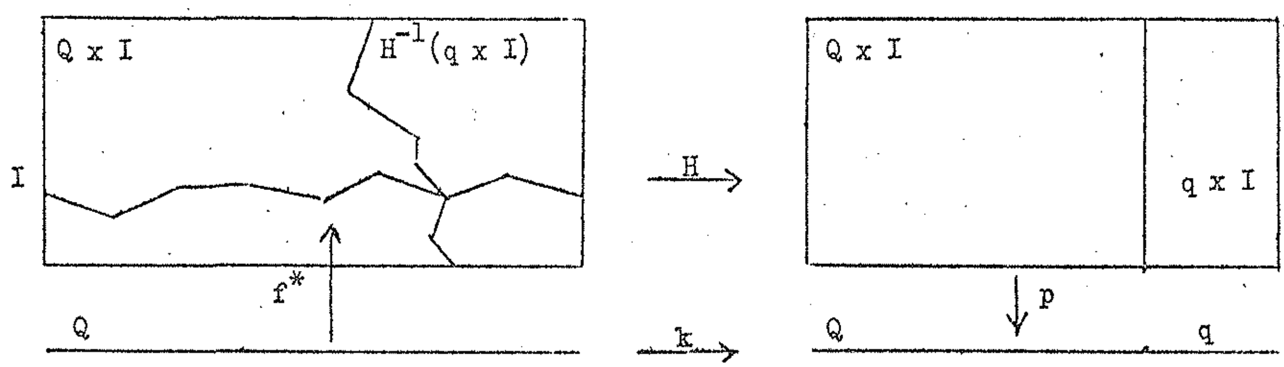
$$W_\delta = \left\{ f \in W \; ; \; \max f = \min f < \delta \right\}.$$

If $f \in W$, denote by f^* the graph of f , given by

$$f^* = 1 \times f : Q \longrightarrow Q \times I \quad .$$

Then f^* maps each simplex of Q linearly into $E^n \times I$. Let $\varphi(f)$ be the maximum angle that any simplex of $f^* Q$ makes with the horizontal . Given $\varepsilon > 0$, there exists $\delta > 0$, such that if $f \in W_\delta$, then $\varphi(f) < \varepsilon$, for choose δ sufficiently small compared with the 1-simplexes of Q . Choose $\varepsilon < \frac{\pi}{2} - \theta$, and choose δ accordingly .

Now let f be a map in W_δ, and q a point of Q . Consider the intersections of the arc $H^{-1}(q \times I)$ with $f^* Q$; we claim there is exactly one intersection .



For since f^* is a graph , $f^* Q$ separates the complement $(Q \times I) - f^* Q$ into points above and below the graph . If there were no intersection , then the arc would connect the below-point $H^{-1}(q,0)$ to the above-point $H^{-1}(q,1)$, contradicting their separation . At each intersection, since $\varphi(f) + \theta < \frac{\pi}{2}$

the arc , oriented by I , passes from below to above . Hence there can be at most one intersection .

Let $p : Q \times I \longrightarrow Q$ denote the projection onto the first factor . Then

$$k = p\,H\,f^* : Q \longrightarrow Q$$

is a $1 - 1$ map by the above claim , and so is a (piecewise linear) homeomorphism of Q .

By the compactness of Q and I , choose a sequence of maps $f_0, f_1, \ldots, f_n$ in W_δ , such that $f_0(Q) = 0$, $f_n(Q) = 1$, and for each , f_{i-1} and f_i agree on all but one , v_i say , of the vertices of Q . Define $k_i = p\,H\,f_i^*$. Then $k_0 = H_0 =$ the identity, and $k_n = H_1$. Define $h_i = k_i k_{i-1}^{-1}$. Then h_i is a homeomorphism of Q supported by the ball $k_i\,(\overline{st}(v_i,Q))$, keeping $k_i\,(lk(v_i,Q))$ fixed , and so is a move . Therefore $H_1 = h_n h_{n-1} \ldots h_1$, a composition of moves .

If H keeps Y fixed , then $k_i \mid Y = k_0 \mid Y$, for each i , and so each move h_i keeps Y fixed . In particular the moves keep $Q - X$ fixed , and are supported by X .

Suppose now that Q is a compact manifold ; let $T \longrightarrow Q$ be a triangulation in the structure . We have proved the theorem for T and so it also follows for Q .

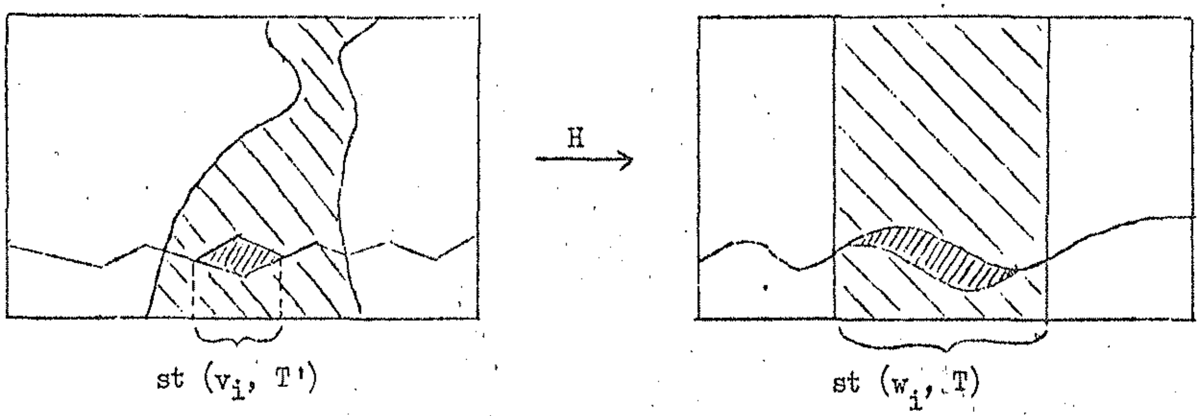
Suppose now that Q is non-compact . Let N be a regular neighbourhood of X in Q . Then N is a compact submanifold , and $N \cap (\overline{Q - N}) \subset Y$. Therefore $H \mid N \times I$ is an ambient isotopy of N keeping $N \cap Y$ fixed , and by the compact case $H_1 \mid N$ is a composition of moves supported by X keeping $N \cap Y$ fixed . The moves can be extended by the identity to moves of Q keeping Y fixed , and so H_1 is composition of moves of Q . The proof of Theorem 11 is complete .

Proof of the Addendum to Theorem 11

We are given a triangulation $T \to N$ of a neighbourhood of X, and have to show that the moves chosen to be supported by the vertex stars of T. Without loss of generality we can assume N is a regular neighbourhood, because any neighbourhood contains a regular neighbourhood. Therefore T is a combinatorial manifold. Let β denote the open covering of $N \times I$:

$$\beta = \{ st(w, T) \times I ; w \in T \},$$

where w runs over the vertices of T. Let λ be the Lebesgue number of the covering $H^{-1}\beta$ of $N \times I$. Choose a subdivision T' of T such that the mesh of the star covering of T' is less than $\lambda/2$. In the above proof of Theorem 11 use T' instead of Q, and choose δ with the additional restriction that $\delta < \lambda/2$.



Continuing with the same notation as in the proof of Theorem 11, for each i the ball $f_i^*(st(v_i, T'))$, is of diameter less than λ, and so is contained in $H^{-1}(st(w_i, T) \times I)$ for some vertex $w_i \in T$. Therefore

$$\text{support } h_i \subset k_i\,(\overline{st}(v_i, T')) \subset st(w_i, T)$$

as desired.

Proof of the Corollary to Theorem 11 .

Given $M \xrightarrow{g} M \xrightarrow{f} Q$, where g is isotopic to the identity keeping $\dot{M}$ fixed , we leave to cover g by a homeomorphism h of Q .

Let N be a regular neighbourhood of fM in Q , and choose triangulations of M , N - call them by the same names - such that $f : M \to N$ is simplicial . By the Addendum we can write

$$g = g_1 g_2 \cdots g_n \quad ,$$

where g_i is supported by the ball $B_i^m = \overline{st}(v_i, M)$, for some vertex $v_i \in M$. Let $B_i^q = \overline{st}(fv_i, Q)$. Then the ball pair $(B_i^q , f B_i^m)$ is unknotted , because f is locally unknotted , and therefore the homeomorphism $fg_i f^{-1}$ of the smaller ball can be suspended to a homeomorphism , h_i say, of the larger ball . Since g keeps $\dot{M}$ fixed , g_i keeps $\dot{B}_i^m$ fixed , and so h_i keeps $\dot{B}_i^q$ fixed. Therefore h_i extends to a move of Q keeping $\dot{Q}$ fixed . The composition $h = h_1 h_2 \cdots h_n$ covers g .

C o l l a r s

Before proving Theorem 12, it is necessary to prove a couple of theorems about collars of compact manifolds . The theorems can be generalised to non-compact manifolds, but we shall only need the compact case . Define a collar of M to be an embedding

$$c : \dot{M} \times I \longrightarrow M$$

such that $c(x,0) = x$, all $x \in \dot{M}$.

Let $f : M \to Q$ be a proper locally-unknotted embedding between two compact manifolds, and let c , d be collars of M , Q . We say c , d are *compatible with f* if the diagram

$$\begin{tikzcd} \dot{Q} \times I \arrow[r, "d"] & Q \\ \dot{M} \times I \arrow[u, "f \times I"] \arrow[r, "c"] & M \arrow[u, "f"'] \end{tikzcd}$$

is commutative and $\operatorname{im} d \cap \operatorname{im} f = \operatorname{im} fc$.

Lemma 24 . *Given a proper locally unknotted embedding between compact manifolds, there exist compatible collars* .

Corollary . *Any compact manifold has a collar* . (For in the lemma choose the smaller manifold to be a point) .

Proof of Lemma 24 .

Let M^+ denote the mapping cylinder of $\dot{M} \subset M$. Then $M^+ = \dot{M} \times I \cup M$, with the identification $(x, 1) = x$, and the induced structure . Then M^+ has a natural collar . The given proper embedding $f : M \longrightarrow Q$ induces a proper embedding $f^+ : M^+ \longrightarrow Q^+$ with which the natural collars are compatible .

Let ρ denote the retraction maps of the mapping cylinders , shrinking the collars ; then the diagram

$$\begin{tikzcd} Q^+ \arrow[r, "\rho"] & Q \\ M^+ \arrow[u, "f^+"] \arrow[r, "\rho"] & M \arrow[u, "f"'] \end{tikzcd}$$

is commutative . We shall produce homeomorphisms c,d (not merely maps) such that

$$\begin{tikzcd} Q^+ \arrow[r, "d"] & Q \\ M^+ \arrow[u, "f^+"] \arrow[r, "c"] & M \arrow[u, "f"'] \end{tikzcd}$$

is commutative, and such that c,d agree with ρ on the boundaries . The restrictions of c,d to the natural collars will prove the lemma .

Choose triangulations of M,Q - call them by the same names - such that f is simplicial, and let M' , Q' denote the barycentric derived complexes . For each p-simplex $A \in \dot{M}$, let A^* denote its dual in $\dot{M}$; more precisely A^* is the $(m-1-p)$-ball in $\dot{M}'$ given by

$$A^* = \bigcap_{v \in A} \text{st}\,(v,\dot{M}') \quad .$$

Using the linear structure of the prisms $A \times I$, $A \in \dot{M}$, define the m-ball $A^+ \subset \dot{M} \times I$ to be the join

$$A^+ = (A \times 0)\ (A^* \times 1) \quad .$$

The set of all such balls cover $\dot{M} \times I$ and determine a triangulation of $\dot{M} \times I$; the latter agrees with M' on the overlap , and so together with M' determines a triangulation of M^+ .

Order the simplexes $A_1, A_2, \ldots, A_r$ of K in an order of locally increasing dimension (i.e. of $A_i < A_j$ then $i \leq j$) . Similarly order the simplexes B_1 , $B_2, \ldots, B_s$ of L such that $fA_i = B_i$, $1 \leq i \leq r$. Define inductively ,

$$M_0 = M', \quad M_i = M_{i-1} \cup A_i^+$$

$$Q_0 = Q', \quad Q_i = Q_{i-1} \cup B_i^+ \quad .$$

We have ascending sequences of subcomplexes

$$M' = M_0 \subset M_1 \subset \ldots \subset M_r = M_{r+1} = \ldots = M_s = M^+$$

$$Q' = Q_0 \subset Q_1 \subset \ldots\ldots\ldots\ldots\ldots\ldots\ldots \subset Q_s = Q^+ ,$$

such that $f^+ M_i \subset Q_i$, for each i. We shall show inductively there exist homeomorphisms c_i, d_i such that

$$\begin{tikzcd} Q_i \arrow[r, "d_i"] & Q \\ M_i \arrow[u, "f^+"] \arrow[r, dash, "c_i"] & M \arrow[u, "f"'] \end{tikzcd}$$

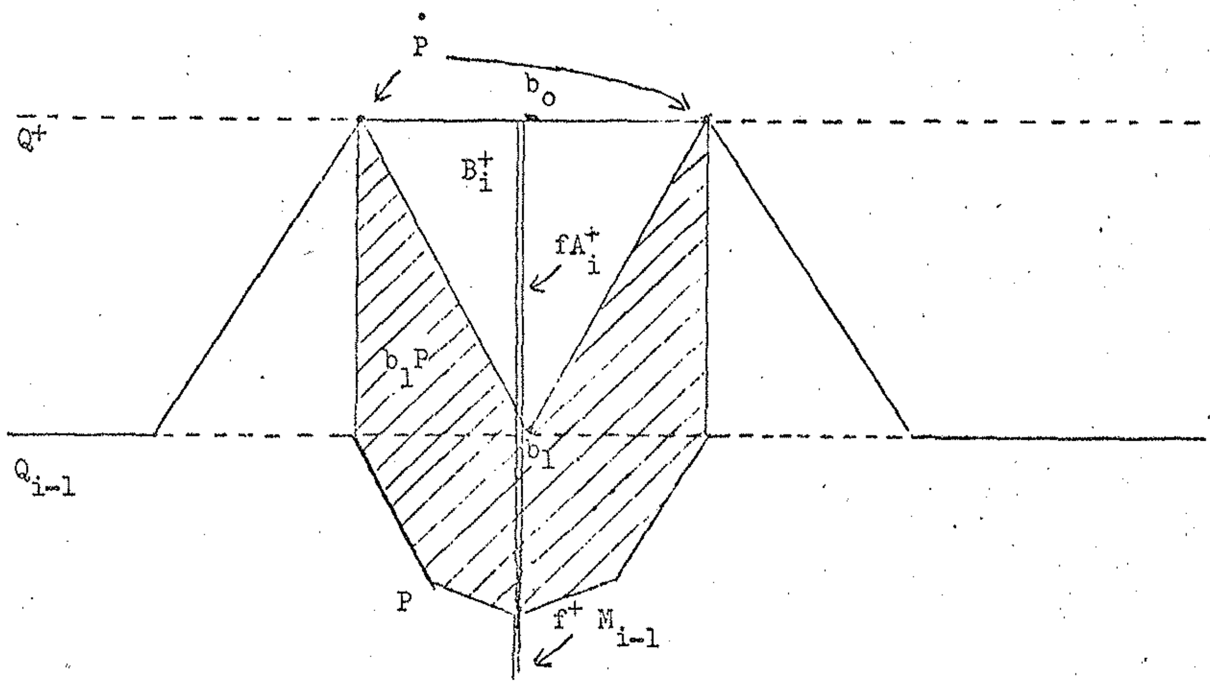
is commutative, and such that c_i, d_i agree with ρ on the boundaries. The induction begins trivially with $c_o = d_o =$ identities, and ends with what we want.

For the inductive step, fix i, and assume c_{i-1}, d_{i-1} to be defined. There are two cases.

<u>Case (i), $i \leqslant r$</u>. For $j = 0, 1$ let a_j denote the barycentre of $A_i \times j$, and let $b_j = f^+ a_j$. Let P denote the $(q-1, m-1)$ ball pair

$$P = \Big(\operatorname{lk}(b_1, Q_{i-1}), \operatorname{lk}(b_1, f^+ M_{i-1}) \Big),$$

which is unknotted because by hypothesis $f : M \to Q$ is locally unknotted, and so by induction $f^+ : M_{i-1} \to Q_{i-1}$ is also. Then $b_1 P$ is the cone pair on P, and $b_0 b_1 \dot{P}$ the cone pair on $b_1 \dot{P}$.



<u>Sublemma</u>. <u>There exists a homeomorphism $h : b_0 b_1 \dot{P} \cup b_1 P \to b_1 P$ that maps $b_0 \to b_1$, is the identity on P, and maps $b_0 \dot{P} \to b_1 \dot{P}$ linearly</u>.

<u>Proof</u>. If P were a standard ball pair, and $b_0 P$ a cone on P, and b_1 the barycentre of (the smaller of the pair) $b_0 P$, then the proof would be trivial by linear projection. An unknotting homeomorphism from P onto a standard pair maps the given set-up onto the standard set-up, and the sublemma follows by composition.

Returning to the proof of the lemma, notice that $b_0 b_1 \dot{P}$ is none other than the ball pair $(B_i^+, f^+ A_i^+)$, and so h extends by the identity to a homeomorphism of manifold pairs

$$h : (Q_i, f^+ M_i) \longrightarrow (Q_{i-1}, f^+ M_{i-1}) .$$

Define $c_i = c_{i-1}(f^+)^{-1}hf^+$ and $d_i = d_{i-1}h$. Then c_i, d_i are homeomorphisms satisfying the commutativity condition.

Finally we have to show that c_i, d_i agree with ρ on the boundaries. For points outside A_i^+, B_i^+ this follows by induction. For points in A_i^+, B_i^+ it follows from the diagram

$$\begin{tikzcd} b_0\dot{P} \arrow[rr, "h"] \arrow[dr, "d_i = \rho"'] & & b_1\dot{P} \arrow[dl, "\rho = d_{i-1}"] \\ & b_1(\rho\dot{P}) & \end{tikzcd}$$

which is commutative by the linearity of the sublemma.

<u>Case (ii) $i > r$</u>. This case is simplex, because only the larger manifold Q is concerned. In case (i) ignore the smaller ball; the proof gives a homeomorphism $h : Q_i \longrightarrow Q_{i-1}$ keeping $M_i = M_{i-1}$ fixed. Define $c_i = c_{i-1}$ and $d_i = d_{i-1}h$. The proof of Lemma 24 is complete.

Our next task is to improve Lemma 24 in Theorem 14 to the extent of moving the smaller collar from thesis to hypothesis. First it is necessary to show, in Theorem 13, that any two collars of the same manifold are ambient isotopic, and for this we need three lemmas. Lemma 24 is about shortening a collar; Lemma 25 is about isotoping a homeomorphism which is not level preserving into one which is level preserving over a small subinterval; and Lemma 26 is about isotoping an isotopy. In each lemma an isotopy is constructed, and we must be careful to avoid the standard mistake and make sure that it is a polymap (i.e. piecewise linear).

<u>Notation</u>. Suppose $0 < \varepsilon \leq 1$. Let I_ε denote the interval $[0, \varepsilon]$. Given a collar c of M, define the <u>shortened collar</u>.

$$c_\varepsilon : \dot{M} \times I \longrightarrow M$$

by $c_\varepsilon (x,t) = c(x, \varepsilon t)$, all $x \in \dot{M}$, $t \in I$.

Lemma 25. The collars c, c_ε are ambient isotopic keeping $\dot{M}$ fixed.

Proof. First lengthen the collar c as follows. The image of c is a submanifold of M, and so the closure of the complement is also a submanifold (by Lemma 17), with boundary $c(\dot{M} \times 1)$. Therefore the latter has a collar, which we can add to c to give a collar, d say, of M such that $c = d_{1/2}$. Then $c_\varepsilon = d_{\varepsilon/2}$.

By Lemma 16 there is an ambient isotopy G of I, keeping $\dot{I}$ fixed, and finishing with the homeomorphism that maps $[0, 1/2]$, $[1/2, 1]$ linearly onto $[0, \varepsilon/2]$, $[\varepsilon/2, 1]$. Let $1 \times G$ be the ambient isotopy of $\dot{M} \times I$, and let H be the image of $I \times G$ under d. Since $1 \times G$ keeps $\dot{M} \times \dot{I}$ fixed, we can extend H by the identity to an ambient isotopy H of M keeping $\dot{M}$ fixed. Then $H_1 c = c_\varepsilon$, proving the lemma.

Lemma 26. Let X be a polyhedron, and $f : X \times I_\varepsilon \longrightarrow X \times I$ an embedding such that $f | X \times 0$ is the identity. Then there exists δ, $0 < \delta < \varepsilon$, and an embedding $g : X \times I \longrightarrow X \times I$ such that:

(i) g is level preserving in I_δ;

(ii) g is ambient isotopic to f keeping $X \times \dot{I}$ fixed;

(iii) If $Y \subset X$, and $f | Y \times I_\varepsilon$ is already level preserving, then we can choose g to agree with f on $Y \times I_\varepsilon$, and the ambient isotopy to keep $f(Y \times I_\varepsilon)$ fixed.

Proof. Let K, L be triangulations of $X \times I_\varepsilon$, $X \times I$ such that $f : K \longrightarrow L$ is simplicial (in fact a simplicial embedding). Choose δ,

$0 < \delta < \varepsilon$, so small that no vertices of K or L lie in the interval $0 < t \leq \delta$. Choose first deriveds K' , L' of K , L according to the rule : if the interior of a simplex meets the level $X \times \delta$, then star it at a point on $X \times \delta$; otherwise star it barycentrically . Let $g : K' \longrightarrow L'$ be the first derived map . We verify the three properties :

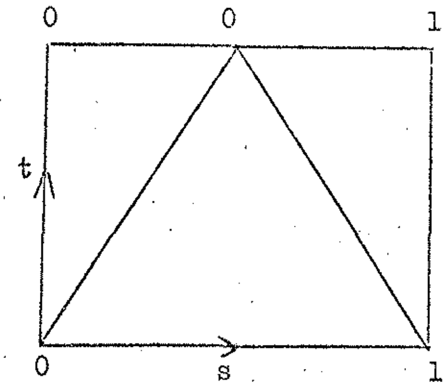
Property (i) holds because by construction g is level preserving at the levels 0 and δ , and any point in between these two levels lies on a unique interval that is mapped linearly onto another interval , both intervals beginning (at the same point) in $X \times 0$ and ending in $X \times \delta$.

To prove property (ii) define another first derived L'' of L by the rule : if a simplex lies in fK then star it so that $f : K' \longrightarrow L''$ is simplicial: otherwise star it barycentrically . The isomorphism $L'' \longrightarrow L'$ is isotopic to the identity by Lemma 15 Corollary 1 , and so f , g are ambient isotopic . The isotopy keeps fixed any subcomplex of L on which L' and L'' agree , and in particular keeps $X \times \dot{I}$ fixed .

To prove property (iii) we put extra conditions on the choices of K and L' . Choose K so as to contain $Y \times I_{\varepsilon}$ as a subcomplex . Having chosen K , K' , and therefore L'' , then choose L' so as to agree with L'' on $f(Y \times I_{\varepsilon})$, this being compatible with the condition of starring on the δ-level because $f | Y \times I_{\varepsilon}$ is already level preserving . Therefore H keeps $f(Y \times I_{\varepsilon})$ fixed .

<u>Lemma 27</u> . <u>Let $g : X \times I \longrightarrow X \times I$ be an ambient isotopy of X . Let h be the ambient isotopy of X that consists of the identity for half the time followed by g at twice the speed . Then g , h are ambient isotopic keeping $X \times \dot{I}$ fixed</u> .

<u>Proof</u>. Triangulate the square I^2 as shown, and let $u : I^2 \longrightarrow I$ be the simplicial map determined by mapping the vertices to 0 or 1 as shown. Define $G : (X \times I) \times I \longrightarrow (X \times I) \times I$ by



$$G\Big((x,s),t\Big) = \Big((g_{u(s,t)}x, s), t\Big) .$$

Then (i) G is a level preserving homeomorphism, and

(ii) G is piecewise linear, because the graph ΓG of G is the intersection of two subpolyhedra of $(X \times I^2)^2$:

$$\Gamma G = \Big((1 \times u)^2\Big)^{-1} \Gamma g \cap \Big(X^2 \times \Gamma 1\Big) ,$$

where $(1 \times u)^2$ denotes the map $(X \times I^2)^2 \longrightarrow (X \times I)^2$, where Γg is the graph of g, and $\Gamma 1$ is the graph of the identity on I^2.

Therefore G is an isotopy of $X \times I$ in itself. By the construction of u, G moves g to h keeping $X \times \dot{I}$ fixed. Therefore g, h are ambient isotopic keeping $X \times \dot{I}$ fixed.

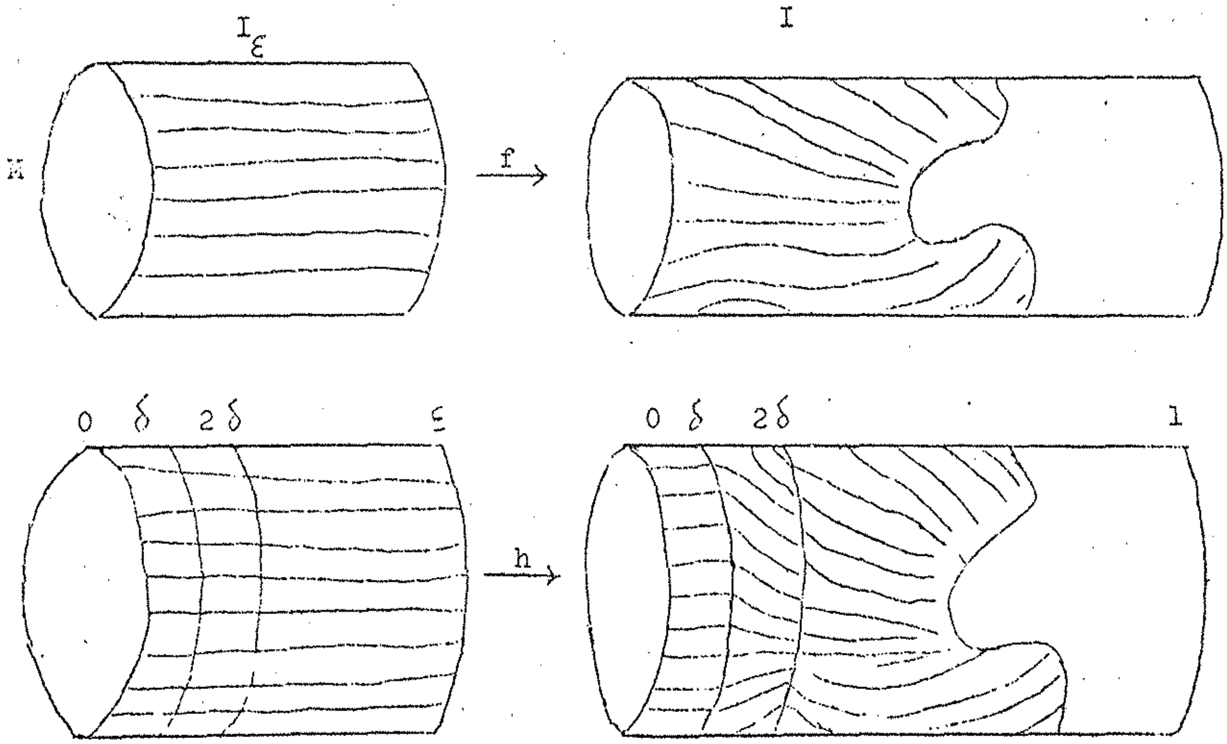
<u>Theorem 13</u>. <u>If M is compact, then any two collars of M are ambient isotopic keeping $\dot{M}$ fixed</u>.

<u>Proof</u>. Given two collars, the idea is to (i) ambient isotope one of them until it is level preserving relative to the other on a small interval, (ii) isotope it further until it agrees with the other on a smaller interval, and then (iii) isotope both onto this common shortened collar.

Let $c,d : \dot{M} \times I \longrightarrow M$ be the two given collars. Since each maps onto a neighbourhood of $\dot{M}$ in M, we can choose $\varepsilon > 0$, such that $c(\dot{M} \times I_{\varepsilon}) \subset d(\dot{M} \times I)$. Since c,d are embeddings, we can factor $c = d\,f$, where f is an embedding such that the diagram

$$\begin{tikzcd}
\dot{M} \times I_{\varepsilon} \arrow[d, "f"'] \arrow[dr, "c"] & \\
\dot{M} \times I \arrow[r, "d"'] & M
\end{tikzcd}$$

is commutative, and $f | \dot{M} \times 0$ is the identity.



By Lemma 26 there exists δ, $0 < 2\delta < \varepsilon$, and an ambient isotopy F of $\dot{M} \times I$ moving f to g keeping $\dot{M} \times \dot{I}$ fixed, and such that g is level preserving for $0 \leqslant t \leqslant 2\delta$. The reason for making g level preserving is that we can now apply Lemma 27 and obtain an ambient isotopy G of $\dot{M} \times I_{2\delta}$ moving $g | \dot{M} \times I_{2\delta}$ to h keeping $\dot{M} \times \dot{I}_{2\delta}$ fixed, and such that h is the identity for $0 \leqslant t \leqslant \delta$. Extend h to an embedding $h : \dot{M} \times I_{\varepsilon} \rightarrow \dot{M} \times I$ by by making it agree with g outside $\dot{M} \times I_{2\delta}$, and extend G by the identity to an ambient isotopy of $\dot{M} \times I$.

Then $G\,F$ is an ambient isotopy moving f to h keeping $\dot{M} \times \dot{I}$ fixed. Let H be the image of $G\,F$ under d. Since $G\,F$ keeps $\dot{M} \times 1$ fixed, we can extend H by the identity to an ambient isotopy H of M keeping $\dot{M}$ fixed. Let $e = H_1 c$. Then e is a collar ambient isotopic to c, and agreeing with the beginning of d, because of $x \in \dot{M}$ and $t \in I$ then

$$
\begin{aligned}
e_{\delta}(x,t) &= e(x, \delta t) \\
&= H_1 c(x, \delta t) \\
&= d\, G_1 F_1 d^{-1} c(x, \delta t) \\
&= d\, G_1 F_1 f(x, \delta t) \\
&= d\, h\, (x, \delta t) \\
&= d(x, \delta t) \\
&= d_{\delta}(x, t)\,.
\end{aligned}
$$

Therefore $e_{\delta} = d_{\delta}$, and so by Lemma 25 there is a sequence of ambient isotopic collars c, e, e_{δ}, d. The proof of Theorem 13 is complete.

<u>Theorem 14</u>. <u>Given a proper locally unknotted embedding $f : M \rightarrow Q$ between compact manifolds, and a collar c of $\dot{M}$, then there exists a compatible collar d of Q</u>.

<u>Proof</u>. Lemma 24 furnishes compatible collars, c^*, d^* say, of M, Q. By Theorem 13 there is an ambient isotopy G of M keeping $\dot{M}$ fixed, such that $G_1 c^* = c$. By Corollary to Theorem 11 we can cover G_1 by a homeomorphism h of Q keeping $\dot{Q}$ fixed. Let $d = hd^*$.

Then the commutativity of the diagram

$$\begin{tikzcd}
\dot{Q} \times I \arrow[rr, "d"] \arrow[dr, "d^*"] & & Q \\
& Q \arrow[ur, "h"] & \\
& M \arrow[u, "f"] \arrow[dr, "G_1"] & \\
\dot{M} \times I \arrow[uuu, "f \times 1"] \arrow[ur, "c^*"] \arrow[rr, "c"'] & & M \arrow[uuu, "f"']
\end{tikzcd}$$

and the fact that

$$\begin{aligned}
\operatorname{im} d \cap \operatorname{im} f &= \operatorname{im} h d^* \cap \operatorname{im} h f \\
&= h (\operatorname{im} d^* \cap \operatorname{im} f) \\
&= h (\operatorname{im} f c^*) \\
&= \operatorname{im} fc ,
\end{aligned}$$

ensure that the collars c , d are compatible with f . The proof of Theorem 14 is complete .

We now prove the critical lemma for the covering isotopy theorem , Theorem 12 .

Lemma 28 . Let M, Q be compact, and F a locally unknotted isotopy of M in Q keeping $\dot{M}$ fixed . Then there exists $\varepsilon > 0$, and a short ambient isotopy $H : Q \times I_\varepsilon \longrightarrow Q \times I_\varepsilon$ of Q that keeps $\dot{Q}$ fixed and covers the beginning of F . In other words the diagram

$$\begin{tikzcd}
Q \times I_\varepsilon \arrow[dr, "H"] & \\
& Q \times I_\varepsilon \\
M \times I_\varepsilon \arrow[uu, "F_0 \times 1"] \arrow[ur, "F"'] &
\end{tikzcd}$$

<u>is commutative</u> .

Proof. For the convenience of the proof of this lemma we assume that $F_0 = F_1$. For, if not, replace F by F^* where F

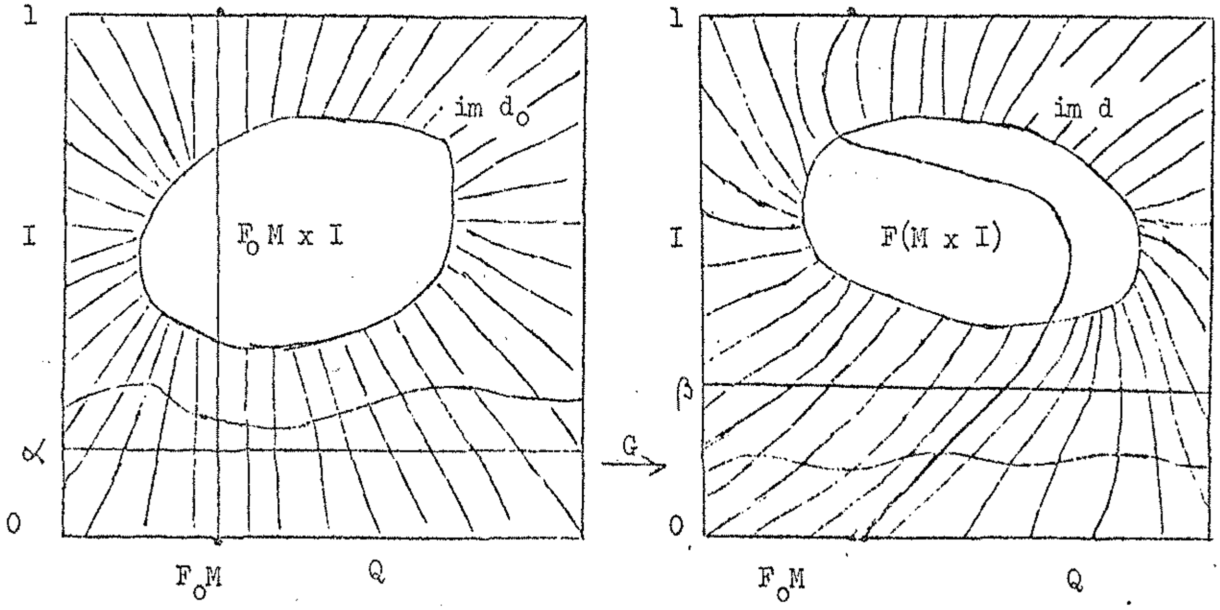
$$F^*_t = \begin{cases} F_t & t \leqslant 1/2 \\ F_{1-t} & t \geqslant 1/2 \end{cases} .$$

Then, since $F^*_0 = F^*_1$, the proof below gives H covering the beginning of F^*, which is the same as the beginning of F if $\varepsilon \leqslant 1/2$.

Therefore assume $F_0 = F_1$. This means that the two proper embeddings F, $F_0 \times 1$ of $M \times I$ in $Q \times I$ agree on the boundary $(M \times I)^{\cdot}$, because F keeps $\dot{M}$ fixed. Choose a collar c of $M \times I$, and then by Theorem 14 choose collars d, d_0 of $Q \times I$ such that c, d are compatible with F, and c, d_0 are compatible with $F_0 \times 1$. We have a commutative diagram of embeddings

$$\begin{tikzcd}
& (Q \times I)^{\cdot} \times I \arrow[dl, "d_0"'] \arrow[dr, "d"] & \\
Q \times I & (M \times I)^{\cdot} \times I \arrow[u, "F \times 1"'] \arrow[d, "c"] & Q \times I \\
& M \times I \arrow[ul, "F_0 \times 1"] \arrow[ur, "F"'] &
\end{tikzcd}$$

Notice that both the collars d, d_0 maps $(Q \times 0) \times 0$ to $Q \times 0$. Therefore im d contains a neighbourhood of $Q \times 0$ in $Q \times I$, and so contains $Q \times I_\beta$, for some $\beta > 0$. Similarly $d_0 d^{-1}(Q \times I_\beta)$ contains a neighbourhood of $Q \times 0$, and so contains $Q \times I_\alpha$, for some α, $0 < \alpha \leqslant \beta$.



Let $G = d\, d_o^{-1} : Q \times I_\alpha \longrightarrow Q \times I_\beta$. Then G has the properties :

(i) $G \mid \dot{Q} \times I$ - identity, because d , d_o agree on $(Q \times I)^{\bullet} \times 0$;

(ii) $G \mid Q \times 0 =$ identity .

(iii) G covers the beginning of F in the sense that the diagram

$$\begin{tikzcd} Q \times I_\alpha \arrow[dr, "G"] & \\ & Q \times I_\beta \\ M \times I_\alpha \arrow[uu, "F_o \times 1"] \arrow[ur, "F"'] & \end{tikzcd}$$

is commutative . For if $x \in M$ and $t \in I_\alpha$, then by compatibility

$$(F_o x,\ t) \in \text{im}\ (F_o \times 1) \cap \text{im}\ d_o = \text{im}\ (F_o \times 1)\ c \ .$$

Therefore for some $y \in (M \times I)^{\bullet} \times I$,

$$(F_o x,\ t) = (F_o \times 1)\ cy = d_o\ (F \times 1)' y \ .$$

Therefore

$$\begin{aligned} G\,(F_o x1)(x,t) &= (dd_o^{-1})\; d_o\;(F \times 1)\; y \\ &= d(F \times 1)\; y \\ &= F\; cy \\ &= F(F_o x1)^{-1}\;(F_o x1)\; cy \\ &= F(F_o x1)^{-1}\;(F_o x,t) \\ &= F(x,t)\;. \end{aligned}$$

In other words $G(F_o x1) = F$, which proves (iii) .

By Lemma 26 there is an ε , $0 < \varepsilon < \alpha$ and an embedding $H : Q \times I_\alpha \to Q \times I_\beta$ ambient isotopic to G , such that $H \mid Q \times 0 =$ identity , and H is level preserving in I_ε . Further, since G is already level preserving on $(\dot{Q} \cup F_o M) \times I_\alpha$, we can by Lemma 26 (iii) choose H to agree with G on this subpolyhedron . In other words , the restriction $H : Q \times I_\varepsilon \to Q \times I_\varepsilon$ is a short ambient isotopic covering the beginning of F and keeping $\dot{Q}$ fixed .

Proof of Theorem 12 , the covering isotopy theorem .

We are given a locally unknotted isotopy $F : M \times I \to Q \times I$ keeping $\dot{M}$ fixed , and a subdivision N of the track of F , and we have to cover F by an ambient isotopy H of Q supported by N keeping $\dot{Q}$ fixed . We are given that M is compact , and we first consider the case when Q is also compact and $N = Q$.

If $0 < t < 1$, the definition of locally unknotted isotopy ensures that the restrictions of F to $[0 , t]$ and $[t , 1]$ are locally unknotted embeddings , and therefore we can apply Lemma 7 to both sides of the level t , and cover F in the neighbourhood of t . More precisely, for each $t \in I$, there exists a neighbourhood $J^{(t)}$ of t in I , and a level preserving homeomorphism $H^{(t)}$ of $Q \times J^{(t)}$ such that $H^{(t)}$ keeps $\dot{Q}$ fixed , $H^{(t)}_t = 1$, and such that the diagram

$$\begin{tikzcd}
Q \times J^{(t)} \arrow[dr, "H^{(t)}"] & \\
& Q \times J^{(t)} \\
M \times J^{(t)} \arrow[uu, "F_t \times 1"] \arrow[ur, "F"'] &
\end{tikzcd}$$

is commutative . By compactness of I we can cover I by a finite number of such intervals $J^{(t)}$. Therefore we can find values $t_1, t_2, \ldots, t_n$ and $0 = s_1 < s_2 < \ldots < s_{n+1} = 1$, such that for each i , $[s_i , s_{i+1}] \subset J^{(t_i)}$. Write $H^i = H^{(t_i)}$.

We now define H by induction on i , as follows . Define $H_o = 1$. Suppose $H_t : Q \to Q$ has been defined for $0 \leqslant t \leqslant s_i$ such that $H_t F_o = F_t$. Then define

$$H_t = H^i_t (H^i_{s_i})^{-1} H_{s_i} , \quad \text{for} \quad s_i \leqslant t \leqslant s_{i+1} .$$

Therefore

$$\begin{aligned} H_t F_o &= H^i_t (H^i_{s_i})^{-i} H_{s_i} F_o \\ &= H^i_t (H^i_{s_i})^{-1} F_{s_i} \\ &= H^i_t F_{t_i} \\ &= F_t . \end{aligned}$$

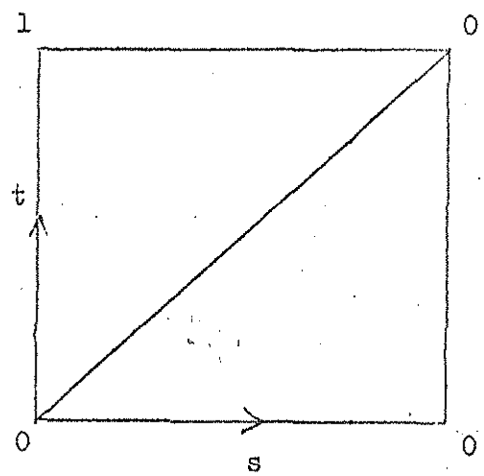
At the end of the induction we have H_t defined and $H_t F_o = F_t$, all $t \in I$.

Moreover H is piecewise linear, because it is composed of a finite number of piecewise linear pieces, and H keeps $\dot{Q}$ fixed because H^i does. Therefore the proof is complete for the case when Q is compact and $N = Q$.

We now extend the proof to the general case, when Q is not necessarily compact, and $N \subset Q$. We may assume that N is a regular neighbourhood of the tracl, because any neighbourhood contains a regular neighbourhood. Therefore N is a compact submanifold of Q. By the compact case, cover F by an ambient isotopy of Q covering F supported by N and keeping $\dot{Q}$ fixed. The proof of Theorem 12 is complete.

<u>Proof of Addendum to Theorem 12</u>.

We have to extend a given ambient isotopy H of $\dot{Q}$ with compact support X to an ambient isotopy H^* of Q supported by a given neighbourhood N of X in Q. (H is not a corollary because the embedding $\dot{Q} \times I \to Q \times I$ induced by H is not proper). Without loss of generally we may assume the neighbourhood N of X to be regular, and therefore a compact manifold. Restrict H to X and extend by the identity to an ambient isotopy, G say, of $\dot{N}$ keeping $\dot{N} - X$ fixed.



Triangulate the square I^2 as shown, and let $n : I^2 \to I$ be the simplicial map determined by mapping the vertices to 0 or 1 as shown.

Define $G^* : (\dot{N} \times I) \times I \longrightarrow (\dot{N} \times I) \times I$ by

$$G^*\Big((x,s),t\Big) = \Big((G_{u(s,t)}x,s),t\Big).$$

As in the proof of Lemma 27, it follows that G^* is an ambient isotopy of $\dot{N} \times I$ keeping $(\dot{N} \times I) \cup (\dot{N} - X) \times I$ fixed.

Choose a collar $c : \dot{N} \times I \longrightarrow N$ and let H^* be the image of G^* under c. Since G^* keeps $\dot{N} \times 1$ fixed, H^* can be extended by the identity to an ambient isotopy of N; and since G^* keeps $(\dot{N} - X) \times 0$, H^* keeps the frontier of N fixed, and so can be further extended to an ambient isotopy H^* of Q supported by N. By construction H^* extends H, as desired.

Proof of Corollary 1 to Theorem 12.

Corollary 1 is concerned with the case when the isotopy F of M in Q does not keep $\dot{M}$ fixed. Let T be the track of F in Q, which is compact since M is compact. Let $\dot{F} : \dot{M} \times I \longrightarrow \dot{Q} \times I$ denote the restriction of F to the boundary, which is locally unknotted because F is. Let X be a regular neighbourhood of the track $T \cap \dot{Q}$ of $\dot{F}$ in $\dot{Q}$, and let N_o be a regular neighbourhood of X in Q. By choosing X, N_o sufficiently small, we can ensure that the given neighbourhood N of T is also a neighbourhood of N_o.

Use Theorem 12 to cover $\dot{F}$ by an ambient isotopy of $\dot{Q}$ supported by X, and by the Addendum extend the latter to an ambient isotopy, G say, of Q supported by N_o. Then $G^{-1}F$ is an isotopy of M in Q keeping $\dot{M}$ fixed, whose track is contained in $T \cup N_o$. But N is a neighbourhood of $T \cup N_o$, and so we can again use Theorem 12 to cover $G^{-1} F$ by an ambient isotopy, H say, of Q supported by N. Therefore GH covers F and is supported by N.

Recall Lemma 16. Any homeomorphism of a ball keeping the boundary fixed is isotopic to the identity keeping the boundary fixed.

Corollary. Any homeomorphism of a ball keeping a face fixed is isotopic to the identity keeping the face fixed. For by Theorem 2 the ball is homeomorphic to a cone on the complementary face . First isotope the complementary face back into position, and extend the isotopy conewise to the ball ; then isotope the ball .

Proof of Corollary 2 to Theorem 12 .

We have to show the equivalence of

(1) isotopic,

(2) ambient isotopic,

(3) ambient isotopic by an ambient isotopy with compact support,

and

(4) isotopic by moves .

(1) implies (3) by Theorem 12 and Corollary 1, because we can choose the neighbourhood N to be compact. (3) implies (4) by Theorem 11 . (4) implies (2) by Lemma 16 and Corollary, because then each move is ambient isotopic to the identity . Finally (2) implies (1) trivially .

Proof of Corollary 3 to Theorem 12 .

If F is a locally trivial isotopy , then by definition each point in $M \times I$ has a neighbourhood which is locally unknotted ; therefore F is locally unknotted . Conversely if F is a locally unknotted isotopy , then the level F_0 is a locally unknotted embedding , and so the constant isotopy $F_0 \times 1$ is locally trivial . By Theorem 12 cover F by H ; then $F = H(F_0 \times 1)$ is locally trivial , because the homeomorphism H preserves local triviality . This completes the proofs of the theorems and corollaries stated at the beginning of the chapter .

Remarks on combinatorial isotopy.

We have framed the definitions of isotopy and proved the theorems in the polyhedral category, because that is the spirit of these seminars. In other words, there is no reference to any specific triangulations of either of the manifolds concerned. However there is a definition of isotopy in the combinatorial category when the receiving manifold Q happens to be Euclidean space, by virtue of the linear structure of Euclidean space. The manifold M is given a fixed triangulation, K say, and the isotopy is defined by moving the vertices of K. At each moment the embedding of M is uniquely determined by the positions of the vertices, and by the linear structure of Euclidean space. But a general polyhedral manifold Q has only a piecewise linear structure, not a linear structure, and so the positions of the vertices of K do not determine a unique embedding of M. It is no good picking a fixed triangulation L of Q, and considering linear embeddings $K \to L$, because this has the effect of trapping M locally, and preventing the movement of any simplex of K across the boundary of any simplex of Q. Therefore to obtain any useful form of isotopy it is essential to retain the polyhedral structure of Q, even though we may descend to the combinatorial structure of M. We now give a definition in these terms, which looks at first sight much more special than the definitions of isotopy above, but in fact turns out to be equivalent; we state the theorem without proof. The moral of the story is: stick to the polyhedral category and don't tinker about with the combinatorial category; keep the latter out of definitions and theorems, and use it only as it ought to be used, as an inductive tool for proofs.

Linear moves with respect to a triangulation

Let Δ^q be the standard q-simplex , and Δ^m an m-dimensional face , $q > m$. Let x be the barycentre of Δ^q, and y a point between x and the barycentre of Δ^m . Let $\sigma : \Delta^q \to \Delta^q$ be the homeomorphism throwing x to y , mapping the boundary by the identity , and joining linearly .

Let M be closed , K a triangulation of M , and let $f , g : M \to Q$ be proper embeddings . We say there is a <u>move from f to g linear with respect to K</u> if the following occurs :

There is a closed vertex star of K , $A = \overline{st}\,(v,K)$ say , and a q-ball $B \subset Q$, and a homeomorphism $h : B \to \Delta^q$ such that

(i) f,g agree on $K - \overset{o}{A}$,

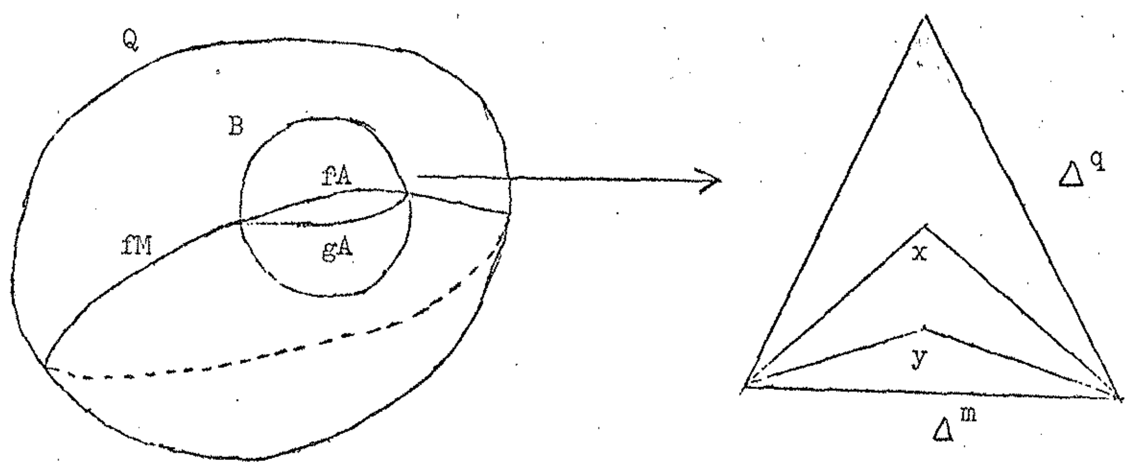
(ii) $A = f^{-1}B = g^{-1}B$,

(iii) hf maps $lk(v,K) \longrightarrow \dot{\Delta}^m$, homeomorphically ,

$$v \longrightarrow x \quad ,$$

$$A \longrightarrow x \quad {}^{m} \text{, by joining linearly .}$$

(iv) $g \mid A = h^{-1} \sigma\, h(f \mid A)$.



Roughly speaking, h is a local coordinate system, chosen so that the move from f to g looks as simple as possible, just moving one vertex of K linearly in the most harmless fashion, like a move of classical knot theory.

Addendum (stated without proof). Let M be closed, and K an arbitrary fixed triangulation of M. Let $f, g : M \to Q$ be proper embeddings that are locally unknotted and ambient isotopic. If codimension > 0, then f, g are isotopic by moves linear with respect to K.

The addendum becomes surprising if we imagine embeddings of a 2-sphere in a manifold, and choose K to be the boundary of a 3-simplex, with exactly 4 vertices. Then we can move from any embedding to any other isotopic embedding by assiduously shifting just those 4 vertices linearly back and forth. All the work is secretly done by judicious choice of the balls, or local coordinate systems in the receiving manifold, in which the moves are made.

Remark. Notice the restriction codimension > 0 that occurs in the addendum (but not in Theorem 11 for example). It is an open question as to whether or not the restriction is necessary. In particular we have the problem: is a homeomorphism of a ball that keeps the boundary fixed isotopic to the identity by linear moves?

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1963

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Chapter 6 : GENERAL POSITION

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Chapter 6: GENERAL POSITION

General position is a technique applied to (poly) maps from polyhedra into manifolds. The idea is to use the homogeneity of the manifold to minimise the dimension of intersections. Throughout this chapter X, Y will denote polyhedra and M a compact manifold. The small letters x, y, m will always denote the dimensions of X, Y, M respectively, and we shall assume $x, y < m$. In particular we tackle the following two situations.

Situation (1) Let $f : X \to M$ be an embedding and let Y be a subpolyhedron of M. In Theorem 15 we show it is possible to move f to another embedding g such that $gX \cap Y$ is of minimal dimension, namely $\leq x + y - m$. We describe the move $f \to g$ by saying ambient isotope f into general position with respect to Y. There are refinements such as keeping a subpolyhedron X_0 of X fixed, and moving $f|X - X_0$ into general position.

Situation (2) Let $f : X \to M$ be a map, not necessarily an embedding. First we show in Lemma 32 that f is homotopic to a non-degenerate map, g say, where non-degenerate means that in any triangulation with respect to which g is simplicial each simplex is mapped non-degenerately (although of course many simplexes of X may be mapped onto one simplex of M).

Next we show in Theorem 17 that g is homotopic to a map, h say, for which the self-intersections are of minimal dimension, namely $\leq 2x - m$. Not only the double points but also the sets of triple points, etc., we wish to make minimal. We describe the composite homotopy $f \to g \to h$ by saying <u>move f into general position</u>. There are refinements such as keeping X_0 fixed if $f|X_0$ happens already to be in general position, and arranging also for $f|X_i$ to be in general position for a finite family $\{X_i\}$ of subpolyhedra (notice that the general position of f does not imply the general position of $f|X_i$ unless $x = x_i$).

We observe that situation (2) is part of a general programme of "improving" maps, and is an essential step in passing from algebra to geometry. For example suppose that an algebraic hypothesis tells us there exists a continuous-map $X \to M$ (we use the hyphenated "continuous-map" to avoid confusion with our normal usage map = polymap), and that we want to deduce as a geometrical thesis the existence of a homotopic embedding $X \subset M$. Then the essential steps are:

local homotopy	continuous map		
	$\downarrow$	simplicial approximation	
	map		
	$\downarrow$	general position	Chapter 6
	non-degenerate map		
	$\downarrow$	general position	
	map in general position		
global homotopy	$\downarrow$	engulfing piping	Chapters 7 and 8.
	embedding		

<u>Remark on homotopy</u> The general programme is to investigate criteria for

(1) an arbitrary continuous-map to be homotopic to a polyembedding, and

(2) for two polyembeddings to be polyisotopic. Therefore although we are very careful to make our <u>isotopies</u> piecewise linear (in situation (1)) we are not particularly interested in making our <u>homotopies</u> piecewise linear (in situation (2)). We regard isotopy as geometric, and homotopy as algebraic-topological.

<u>Invariant definition</u> If M is Euclidean space, then general position is easy because of linearity; it suffices to move the vertices of some triangulation of X into "general position", and then the simplexes automatically intersect

minimally. However in a manifold we only have <u>piecewise</u> linearity, and the problem is complicated by the fact that the positions of the vertices do not uniquely determine the maps of the simplexes; therefore the moving of the vertices into "general position" does not guarantee that the simplexes intersect minimally. In fact defining general position in terms of a particular triangulation of X leads to difficulties.

Notice that the definitions of general position we have given above depend only on dimension, and so are <u>invariant</u> in the sense that they do not depend upon any particular triangulation of X or M. The advantages of an invariant definition are considerable in practice. For example, having moved a map f into general position, we can then triangulate f so that f is both simplicial <u>and</u> in general position (a convenient state of affairs that was not possible in the more naïve Euclidean space approach). The closures of the sets of double points, triple points, etc. will then turn out to be a descending sequence of subcomplexes.

<u>Transversality</u> In differential theory the corresponding transversality theorems of Whitney and Thom serve a different purpose, because they assume X, Y to be manifolds. Whereas in our theory it is essential that X, Y be more general polyhedra than manifolds. For general polyhedra the concept of "transversality" is not defined, and so our theorems aim at minimising dimension rather than achieving transversality.

When X,Y are manifolds then transversality is well defined in combinatorial theory, but the general position techniques given below are not sufficiently delicate to achieve transversality, except in Theorem 16 for the special case of 0-dimensional intersections $(x+y=m)$.

When $x+y>m$ the difficulty can be pinpointed as follows. The basic idea of the techniques below is to reduce the intersection dimension of two cones in euclidean space by moving their vertices slightly apart. However this is no good for transversality, because if two spheres cut combinatorially transversally in E^n, then the two cones on them in E^{n+1}, with vertices in general position, do <u>not</u> in general cut transversally: there is trouble at the boundary.

The use of cones is a primitive tool compared with the function space techniques used in differential topology, but is sufficient for our purposes because the problems are finite. It might be more elegant, but probably no easier, to work in the combinatorial function space.

<u>Wild embeddings</u> Without any condition of local niceness, such as piecewise linearity or differentiability, then it is not possible to appeal to general position to reduce the dimension of intersections. For consider the following example. It is possible to embed an arc and a disk in E^4 (and also in E^n, $n\geq 4$) intersecting at one point in the interior of each, and to choose $\varepsilon>0$, such that is is impossible to ε-shift the disk off

the arc (although it is possible to shift the arc off the disk). The construction is as follows: Let A be a wild arc in E^3, and let D be a disk cutting A once at an interior point of each, such that $\dot{D}$ is essential in $E^3 - A$. If we shrink A to a point x, and then multiply by a line, the result is 4-space, $(E^3/A) \times R = E^4$ (by a theorem of Andrews and Curtis). If D' denotes the image of D in E^3/A, then $D' \times 0$ meets $x \times R$ in one point $x \times 0$, and $\dot{D}' \times 0$ is essential in $E^4 - (x \times R)$. Therefore if ε is less than the distance between $\partial\dot{D}' \times 0$ and $x \times R$, it is impossible to ε-shift the disk $D' \times 0$ off the arc $x \times R$.

<u>Compactness</u> We restrict ourselves to the case when X is a polyhedron and therefore compact. Consequently we can assume that M is also compact, for, if not, replace M by a regular neighbourhood N of fX in M. Then N is a compact manifold of the same dimension, and moving f into general position in N a fortiori moves f into general position in M.

<u>General position of points in Euclidean space</u> Before we can move maps into general position we need a precise definition of the general position of a point in Euclidean space E^n with respect to other points, as follows. Let X be a countable (finite or denumerable) subset of E^n. Each point is, trivially, a linear subspace of E^n, and the set X generates a countable sublattice, $L(X)$ say, of the lattice of all linear subspaces of E^n. Let $\Omega(X)$ be the set union of all <u>proper</u> linear subspaces in $L(X)$. Since $L(X)$ is countable, the complement $E^n - \Omega(X)$

is everywhere dense. Define $y \in E^n$ to be in <u>general position with respect to X</u> if $y \notin \Omega(X)$.

Now let Δ be an n-simplex and X a finite set of points in Δ . We say $y \in \Delta$ is in <u>general position with respect to X</u> if the same is true for some linear embedding $\Delta \subset E^n$ (the definition being independent of the embedding). Let Δ' be a subdivision of Δ , with vertices $x_1, x_2, \ldots, x_r$ say. We define an ordered sequence $(y_1, \ldots, y_s) \subset \Delta$ to be in <u>general position with respect to Δ'</u> if, for each i, $1 \leq i \leq s$, y_i is in general position with respect to the set $(x_1, \ldots, x_r, y_1, \ldots, y_{i-1})$.

<u>Lemma 29</u> <u>Given a subdivision Δ' of Δ and a sequence $(v_1, \ldots, v_s)$ of vertices of Δ' (not necessarily distinct), then it is possible to choose a sequence $(y_1, \ldots, y_s) \subset \Delta$ in general position with respect to Δ', such that y_i is arbitrarily close to v_i</u> , $1 \leq i \leq s$.

<u>Proof</u> Inductively, the complement $\Delta - \Omega(x_1, \ldots, y_{i-1})$ is dense at v_i , enabling us to choose y_i arbitrarily near v_i .

<u>Remark 1</u> Notice that all the y_i have to be interior to Δ .

<u>Remark 2</u> This is the first time we have used the <u>reals</u>: previously all our theory would work over the <u>rationals</u>, and even now it would suffice to use smaller field, like the algebraic number field.

<u>Remark 3</u> There is an intrinsic inelegance in our definition of a sequence of points being in general position, because if the

order is changed they may no longer be so. To construct a counter-example in E^2, choose 4 points X such that $\Omega(X)$ contains all rationals on the real axis (regarding E^2 as the complex numbers), and then add π, $\sqrt{\pi}$ in that order. To get rid of this inelegance, and at the same time preserve the lattice property would be more trouble than it is worth; because all we need is some gadget to make Lemmas 30 and 34 work.

<u>ISOTOPING EMBEDDINGS INTO GENERAL POSITION</u> We consider situation (1) of the introduction. Let $X_0 \subset X$, $Y \subset M$ be polyhedra, and let M be a manifold. Let $x = \dim X - X_0$, $y = \dim Y$, $m = \dim M$. Let $g : X \to M$ be a map. We say that $g|X - X_0$ is in <u>general position with respect to Y</u> if

$$\dim (g(X - X_0) \cap Y) \leq x + y - m .$$

<u>Theorem 15</u> <u>Given</u> $X_0 \subset X$ <u>and</u> $Y \subset M$, <u>and an embedding</u> $f : X \to M$ <u>such that</u> $f(X - X_0) \subset \mathring{M}$, <u>then we can ambient isotope</u> f <u>into</u> g <u>by an arbitrarily small ambient isotopy keeping</u> $\dot{M}$ <u>and the image of</u> X_0 <u>fixed, such that</u> $g|X - X_0$ <u>is in general position with respect to</u> Y.

<u>Remark 1</u> In the theorem we say nothing about $f|X_0$ being in general position. In fact in many applications for engulfing in the next two chapters, $f|X_0$ will definitely not be in general position with respect to Y. The intuitive idea is to think of X_0 and Y as large high-dimensional blocks, and $\overline{X - X_0}$ as a little low-dimensional feeler attached to X_0 by its

frontier $X_0 \cap \overline{X - X_0}$. The theorem says we can ambient isotope the feeler keeping its frontier fixed so that the interior of the feeler meets Y minimally (although its frontier may not). In other applications we may already have $f|X_0$ in general position, as in the following three corollaries.

<u>Corollary 1</u> <u>If</u> $f|X_0$ <u>is already in general position, or if</u> fX_0 <u>does not meet</u> $f(X - X_0)$, <u>then Theorem 15 is true for maps as well as embeddings</u>.

<u>Proof</u> Apply the theorem to the embedding of the image $fX \subset M$, and ambient isotope fX into general position with respect to Y keeping fX_0 fixed. (Notice the extra hypothesis is necessary, otherwise having to keep fX_0 fixed may prevent us from moving awkward pieces of $X - X_0$ that overlap X_0).

<u>Corollary 2</u> <u>(Interior Case)</u> <u>Given a map</u> $f : X \to \mathring{M}$ <u>and</u> $Y \subset M$ <u>then we can ambient isotope</u> f <u>into general position with respect to</u> Y <u>keeping</u> $\dot{M}$ <u>fixed</u>.

For put $X_0 = \emptyset$ in Corollary 1.

<u>Corollary 3</u> <u>(Bounded Case)</u> <u>Given a map</u> $f : X \to M$ <u>and</u> $Y \subset M$, <u>let</u> $X_0 = f^{-1}\dot{M}$, $Y_0 = Y \cap \dot{M}$. <u>Then we can ambient isotope</u> f <u>to</u> g <u>such that</u> $g|X_0$ <u>is in general position in</u> $\dot{M}$ <u>with respect to</u> Y_0, <u>and</u> $g|X - X_0$ <u>in general position in</u> $\mathring{M}$ <u>with respect to</u> Y.

<u>Proof</u> First apply Corollary 2 to the boundary, and extend the ambient isotopy of $\dot{M}$ to M by Theorem 12 Addendum; then apply Corollary 1.

For the proof of Theorem 15 we shall use a sequence of special moves which we call t-shifts, and which we construct below. The parameter t concerns dimension, with $0 \leq t \leq x$. The construction involves choices of local coordinate systems (i.e. replacing the piecewise linear structure by local linear structures) and choices of points in general position.

<u>The t-shift of an embedding</u> By Theorem 1 choose triangulations of X, X_0 and M, Y with respect to which $f : X \to M$ is simplicial. Let K, L denote the triangulations of X, M. Let K', L' denote the barycentric derived complexes modulo the $(t-1)$-skeletons of K, L (obtained by starring all simplexes of dimension $\geq t$, in some order of decreasing dimension). Then $f : K' \to L'$ remains simplicial because f is non-degenerate (it is an embedding).

Let A be a t-simplex of K, and $B = fA$ the image t-simplex of L. Let a, b be the barycentres of A, B (with $fa = b$). Then

$$\overline{st}(a,K') = a\dot{A}P \qquad\qquad \overline{st}(b,L') = b\dot{B}Q$$

where P, Q are subcomplexes of K', L'. If $A \not\subset X_0$, then $\dim P \leq x - t - 1$, and Q is an $(m-t-1)$-sphere because $fA \subset \mathring{M}$. Let

$$f_A : a\dot{A}P \to b\dot{B}Q$$

denote the restriction of f. Then f_A is the join of three maps $a \to b$, $\dot{A} \to \dot{B}$ and $P \to Q$, and therefore embeds the frontier $\mathring{A}P$ of $a\mathring{A}P$ in the boundary $\dot{B}Q$ of the m-ball $b\dot{B}Q$.

The idea is to construct another embedding

$$g_A : a\dot{A}P \to b\mathring{B}Q$$

that agrees with f_A on the frontier $\dot{A}P$, and is ambient isotopic to f_A keeping the boundary $\mathring{B}Q$ fixed. We shall call the move $f_A \to g_A$ a <u>local shift</u>, and give the explicit construction below. From the construction it will be apparent that g_A can be chosen to be arbitrarily close to f_A, and the ambient isotopy be made arbitrarily small.

Now let A run over all t-simplexes of K; for each $A \subset X - X_0$ construct a local shift $f_A \to g_A$, and for each $A \subset X_0$ define $g_A = f_A$. The closed stars $\{\overline{st}(a,K')\}$ cover X and overlap only in their frontiers, on which the $\{g_A\}$ agree with f, and therefore with each other. Therefore the $\{g_A\}$ combine to give a global embedding $g : X \to M$ arbitrarily close to f. Also since the stars $\{\overline{st}(b,L')\}$ overlap only in their boundaries which the local ambient isotopies keep fixed, the latter combine to give an arbitrarily small global ambient isotopy from f to g. Moreover the ambient isotopy is supported by the simplicial neighbourhood of $f(X - X_0)$ in L', and so in particular keeps $fX_0 \cup \dot{M}$ fixed.

We call the move $f \to g$ a <u>t-shift with respect to</u> Y <u>keeping</u> X_0 <u>fixed</u>. Notice that Y entered into the construction when choosing the triangulation L of M so as to have Y a subcomplex.

<u>Local shift of an embedding</u> We are given a simplicial embedding

$$f : a\dot{A}P \to \dot{b}BQ$$

which is the join of the three maps $a \to b$, $\dot{A} \to \dot{B}$ and $P \to Q$, and we want to construct

$$g : a\mathring{A}P \to \dot{b}BQ .$$

(We drop the subscript A from f_A and g_A.)

Now Q is an (m-t-1)-sphere, and by construction $Y \cap Q$ is a subcomplex of Q, and by hypothesis both $Y \cap Q$ and fP are of lower dimension than Q. Therefore, if Δ is an (m-t)-simplex with an (m-t-1) face Γ, we can choose a homeomorphism

$$h : Q \to \dot{\Delta}$$

throwing $fP \cup (Y \cap Q)$ into the face Γ. Let v be the barycentre of Δ, and extend $h : Q \to \dot{\Delta}$ to $h : bQ \to \Delta$ by mapping $b \to v$ and joining linearly. Choose subdivisions such that

$$h : (bQ)' \to \Delta'$$

is simplicial. Choose v_1 near v in Δ in general position with respect to Δ'. Then in particular $v_1 \neq v$ because v is a vertex of Δ'. Define the homeomorphism

$$k_1 : \Delta \to \Delta$$

to be the join of the identity on $\dot{\Delta}$ to the map $v \to v_1$. Define

$$k : b\dot{B}Q \to b\dot{B}Q$$

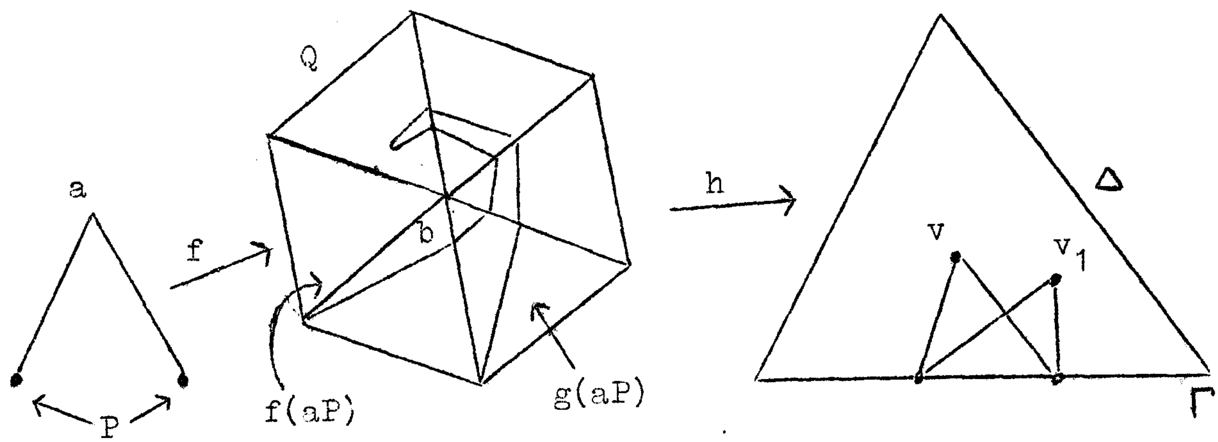
to be the join of the identity on $\mathring{B}$ to the homeomorphism

$$h^{-1}k_1h : bQ \to bQ .$$

Then k is a homeomorphism of the ball $b\mathring{B}Q$ keeping the boundary fixed. Define

$$g = kf : a\dot{A}P \to b\dot{B}Q \; .$$

Then g is ambient isotopic to f keeping the boundary fixed by Lemma 15. We can make g arbitrarily near f, and the isotopy arbitrarily small, by choosing v_1 sufficiently near v. This completes the definition of the local shift.



Remark Since f, k are joins, it follows that $g : a\dot{A}P \to b\dot{B}Q$ is the join of $g : aP \to bQ$ to $f : \mathring{A} \to \mathring{B}$. However $g : aP \to bQ$ is not a join with respect to the simplicial structure of bQ, as the diagram shows, but is a join with respect to the linear structure induced on bQ from Δ by h^{-1}.

Lemma 30 Given the hypothesis of Theorem 15, let $f \to g$ be a t-shift with respect to Y keeping X_0 fixed.

(i) If f is in general position with respect to Y, then so is g.

(ii) On the other hand if f is not in general position, and if $\dim(f(X - X_0) \cap Y) = t > x + y - m$ then $\dim(g(X - X_0) \cap Y) = t - 1$

<u>Proof</u> (i) It suffices to examine the local shift from f to $g : a\dot{A}P \to b\dot{B}Q$, for a t-simplex $A \subset X - X_0$. Since g agrees with f on the frontier $\mathring{A}P$, we have

$$\dim(g(X - X_0) \cap Y \cap \mathring{B}Q) \leq \dim(f(X - X_0) \cap Y) \leq x + y - m .$$

Therefore it suffices to examine the intersection of $g(X - X_0) \cap Y$ with the interior of the ball $b\dot{B}Q$.

Since Y is a subcomplex of L, Y meets the interior of $b\dot{B}Q$ only if $B \subset Y$, and so we assume this to be the case. Therefore

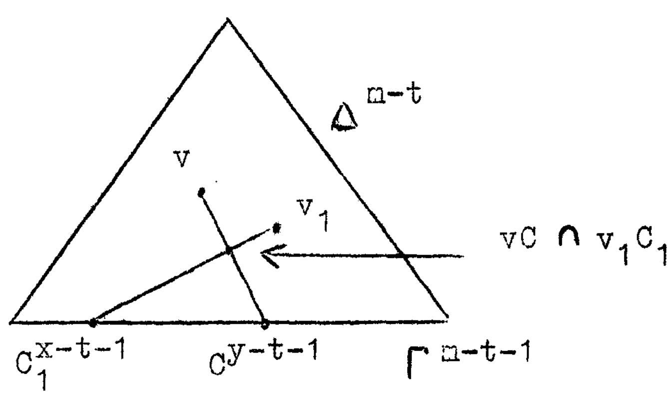
$$g(X - X_0) \cap Y \cap b\dot{B}Q = \dot{B}h^{-1}(v_1 hfP \cap vh(Q \cap Y))$$

and so

$$\dim(g(X - X_0) \cap Y \cap \operatorname{int}(b\dot{B}Q)) = t + \max \dim(v_1 C_1 \cap vC)$$

where the maximum is taken over all pairs of simplexes C_1, C of Δ' such that $C_1 \subset hfP$, $C \subset h(Q \cap Y)$ and such that $v_1 C_1 \cap vC$ meets the interior of Δ.

Since BC_1, BC are in the images of $X - X_0$, Y under hf, h respectively, we have $\dim C_1 \leq x - t - 1$, $\dim C \leq y - t - 1$.



Regard Δ as embedded in E^{m-t}, and let $[C]$ denote the linear subspace spanned by C. There are two possibilities, according as to whether or not $[C]$ and $[C_1]$ span $[\Gamma]$.

<u>Case (a)</u> $[C]$ and $[C_1]$ span $[\Gamma]$. Therefore $[vC]$ and $[v_1C_1]$ span $[\Delta]$, and so

$$\begin{aligned}\dim(v_1C_1 \cap vC) &\leq \dim v_1C_1 + \dim vC - \dim \Delta \\ &= (x-t) + (y-t) - (m-t) .\end{aligned}$$

Therefore

$$t + \dim(v_1C_1 \cap vC) \leq x+y-m .$$

<u>Case (b)</u> $[C]$ and $[C_1]$ do not span $[\Gamma]$. Therefore $[vC]$ and $[C_1]$ span a proper subspace of $[\Delta]$, which does not contain v_1, by our choice of v_1 in, and the definition of, general position in Δ with respect to Δ'.
(Note that this application was the reason for our definition of the general position of a point off the lattice of subspaces generated by the vertices of Δ'; a more complicated application of the same kind occurs in Lemma 34 below.) Therefore

$$v_1C_1 \cap vC = C_1 \cap C$$

which does not meet the interior of Δ, contradicting our assumption that it did. Therefore case (b) does not apply, and the proof of part (i) of Lemma 30 is complete.

(ii) We are given $\dim(f(X-X_0) \cap Y) = t > x+y-m$, and have to show that the t-shift drops this dimension by one. Again it suffices to examine the local shift. Since $f(X-X_0) \cap Y$ is contained in the t-skeleton of L, and since

$$b\dot{B}Q \cap (\text{t-skeleton of } L) = B \quad ,$$

we have

$$g(X-X_0) \cap Y \cap \dot{B}Q = f(X-X_0) \cap Y \cap \dot{B}Q \subset \dot{B} \quad ,$$

which is of dimension $t-1$. Moreover $f(X-X_0) \cap Y \supset B$, for some B, and so $\dim(g(X-X_0) \cap Y) \geq t-1$. Conversely, to show $\dim(g(X-X_0) \cap Y) \leq t-1$, it suffices to show that $g(X-X_0) \cap Y$ does not meet the interior of $b\dot{B}Q$, for any B. If $B \not\subset Y$ this is trivially true, and so assume $B = fA \subset Y$. Again there are two cases, and, as above, only case (a) applies. In case (a),

$$\dim(v_1C_1 \cap vC) \leq (x+y-m) - t < 0 \quad ,$$

and so $v_1C_1 \cap vC$ is empty. Therefore $g(X-X_0) \cap Y$ does not meet the interior of $b\dot{B}Q$, and the proof of Lemma 30 is complete.

<u>Proof of Theorem 15</u> Given $f : X \to M$, let $s = \dim(f(X-X_0) \cap Y)$ and assume $s > x+y-m$, otherwise the theorem is trivial. Perform t-shifts for $t = s, s-1, \ldots, x+y-m+1$, in that order; by Lemma 30 (ii) each t-shift knocks the dimension of the intersection down by 1, until we are left with an embedding in general position with respect to Y. Each t-shift, and

therefore also their composition, can be realised by an arbitrarily small ambient isotopy keeping $fX_0 \cup \dot{M}$ fixed. The proof of Theorem 15 is complete.

<u>0-DIMENSIONAL TRANSVERSALITY</u> Let X, Y, M be manifolds such that $x + y = m$, and let $f : X \to M$ and $Y \subset M$ be proper embeddings such that f is in general position with respect to Y. Therefore $fX \cap Y$ is a finite set of points interior to M. Given $v \in fX \cap Y$ we say f <u>is transversal to</u> Y <u>at</u> v if, for some (and hence for any) triangulation of

$$\begin{array}{ccc} X & \searrow^{f} & \\ & & M \\ Y & \nearrow_{\subset} & \end{array}$$

containing v as vertex, there is a homeomorphism

$$st(v,M) \to E^m = E^x \times E^y$$

throwing $st(v,fX)$, $st(v,Y)$ onto E^x, E^y respectively. We say f <u>is transversal to</u> Y if it is transversal at each point of $fX \cap Y$.

<u>Example</u> Let M be a 4-ball with boundary S^3, and let X, Y be two locally unknotted disks in M formed by joining the centre to two unknotted curves in S^3 that link more than once. Then $f : X \subset M$ is in general position with respect to Y, but not transversal.

Theorem 16 Let X, Y, M be manifolds such that $x+y=m$, and let $f : X \to M$ and $Y \subset M$ be proper embeddings. Then we can ambient isotope f into g by an arbitrarily small ambient isotopy such that g is transversal to Y.

Proof First ambient isotope f into general position, and then perform a 0-shift with respect to Y. Since $fX \cap Y$ does not meet the boundaries of each local shift it suffices to examine the interior of one local shift $g_A : AP \to BQ$ (A,B = points because $t=0$). As in the proof of Lemma 30, only case (a) applies, and the intersection of the two cones in Δ^{x+y} consists of a finite number of points where an x-simplex crosses a y-simplex at point interior to both; such a crossing is transversal. Therefore g_A is transversal to $BQ \cap Y$, and so g is transversal to Y.

SINGULAR SETS We now pass onto situation (2) of the introduction, to the homotoping of maps into general position. Let $f : X \to M$ be a map between polyhedra (for this definition it is not necessary that M be a manifold).

The singular set $S(f)$ of f is defined by:

$$S(f) = \text{closure}\{ x \in X ;\ f^{-1}fx \neq x \} .$$

Then $S(f) = \emptyset$ if and only if f is an embedding.

The branch set $Br(f)$ of f is a subset of $S(f)$ defined by:

$$Br(f) = \{ x \in X ;\ \text{no neighbourhood of } x \text{ is embedded by } f \}.$$

We deduce that $Br(f)$ is closed, and that $Br(f) = \emptyset$ if and only if g is an immersion.

The r^{th} singular set $S_r(f)$ of f is defined by:

$$S_r'(f) = \{ x \in X ;\ f^{-1}fx \text{ contains at least } r \text{ points} \} .$$
$$S_r(f) = \text{closure } S_r'(f) .$$

Thus $S_2(f)$ is the closure of the double points, $S_3(f)$ the triple points, etc.. We deduce

$$X = S_1(f) \supset S_2(f) \supset \ldots\ldots \supset S_\infty(f) .$$
$$S(f) = S_2(f) = S_2'(f) \cup Br(f) .$$

To prove the last statement it suffices to show $S_2 - S_2' \subset Br$; therefore suppose $x \in S_2 - S_2'$. Then there is a sequence $x_n \to x$, that is identified with a disjoint sequence $\{y_n\}$, which tends to a limit $y_n \to y$ because X is compact. Therefore $fx = fy$, and so $x = y$ because $x \notin S_2'$. Consequently any neighbourhood of x contains $x_n \neq y_n$, for some n, and so it is not embedded. Hence $x \in Br$.

Notice that although $S_2 - S_2' \subset Br$, in general

$$Br \not\subset S_2 - S_2' , \quad \text{and}$$
$$S_r - S_r' \not\subset Br , \quad \text{for } r > 2 .$$

The singular sets have been defined invariantly, without reference to any triangulation. Now choose triangulations K, L of X, M such that $f : K \to L$ is simplicial.

<u>Lemma 31</u> (i) <u>There is an integer</u> s, <u>and a decreasing sequence of subcomplexes</u>

$$K = K_1 \supset K_2 \supset \dots \supset K_s = K_{s+1} = \dots = K_\infty$$

<u>such that</u> $|K_r| = S_r(f)$.

(ii) $S_\infty(f) = \emptyset$ <u>if and only if</u> f <u>maps every simplex of</u> K <u>non-degenerately</u>.

(iii) <u>There is a subcomplex</u> L, $K_2 \supset L \supset K_\infty$, <u>such that</u> $|L| = Br(f)$ <u>and</u> $\dim (L - K_\infty) < \dim K_2$.

<u>Proof</u> (i) Let A be a p-simplex of K. We shall show that if $\mathring{A}$ meets $S_r'(f)$ then $\mathring{A} \subset S_r'(f)$. For $f^{-1}f\mathring{A}$ is the disjoint union of open simplexes of K, and must contain either a simplex of dimension $> p$, or at least r simplexes of dimension p. In either case, each point of $\mathring{A}$ is identified under f with at least $r-1$ other points, and so $\mathring{A} \subset S_r'(f)$. Therefore $S_r'(f)$ is the union of open simplexes, and so the closure $S_r(f)$ is a subcomplex, K_r say.

Let n be the number of simplexes of K. If $A \in K_r - K_\infty$, then $\mathring{A}$ is identified with at least $r-1$ other simplexes, and so $r \leq n$. Therefore $K_r = K_\infty$ for $r > n$. Define s to be the least r such that $K_r = K_\infty$.

(ii) If a simplex is mapped degenerately then a continuum is shrunk to a point and $S_\infty(f) \neq \emptyset$. Conversely if every simplex is mapped non-degenerately then at most n points can be identified, and so $S_\infty(f) = \emptyset$.

(iii) If $A \in K_\infty$, then A faces some simplex mapped degenerately and so $A \subset Br(f)$. If $A \notin K_\infty$, then $st(A,K)$ is mapped non-degenerately, and either $\mathring{A} \subset X - Br(f)$ or $A \subset Br(f)$ according as to whether or not $lk(A,K)$ is embedded. Therefore $Br(f)$ is the union of closed simplexes, and is therefore a subcomplex, L say.

If $A \in L - K_\infty$, there is a vertex $x \in S_2(f|lk(A,K))$, and so $xA \in K_2$. Therefore $\dim A < \dim K_2$, and so $\dim (L - K_\infty) < \dim K_2$.

<u>NON-DEGENERACY</u> Define $f : X \to M$ to be <u>non-degenerate</u> if $S_\infty(f) = \emptyset$. The justification for the definition is Lemma 31 (ii). We recall that this is equivalent to the more general definition given in Chapter 2.

<u>Lemma 32</u> <u>Given a map</u> $f : X^x \to M^m$ <u>from a polyhedron</u> X <u>to a manifold</u> M, <u>with</u> $x \leq m$, <u>then we can homotope</u> f <u>to a non-degenerate map</u> g <u>by an arbitrarily small homotopy</u>. <u>If, further,</u> $X_0 \subset X$ <u>and</u> $f|X_0$ <u>is already non-degenerate, we can keep</u> X_0 <u>fixed during the homotopy</u>.

<u>Proof</u> Choose triangulations of X, M with respect to which f is simplicial; let K be a first derived complex of the triangulation of X, and let $B_1, \ldots, B_t$ denote the open vertex stars of the triangulation of M (each B_s is either an open m-cell or a half-open m-cell, according to whether vertex lies in the interior or boundary of M). The set $\{B_s\}$ is an open

covering of M, and f has the property:

(P) for each $A \in K$, $f(\overline{st}(A,K)) \subset$ some B_s. Any map $X \to M$ sufficiently close to f also satisfies (P).

Now order the simplexes $A_1, \dots, A_r$ of K in some order of locally increasing dimension (i.e. if $A_i < A_j$ then $i \leq j$), and let $K_i = \bigcup_1^i A_j$. We construct, by induction on i, starting with $f_0 = f$, a sequence of maps $f_i : X \to M$ such that

(i) $f_i \sim f_{i-1}$ by an arbitrarily small homotopy keeping K_{i-1} fixed,

(ii) f_i satisfies (P),

(iii) $f_i | K_i$ is non-degenerate.*

The end of the induction $g = f_r$ proves the lemma.

For the case when $X_0 \subset X$ and $f | X_0$ is already non-degenerate, we choose the original triangulation so as to contain X_0 as a subcomplex, choose the ordering so that $X_0 = |K_j|$, some j, and then start the induction at j, with $f_j = f$.

We must now prove the inductive step. Assume f_{i-1} defined, and let $A = A_i$, $L = \overline{st}(A,K)$, and let B denote the B_s such that $f_{i-1} L \subset B$. Choose a homeomorphism $h : B \to \Delta$ onto a simplex and define $g = hf_{i-1}$; in other words the diagram

$$\begin{tikzcd} L \arrow[rr, "f_{i-1}"] \arrow[dr, "g"'] & & B \arrow[dl, "h", "\cong"'] \\ & \Delta & \end{tikzcd}$$

is commutative.

---

* We do not claim that $f_i : K_i \to M$ is simplicial, nor do we claim that f_i embeds end simplex of K_i in M, but only that $S_\infty(f_i | K_i) = \emptyset$.

Choose subdivisions L', Δ' of L, Δ such that $g : L' \to \Delta'$ is simplicial, and such that L' has at least one vertex in $\mathring{A}$. Let $x_1, \ldots, x_p$ denote the vertices of L' contained in $\mathring{A}$, and let $x_{p+1}, \ldots, x_q$ denote the remaining vertices of L'. Choose a sequence of points $(y_1, \ldots, y_p) \subset \Delta$ in general position with respect to Δ', such that y_n is arbitrarily close to fx_n, $1 \leq n \leq p$. Define $g' : L' \to \Delta$ to be the linear map determined by the vertex map

$$g'x_n = \begin{cases} y_n, & 1 \leq n \leq p \\ gx_n, & p < n \leq q \end{cases} .$$

Then $g'|\dot{A} = g|\dot{A}$, which is non-degenerate by induction, and so $g|A$ is non-degenerate by our choice of y's because $\dim A \leq \dim \Delta$. Now define $f_i : X \to M$ so that f_i agrees with f_{i-1} outside L, and on L the diagram

$$\begin{tikzcd} L \arrow[rr, "f_i"] \arrow[dr, "g'"'] & & B \arrow[dl, "h"] \\ & \Delta & \end{tikzcd}$$

is commutative. Having defined f_i we must verify the three inductive properties.

Firstly $g' \simeq g$ by straight line paths in Δ, keeping the frontier $\mathrm{Fr}(L,K)$ fixed. Therefore $h^{-1}g' \simeq h^{-1}g$ can be extended to a homotopy $f_i \simeq f_{i-1}$ supported by L. By the choice of ordering of A's, $K_{i-1} \subset \overline{K - L}$, and so the homotopy keeps K_{i-1} fixed. Secondly f_i satisfies (P) provided the homotopy is sufficiently small. Thirdly $f_i | K_i$ is non-degenerate because $K_i = K_{i-1} \cup A$, and f_i is non-degenerate on K_{i-1} by induction and on A by construction. The proof of Lemma 32 is complete.

<u>GENERAL POSITION OF MAPS</u> Consider maps of X^x into M^m, where $x < m$. Define the <u>codimension</u>

$$c = m - x .$$

Define the <u>double point dimension</u>

$$d = d_2 = x - c = 2x - m .$$

More generally define the <u>r-fold point dimension</u>

$$d_r = x - (r-1)c .$$

Define $g : X \to M$ to be <u>in general position</u> if

$$\dim S_r(g) \leq d_r , \quad \text{each } r .$$

Our principal aim is now to show that any map is homotopic to a map in general position.

<u>Remark 1</u> The dimensions are the best possible, as can be seen from linear intersections in euclidean space.

<u>Remark 2</u> If f is in general position then f is non-degenerate and

$$\dim Br(f) < d_2 .$$

The first follows from Lemma 31 (ii), because we are assuming $x < m$, and so $d_r < 0$ for r large; the second then follows from Lemma 31 (iii).

Remark 3 We shall confine ourselves to the interior of manifolds for simplicity. The engulfing theorems are especially tricky at the boundary. In applications the boundary problem can generally be treated independently and more el gantly by the Addendum to Theorem 12.

Remark 4 In applications we frequently have the relative situation of wanting to keep a map fixed on a subpolyhedron X_0 of X which already happens to be in general position (and is often embedded). Therefore before stating the main theorem we introduce a relative definition.

Suppose $X_0 \subset X$. Define $g : X \to M$ to be in general position for the pair (X, X_0) if

(i) g is in general position.

(ii) $g|X_0$ is in general position.

(iii) if $x_0 < x$, then $\dim(S_r(g) \cap X_0) < d_r$, each r.

Remark 1 If $x_0 = x$, then (i) implies (ii), and (iii) is vacuous, and so then general position of g implies general position for (X, X_0). But if $x_0 < x$, then (i) does not imply (ii) or (iii).

Remark 2 Condition (iii) is, surprisingly, the best possible. At first sight it would seem that we ought to be able to make

$$\dim\,(S_r(g) \cap X_0) \leq d_r - (x - x_0)$$

instead of merely $\leq d_r - 1$. But if X is not a manifold, then the non-homogeneity of X may cause certain points of X always to lie in $S_r(g)$ independent of g. It is not the r-fold points

$S_r'(g)$ themselves, but the limit points in the branch set that cause the trouble.

Example Let X be the join of a p-simplex X_0 to an n-dimensional polyhedron not embeddable in 2n-space, and let $m = 2n+1$. Then X cannot be locally embedded at any point of X_0, and so $X_0 \subset Br(f)$ for all g. If g is in general position for (X,X_0) then

$$\dim S_2(g) = d_2 = p+1$$
$$\dim (S_2(g) \cap X_0) = p$$

but $d_2 - (x - x_0) < 0$ for n large.

Remark 3 In applications we shall be primarily concerned with the whole singular set $S(f)$. But in critical cases we shall want to "pipe away" the middles of the top dimensional simplexes of $S(f)$, and in order to do this it will be important that the interiors of such simplexes consist of pure double points, and should avoid the triple point set, the branch set, and a certain subpolyhedron X_0. We summarise this information in a useful form:

Theorem 17 Let $f : X \to \mathring{M}$ be in general position for the pair (X,X_0), where $x_0 < x < m$. Denote the double point dimension by $d = 2x - m$. Let K be a triangulation of X, that contains X_0 as a subcomplex, and such that $f : K \to M$ is simplicial for some triangulation of M.

(i) Then the singularities $S(f)$ of f form a subcomplex of K of dimension $\leq d$.

(ii) _If_ A _is a d-simplex of_ $S(f)$, _then there is exactly one other d-simplex_ A_* _of_ $S(f)$ _such that_ $fA = fA_*$. _If_ U, U_* _denote the open stars of_ A, A_* _in_ K , _then_ U, U_* _are contained in_ $X - X_0$, _and the restrictions_ $f|U$, $f|U_*$ _are embeddings;_ _the images_ fU , fU_* _intersect in_ $f\mathring{A} = f\mathring{A}_*$, _and contain no other points of_ fX .

Proof (i) By Lemma 31 and the definition of general position of f .

(ii) Dim $S_3(f) < d$ by definition of general position, and so $\mathring{A} \not\subset S_3(f)$. Also dim $Br(f) < d$ by Lemma 31 and so $\mathring{A} \not\subset Br(f)$. Therefore $\mathring{A} \subset S_2'(f)$ because

$$A \subset S(f) = S_2'(f) \cup Br(f) .$$

In other words $\mathring{A}$ consists of exactly double points, and so $fA = fA_*$ for exactly one other simplex A_* .

Now $A \not\subset X_0$, because $\dim(S(f) \cap X_0) < d$, by definition of general position for the pair (X, X_0) . Therefore $U = st(A,K) \subset X - X_0$, because X_0 is a subcomplex. Since f is non-degenerate, if $f|U$ was not an embedding, then $\mathring{A} \subset Br(f)$, a contradiction. Similarly for U_* . Finally if $u \in U$, $fu = fu_*$ and $u \neq u_*$, then $u \in S(f) \cap U = \mathring{A}$, and so $u_* \in \mathring{A}_*$. Therefore fU , fU_* meet only in $f\mathring{A} = f\mathring{A}_*$, and contain no other points of fX . The proof of Theorem 17 is complete.

The rest of the chapter is devoted to showing that any map can be moved into general position.

Theorem 18 Let $f : X \to \overset{\circ}{M}$ be a map from a polyhedron X to the interior of a manifold M, where $x < m$. Suppose $f|X_0$ is in general position where X_0 is a subpolyhedron of X. Then $f \sim g$ by an arbitrarily small homotopy keeping X_0 fixed, such that g is in general position for the pair (X,X_0).

(Notice that the theorem is trivial if $x = m$).

Corollary 1 Any map $X \to M$ is homotopic to a map in general position.

Proof First homotope X into the interior, and then put $X_0 = \emptyset$ in the theorem.

Corollary 2 With the hypothesis of Theorem 18, let $\{X_i\}$ be a finite family of polyhedra such that $X_0 \subset X_i \subset X$, each i. Then we can choose g so as to be in general position for every pair (X_i,X_j) for which $X_i \supset X_j$ (including $j = 0$).

Proof Choose a triangulation K of X containing all the X_i as subcomplexes, by Theorem 1, and let K^n be the n-skeleton of K. By induction on n, use Theorem 18 to homotope $f|K^n$ into general position for the pair (K^n,K^{n-1}) keeping K^{n-1} fixed, and extend the homotopy from K^n to K by the homotopy extension theorem. The induction begins with $n = x_0$, by moving $f|K^n$ into general position keeping X_0 fixed. At the end of the induction we have a map g, that is in general position for each adjacent pair of skeletons containing X_0. If $n = x_i$ then $X_i \subset K^n$, and so $g|X_i$ is in general position. If $X_i \supset X_j$ then either $x_i = x_j$ and the general position of $g|X_i$ implies general position for the pair (X_i, X_j) ; or else $x_i > x_j$ and $(X_i, X_j) \subset (K^n, K^{n-1})$, so condition (iii) is satisfied for (X_i, X_j) because it is satisfied for (K^n, K^{n-1}) .

Remark In Corollary 2, if we put $X_0 = \emptyset$ and the family equal to the family of skeletons of a triangulation K of X , we recover, for general position in manifolds, a generalisation of the primitive general position of K in euclidean space.

Corollary 3 Given $f : X \to M$ and $Y \subset M$, we can homotope f to general position g such that for each r ,

$$\dim (g S_r(g) \cap Y) \leq d_r + y - m .$$

Proof Having moved f into general position g by Corollary 1, we then use Theorem 15: by induction on r , starting with r large and $S_r(g) = \emptyset$, ambient isotope $gS_r(g) \subset M$ into general position with respect to Y keeping $gS_{r+1}(g)$ fixed.

The t-shift of a map The proof of Theorem 18 is like that of Theorem 15, and uses a generalisation of the t-shift as follows.

Given $X_0 \subset X$ and $f : X \to M$ such that $f|X_0$ is in general position, then in particular $f|X_0$ is non-degenerate, and so by Lemma 32 we can first homotope f into a non-degenerate map keeping X_0 fixed. Therefore assume f non-degenerate. Choose triangulations K, K_0 of X, X_0 and L of M such that $f : K \to L$ is simplicial. Let K', L' denote the barycentric derived complexes modulo the (t-1)-skeletons of K,L .

Then $f, K' \to L'$ remains simplicial because f is non-degenerate.

Let B be a t-simplex of fK, and let $A_1, \ldots, A_n$ be the simplexes of K that are mapped onto B, which are all t-simplexes since f is non-degenerate. Order the A's so that those in X_0 come last; in other words there is an integer q such that $A_i \in X_0$ if and only if $q < i \leq n$. Let a_i, b be the barycentres of A_i, B (with $fa_i = b$). Then for each i

$$\overline{st}(a_i, K') = a_i \dot{A}_i P_i, \qquad \overline{st}(b, L') = b\dot{B}Q,$$

and if f_i denotes the restriction

$$f_i : a_i \dot{A}_i P_i \to b\dot{B}Q$$

of f, then f_i is the join of the three maps $a_i \to b$, $\dot{A}_i \to \dot{B}$ and $P_i \to Q$. The local shift, given below, determines a new map

$$g_i : a_i \dot{A}_i P_i \to b\dot{B}Q$$

that equals f_i if $i > q$, and is homotopic to f_i keeping the frontier $\dot{A}_i P_i$ fixed if $i \leq q$. Therefore, letting i and B vary, the local maps g_i combine into a global map $g : X \to M$ that is homotopic to f by an arbitrarily small homotopy keeping X_0 fixed. We call the move $f \to g$ a <u>t-shift keeping X_0 fixed</u>.

<u>Local shift of a map</u> The local shift is much the same as before, except that instead of moving one cone away from the centre we have to move several cones away from each other. Although each cone is not in general embedded, the movement of

each cone can be realised by an ambient isotopy as in the local shift of an embedding, but of course the movement of the union of the cones is only a homotopy.

As before, choose a homeomorphism $h : Q \to \dot{\Delta}$ onto the boundary of an (m-t)-simplex, throwing $\bigcup_i fP_i$ into the (m-t-1)-face Γ (which is possible since by hypothesis $x < m$). Extend h to $h : bQ \to \Delta$ by mapping b to the barycentre v of Δ, and joining linearly. Subdivide so that $h : (bQ)' \to \Delta'$ is simplicial. Define $v_i = v$, $q < i \leq n$, and choose a sequence $(v_1, v_2, \dots, v_q) \subset \Delta$ near v in general position with respect to Δ'. Let $k_i : \Delta \to \Delta$ be the homeomorphism joining the identity on $\dot{\Delta}$ to the map $v \to v_i$. In particular $k_i = 1$, $i > q$. For each i, define

$$g_i : a_i\mathring{A}_iP_i \to b\mathring{B}Q$$

to be the join of the maps $f : \mathring{A}_i \to \mathring{B}$ and $h^{-1}k_ihf : a_iP_i \to bQ$. If $i > q$ then $g_i = f_i$, and if $i \leq q$ then g_i is ambient isotopic (and therefore homotopic) to f_i by an arbitrarily small ambient isotopy keeping fixed the boundary $\mathring{B}Q$ (and therefore the frontier $\mathring{A}_iP_i$). This completes the definition of the local shift.

<u>Lemma 33</u> <u>In the local shift</u> $S_r(g_i) = S_r(f_i)$.

<u>Proof</u> The singular sets are unaltered by ambient isotopy.

<u>Corollary</u> <u>A t-shift preserves non-degeneracy</u>.

Lemma 34 *With the hypothesis of Theorem 18 let* $f \to g$ *be a t-shift keeping* X_0 *fixed. If* $\dim S_r(f) \le d_r$ *for all* $r < s$, *then the same is true for* g.

Corollary *A t-shift preserves general position* (for choose s sufficiently large).

Proof of Lemma 34 Suppose not: suppose $d > d_r$, where $d = \dim S_r(g)$ and $r < s$. Since g agrees with f on the frontiers of the local shifts, something must go wrong in the interior of some local shift. Therefore

$$\dim S_r(g \mid \bigcup_{i=1}^{n} (a_i \mathring{\Delta}_i P_i - \dot{\Delta}_i P_i)) = d ,$$

and

$$\dim S_r(g \mid \bigcup_{1}^{n} (a_i P_i - P_i)) = d - t .$$

Therefore if we choose subdivisions $(\bigcup a_i P_i)'$, Δ'' of $\bigcup a_i P_i$, Δ' with respect to which hg is simplicial, then there is a $(d-t)$-simplex $D \in \Delta''$, in the interior of Δ, that is the image under hg of at least r simplexes. Select a set of exactly r simplexes mapping onto D, and, of these, suppose r_i lie in $a_i P_i$, $1 \le i \le q$, and suppose r_0 lie in $\bigcup_{q+1}^{n} a_i P_i$. Therefore we have

$$r = \sum_{0}^{q} r_i , \qquad 0 \le r_i \le r , \quad \text{each } i .$$

We shall now choose certain simplexes $C_i \subset \Gamma$ and $D_i \subset \Delta$, for $i = 0, 1, \ldots, q$, with the properties

$$D_i \supset D$$

$$(*) \qquad \dim D_i \le \dim \Delta - r_i c .$$

(Recall c = codimension = $m - x$.)

Firstly if $r_i = 0$, choose $C_i = \Gamma$ and $D_i = \Delta$, so that (*) is trivially satisfied.

Secondly suppose $r_i \neq 0$ and $i \geq 1$. By Lemma 33 $hg\,S_{r_i}(g|a_iP_i)$ is a subcone of $hg(a_iP_i)$ with vertex v_i and base $hf\,S_{r_i}(f|P_i)$. Therefore there is a simplex $C_i \in \Delta'$ such that

$$C_i \subset hfS_{r_i}(f|P_i) \quad ,$$

$$D_i = v_iC_i \supset D \quad .$$

Therefore

$$\begin{aligned}
\dim C_i &\leq \dim S_{r_i}(f|P_i) \\
&= \dim S_{r_i}(f|a_i\mathring{\Delta}_iP_i) - t - 1 \\
&\leq d_{r_i} - t - 1, \text{ by hypothesis since } r_i \leq r < s, \\
&= m - r_ic - t - 1, \text{ where } c = \text{codimension} .
\end{aligned}$$

Therefore

$$\begin{aligned}
\dim D_i &\leq m - r_ic - t \\
&= \dim \Delta - r_ic ,
\end{aligned}$$

verifying property (*).

Finally suppose $r_0 \neq 0$. By construction g is the same as f on $\bigcup_{q+1}^{n} a_iP_i$, and so there is a simplex $C_0 \in \Delta'$ such that

$$C_0 \subset hfS_{r_0}(f|\bigcup_{q+1}^{n} P_i) \quad ,$$

$$D_0 = vC_0 \supset D \quad .$$

We verify (*) as in the previous case:

$$\begin{aligned} \dim C_0 &\le \dim S_{r_0}\Big(f \Big| \bigcup_{q+1}^{n} P_i\Big) \\ &= \dim S_{r_0}\Big(f \Big| \bigcup_{q+1}^{n} a_i \mathring{\Delta}_i P_i\Big) - t - 1 \\ &\le d_{r_0} - t - 1 \,, \quad \text{since} \quad r_0 \le r < s \,, \quad \text{and so} \end{aligned}$$

$$\dim D_0 \le \dim \Delta - r_0 c \,.$$

As in the proof of Lemma 30, embed Δ in euclidean space, and denote by $[C_i]$ the linear subspace spanned by C_i, etc.. As before there are two possibilities, each leading to a contradiction.

<u>Case (a)</u> For each j, $1 \le j \le q$, $\bigcap_0^{j-1} [D_i]$ and $[D_j]$ span $[\Delta]$.

<u>Case (b)</u> Not (a).

In case (a) we deduce

$$\dim \bigcap_0^{j} [D_i] + \dim \Delta = \dim \bigcap_0^{j-1} [D_i] + \dim D_i$$

Summing for $j = 1,2,\dots,q$, and cancelling, we have

$$\begin{aligned} \dim \bigcap_0^{q} [D_i] &= \sum_0^q \dim D_i - q \dim \Delta \\ &= \dim \Delta + \sum_0^q (\dim D_i - \dim \Delta) \\ &\le (m - t) - \sum_0^q r_i c \,, \quad \text{by } (*) \\ &= m - t - rc \\ &= d_r - t \\ &< d - t \,, \end{aligned}$$

contradicting the fact that $D^{d-t} \subset \bigcap_0^q D_i \subset \bigcap_0^q [D_i]$.

In case (b), there exists some j, $1 \leq j \leq q$, such that $\bigcap_0^{j-1} [D_i]$ and $[D_j]$ do not span $[\Delta]$. Therefore $D_j \neq \Delta$, and so $r_j \neq 0$ and $D_j = v_j C_j$. Also $\bigcap_0^{j-1} [D_i]$ and $[C_j]$ span a proper subspace, π say, of $[\Delta]$. Now the vertices of the C's and D's are all vertices of Δ' , and our choice of v_j in general position in Δ with respect to Δ' ensures that $v_j \notin \pi$, because the definition of the general position of a point involved sufficient lattice operations to cover this eventuality. Therefore $\pi \cap v_j C_j = C_j$, and so

$$D \subset \bigcap_0^q D_i \subset \pi \cap D_j = C_j \subset \Gamma$$

contradicting our choice of D in the interior of Δ . This completes the proof of Lemma 34.

<u>Lemma 35</u> <u>With the hypothesis of Lemma 34 suppose that</u> $\dim S_s(f) = t > d_s$. <u>Then</u> $\dim S_s(g) = t - 1$.

<u>Corollary</u> <u>With the hypothesis of Theorem 18, we can move</u> f <u>into general position by t-shifts keeping</u> X_0 <u>fixed</u>.

<u>Proof</u> By increasing induction on s , starting trivially with $s = 1$, and, for each s , by decreasing induction on t , starting with $t = \dim S_s(f)$, we can reduce each singular set $S_s(f)$ to its correct dimension by Lemma 35, at the same time keeping correct the singular sets $S_r(f)$, $r < s$, by Lemma 34.

<u>Proof of Lemma 35</u> Let $d = \dim S_s(g)$. By Lemma 31 $S_s(f)$ is a t-dimensional subcomplex of the triangulation K of X used in the t-shift $f \to g$. By construction the t-shift keeps the (t-1)-skeleton of K fixed, and so

$$S_s(g) \supset S_s(f) \cap ((t-1)\text{-skeleton of } K)$$

implying that $d \geq t-1$.

Suppose $d > t-1$; then $d > d_s$, and with one modification the proof is exactly the same as that of Lemma 34, substituting s for r. That is to say, we examine the interior of a local shift, and find $D^{d-t}, \subset \mathring{\Delta}$, that is the image of r_0 simplexes in $\bigcup_{q+1}^{n} a_i P_i$, and r_i simplexes in $a_i P_i$, $1 \leq i \leq q$, where

$$s = \sum_0^q r_i$$

$$0 \leq r_i \leq s \qquad \text{for} \qquad 0 \leq i \leq q \ .$$

The modification that we need to prove is

$$r_i < s \qquad \text{for} \qquad 0 \leq i \leq q$$

in order to be able to verify (*) , and therefore achieve a contradiction in each of the two cases. The contradictions establish $d = t-1$.

There remains to prove the modification, and for this we use two pieces of hypothesis that we have not yet used, that $\dim S_s(f) = t$ and $f|X_0$ is already given to be in general position.

Using Lemma 33 and that $S_s(f)$ is contained in the t-skeleton of K, we have for each i,

$$\begin{aligned} S_s(g|a_iP_i) &= S_s(f|a_iP_i) \\ &\subset (a_iP_i) \cap (\text{t-skeleton of } K) \\ &= (a_iP_i) \cap A_i \\ &= a_i \ . \end{aligned}$$

Now trivially $s \geq 2$, because if $s = 1$ then $S_s(f) = X$ and $d_s = x$ and so we could not have $\dim S_s(f) > d_s$. And a_i is the only point of a_iP_i mapped by g to v_i. Therefore $S_s(g|a_iP_i) = \emptyset$, and so $r_i < s$ for $1 \leq i \leq q$.

There remains the case $i = 0$. If $n - q \geq s$, then there are at least s simplexes $A_{q+1}, \ldots, A_n$ of X_0 mapped by f into the t-simplex B, implying $\dim S_s(f|X_0) \geq t > d_s$, and contradicting the hypothesis $f|X_0$ in general position. Therefore $n - q < s$. Let $Z = \bigcup_{q+1}^{n} a_iP_i$. By definition of the t-shift g agrees with f on Z, and so

$$\begin{aligned} S_s(g|Z) &\subset Z \cap (\text{t-skeleton of } K) \\ &= \bigcup_{q+1}^{n} a_i \ . \end{aligned}$$

Since $a_{q+1}, \ldots, a_n$ are the only points of Z mapped by g to v, and since there are less than s of them, we deduce $S_s(g|Z) = \emptyset$, and so $r_0 < s$. The proof of Lemma 35 is complete.

<u>Lemma 36</u> <u>Given</u> $X_0 \subset X$, $x_0 < x$, <u>and</u> $f : X \to M$, <u>suppose that both</u> f <u>and</u> $f|X_0$ <u>are in general position</u>. <u>Let</u> $f \to g$ <u>be a t-shift keeping</u> X_0 <u>fixed</u>.

(i) <u>If</u> $\dim(S_r(f) \cap X_0) < d_r$, <u>for all</u> $r < s$, <u>then the same is true for</u> g.

(ii) <u>If, further,</u> $t = d_s$ <u>then</u> $\dim(S_s(g) \cap X_0) < d_s$.

<u>Corollary</u> <u>A t-shift preserves general position for pairs</u>.

For use the Corollary to Lemma 34, and Lemma 36(i) with s large.

<u>Proof of Lemma 36 (i)</u> Suppose not. Then for some r, $< s$, we have $\dim(S_r(g) \cap X_0) = d_r$, because $\dim S_r(g) \leq d_r$ by Lemma 34 Corollary. As in the proof of Lemma 34 we examine the interior of a local shift, and find a simplex $D \subset \mathring{\Delta}$, of dimension $d_r - t$, in the image of r_0 simplexes of $Z = \bigcup_{q+1}^{n} a_i P_i$, and r_i simplexes of $a_i P_i$, $1 \leq i \leq q$. Also $D \subset fX_0$. But $X_0 \cap a_i P_i = \emptyset$ for $1 \leq i \leq q$, and so $r_0 \neq 0$. Therefore D is in the image of $S_{r_0}(g|Z) \cap X_0$. But $g|Z = f|Z$, and

$$\dim(S_{r_0}(f) \cap X_0) < d_{r_0}$$

by hypothesis, and so in the verification of (*) (as in the proof of Lemma 34) we gain one dimension:

$$\dim D_0 < \dim \Delta - r_0 c .$$

Therefore in case (a) we have

$$\dim \bigcap_0^q [D_i] < d_r - t$$

contradicting the construction $D^{d_r - t} \subset \cap D_i$.
In case (b) the contradiction is unchanged.

(ii) The proof of Lemma 36 part (ii) is the same as for part (i), except for the modification of having to show

$$r_i < s , \quad \text{for } 0 \leq i \leq q ,$$

as in the proof of Lemma 35. Firstly $r_0 \neq 0$ because $D \subset fX_0$, and so $r_i < s$ for $1 \leq i \leq q$. Finally $r_0 < s$, otherwise we should have s simplexes of X_0 mapped onto B , implying

$$\dim S_s(f|X_0) \geq t = d_s ,$$

contradicting the hypothesis $x > x_0$ and the condition

$$\dim S_s(f|X_0) \leq d_s - s(x - x_0)$$

included in the general position of $f|X_0$. The proof of Lemma 36 is complete.

<u>Proof of Theorem 18</u> We are given $f : X \to \overset{o}{M}$ with $f|X_0$ in general position, and we have to move f into general position for the pair (X,X_0) keeping X_0 fixed. Lemma 35 Corollary shows that f can be moved into general position using t-shifts. If $x = x_0$ we are then finished, because the general position of g implies general position for the pair. If $x > x_0$, there remains to achieve condition (iii) for general position of the pair. Lemma 36 shows this can be also using t-shifts, by induction on s putting $t = d_s$, and starting trivially with $s = 1$. The general position of f meanwhile is preserved by Lemma 34 Corollary.

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1963 (Revised 1965)

Seminar on Combinatorial Topology

by E.C. ZEEMAN.

Chapter 7 : ENGULFING

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Chapter 7 : ENGULFING.

The idea of an engulfing theorem is to convert a homotopy statement into a geometrical statement: it is a key step in passing from algebra to geometry.

For example let X be a compact subspace in the interior of a manifold M, and consider the following two statements about X.

(1) X is inessential in M ; that is to say the inclusion map $X \subset M$ is homotopic to a constant.

(2) X is contained in an m-ball in M. The first is a homotopy statement about X , and the second is a geometrical statement. Obviously the second implies the first, because a ball is contractible. The converse is not so obvious, and is given by our first engulfing theorem. For the theorem we shall assume that M is k-connected, that is to say the homotopy groups $\pi_i(M)$ vanish for $i \leqslant k$.

Theorem 19. Let M^m be a k-connected manifold, and X^x a compact subspace in the interior of M, such that $x \leqslant m - 3$ and $2x \leqslant m + k - 2$. Then X is inessential in M if and only if it is contained in a ball in the interior of M.

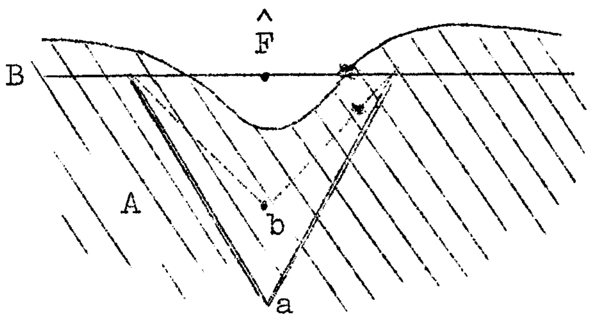
We shall prove a generalisation in Theorem 21 below, from which the above result follows at once. However since the proof of Theorem 21 is long, involving special techniques for the boundary, we give a shorter proof of Theorem 19, which will illustrate the main idea of an engulfing theorem. The proof requires four lemmas, the first three of which are straightforward. The last one, Lemma 48, involves a more complicated technique called "piping", and we postpone this until later in the chapter in order not to interrupt our main flow of thought. In effect the piping lemma is only concerned with winning the last dimension $x = m - 3$.

<u>Lemma 37. Let B^m be a submanifold of $\mathring{M}^m$. If $X \searrow Y$ in $\mathring{M}$ and $Y \subset B$ then we can ambient isotop B until $X \subset B$. In particular if Y is contained in a ball in $\mathring{M}$, then so is X.</u>

<u>Proof</u>. First we may assume that $Y \subset \mathring{B}$, for if not isotop B onto a regular neighbourhood of itself in $\mathring{M}$, by Theorem 8(3). The proof is easy to visualise because as Y expands to X we push B along with it. Notice that we may have to push other bits of B out of the way as we go, which explains why it was necessary to have B in the interior of M.

Now for the details : triangulate a neighbourhood of X in M so that X, Y are subcomplexes. By subdividing if necessary we can ensure that X collapses simplicially to Y. By induction on the number of elementary simplicial collapses, it suffices to consider the case when $X \searrow Y$ is an elementary simplicial collapse. (Notice that we do not say anything about B being a subcomplex of the triangulation, for otherwise this would violate the induction, because during the induction B gets pushed around.)

Suppose, therefore, that $X \searrow Y$ across the simplex $A = aF$ from the face F. Let $\hat{F}$ denote the barycentre of F, and L the link of F in M, which is a sphere because $F \subset X \subset \mathring{M}$. Let Δ be a simplex of dimension $1 + \dim L$, and choose a homeomorphism $h : L \to \dot{\Delta}$. Map $\hat{F}$ to the barycentre $\hat{\Delta}$ of Δ, and extend linearly to a homeomorphism $h : \hat{F}L \to \Delta$.



Since $a\dot{F} \subset Y \subset \mathring{B}$, we can choose a point b in the interior of the segment $a\hat{F}$ such that $ab\dot{F} \subset \mathring{B}$. Let $f : \Delta \to \Delta$ be the homeomorphism defined by mapping $hb \to \hat{\Delta}$, keeping $\dot{\Delta}$ fixed, and joining linearly. The composition $h^{-1}fh : \hat{F}L \to \hat{F}L$ throws b onto $\hat{F}$. Join this composition to the identity on $\dot{F}$ to give a homeomorphism of $\overline{st}(F, M)$ onto itself, which keeps the boundary fixed, and which therefore extends to a homeomorphism $g : M \to M$ ambient isotopic to the identity, keeping $M - st(F, M)$ fixed. The isotopy keeps Y fixed, because Y does not meet $st(F, M)$, and moves $ab\dot{F}$ onto A. Since $\mathring{B} \supset Y \cup ab\dot{F}$, the isotopy moves B to gB where $g\mathring{B} \supset Y \cup A = X$. The proof of Lemma 37 is complete. We shall prove a more delicate version of this result in Lemma 42 below, replacing the manifold B by an arbitrary subspace.

Lemma 38. *Let $S(f)$ denote the singular set of the map $f : X \to Z$. Let Y be such that $X \supset Y \supset S(f)$ and $X \searrow Y$. Then $fX \searrow fY$.*

Proof. Let K, L be triangulations of X, Z such that $f : K \to L$ is simplicial. Let K' be a subdivision of K such that Y is a subcomplex, and K' collapses simplicially to Y. Let

$$K' = K_0 \searrow K_1 \searrow \ldots \searrow K_n = Y$$

denote the sequence of elementary simplicial collapses, and suppose, for each i, $K_{i-1} \searrow K_i$ across the simplex A_i from the face B_i. We claim that $fK_{i-1} \searrow fK_i$ is an elementary collapse across the ball fA_i from the face fB_i (notice that we do not claim it is a simplicial collapse because in general there is no subdivision L' such that $f : K' \to L'$ is simplicial). The reason for our claim is that f maps A_i linearly into some simplex of L, and non-degenerately because $A_i \not\subset S(f)$. Therefore fA_i is a ball and fB_i a face. Also $f\left(\mathring{A}_i \cup \mathring{B}_i\right) \cap fK_i = \emptyset$ because $\mathring{A}_i \cup \mathring{B}_i \subset X - Y \subset X - S(f)$. Therefore $fA_i \cap fK_i$ is the complementary face of fA_i to fB_i. Therefore $fK_{i-1} \searrow fK_i$.

The sequence of elementary collapses gives $fX \searrow fY$.

Lemma 39. *If X^x is inessential in $\mathring{M}^m$, then there exist subspaces $Y^y, Z^z \subset \mathring{M}$, such that $X \subset Y \searrow Z$, $y \leq x + 1$, and $z \leq 2x - m + 2$.*

The proof is trivial if $x > m - 3$, for then choose $X = Y = Z$. Therefore assume $x \leq m - 3$. We first prove a weaker result, namely the same statement except that Z is one dimension higher.

Proof of the weaker result ($z \leq 2x - m + 3$).

Let C be the cone on X. Since X is inessential, we can extend the inclusion $X \subset \overset{\circ}{M}$ to a continuous-map $f:C \to \overset{\circ}{M}$. By the relative simplicial approximation theorem we can make f piecewise linear, keeping $f|X$ fixed. By Theorem 18 we can homotop f into general position keeping $f|X$ fixed. Therefore the singular set $S(f)$ of f will be of dimension $\leq 2(x + 1) - m$.

Let D be the subcone of C through $S(f)$; that is to say D is the union of all rays of C that meet $S(f)$ in some point other than the vertex of the cone. Then $\dim D \leq 2x - m + 3$. Let $Y = fC$, $Z = fD$. Since a cone collapses to any subcone we have $C \searrow D$, and since $D \supset S(f)$ we have $Y \searrow Z$ by Lemma 38. Since $fX = X$, we have $X \subset Y \searrow Z$, and the proof of the weaker result is complete.

Proof of the stronger result ($z \leq 2x - m + 2$).

For this we need the piping lemma (Lemma 48) below. Since the proof of the piping lemma is long we postpone it until later in the chapter.

As in the weaker case, let C be the cone on X, and $f:C \to \overset{\circ}{M}$ a (piecewise linear) extension of the inclusion $X \subset \overset{\circ}{M}$. Triangulate X and let C_0 be the subcone on the $(x - 1)$-skeleton of X. By Theorem 18 we can homotop f into general position for the pair C, $X \cup C_0$ keeping $f|X$ fixed. The triple X^x, $C_0{}^x \subset C^{x+1}$ is cylinder-like (in the sense of the piping lemma) and so by the piping lemma we can homotop f keeping $f|X$ fixed, and choose a subspace $C_1 \subset C$ such that

$$S(f) \subset C_1$$

$$\dim(C_0 \cap C_1) < \dim C_1 \leqslant 2x - m + 2$$

$$C \searrow C_0 \cup C_1 \searrow C_0.$$

Let D be subcone through $C_0 \cap C_1$; $\dim D \leqslant 2x - m + 2$. Then $C_0 \cup C_1 \searrow D \cup C_1$ because $C_0 \searrow D$ and $C_0 \cap C_1 \subset D$. Therefore $C \searrow D \cup C_1$. Define $Y = fC$, $Z = f(D \cup C_1)$ and the result follows by Lemma 38, because $D \cup C_1 \supset S(f)$. The proof of Lemma 39 is complete.

Proof of Theorem 19.

We have X^x inessential in $\overset{\circ}{M}$, and have to show that X is contained in a ball in $\overset{\circ}{M}$. The proof is by induction on x, starting trivially with $x = -1$. Assume the result is true for dimensions less than x.

By Lemma 39 choose $Y, Z \subset \overset{\circ}{M}$ such that $X \subset Y \searrow Z^z$, where

$$\begin{aligned} z &\leqslant 2x - m + 2, && \text{by Lemma 39} \\ &\leqslant k, && \text{by the hypothesis } 2x \leqslant m + k - 2. \end{aligned}$$

Therefore Z is inessential in M. But $z < x$ by the hypothesis $x \leqslant m - 3$. (This is one of the places where codimension $\geqslant 3$ is crucial). Therefore Z is contained in a ball in $\overset{\circ}{M}$ by induction. By Lemma 37 so is Y. Therefore we have put a ball round X, and the proof of Theorem 19 is complete. We deduce some corollaries.

<u>Corollary 1. If M is closed and k-connected, $k \leq m-3$, then any subspace of dimension $\leq k$ is contained in a ball.</u>

The corollary follows immediately from Theorem 19.

<u>Corollary 2.</u> (<u>Weak Poincaré Conjecture</u>). <u>If M is a homotopy m-sphere, $m \geq 5$, then M is topologically homeomorphic to S^m.</u>

<u>Remark.</u> We call this the weak Poincaré Conjecture because although the hypothesis assumes that M has a polyhedral manifold structure (we always assume this), the thesis gives only a topological homeomorphism, not a polyhedral homeomorphism. The reason is that the proof that we give here is Stallings' proof, which depends upon the topological Schönflies Theorem of Mazur-Brown. In Chapter 9 we shall give Smale's proof, using combinatorial handlebody theory, which does not depend upon the Schönflies Theorem, and which gives the stronger result that M is in fact a polyhedral sphere, $m \geq 6$. The stronger result for $m = 5$ is also true, but we shall not give it in these notes, because the only known proof depends upon smoothing, and deep results from differential theory, including $\theta^5 = \Gamma^4 = 0$.

<u>Proof of Corollary 2.</u>

Let $x = [m/2]$ and $x_* = m - x - 1$.

Then since $m \geq 5$ we have both $x, x_* \leq m - 3$. Choose a triangulation of M, and call this complex M also. Let X be the x-skeleton of M, and X_* the dual x_*-skeleton (which is defined to be the largest subcomplex of the barycentric first derived of M, not meeting X). Now a homotopy m-sphere is (m-1)-connected. Therefore by Corollary 1 both X, X_* are contained in balls B, B_*, say.

We can also assume that X, X_* are in the interiors of balls (by taking regular neighbourhoods of B, B_* if necessary).

We now want the interiors of the two balls to cover M, and if they don't already then we stretch them a little until they do, as follows. Let N, N_* be the simplicial neighbourhoods of X, X_* in the second barycentric derived complex of M. Then $M = N \cup N_*$. Now pick a regular neighbourhood of X in $\mathring{B}$, and ambient isotope it onto N. The isotopy carries B into another ball, A say, whose interior contains N. Similarly construct a ball A_*, whose interior contains N_*. Therefore $M = \mathring{A} \cup \mathring{A}_*$.

Now let $C = M - A_*$. Then $\dot{C}$ is a collared (m -1)-sphere in the interior of A) (by the Corollary to Lemma 24). Therefore by the topological Schönflies Theorem of Mazur-Brown C is a topological ball. Therefore $M = A \cup C$ is the union of two topological balls sewn along their boundaries; in other words M is a topological sphere.

<u>Corollary 3. If M is closed and $[m/r]$-connected, $r \geq 2$, then M is the union of r balls. Consequently M is of Lusternick-Schirrelman category $\leq r$.</u>

<u>Proof.</u> Let M be k-connected.

Now $$[m/r] \leq k \iff m/r < k+1$$
$$\iff m < r(k+1)$$
$$\iff m+1 \leq r(k+1).$$

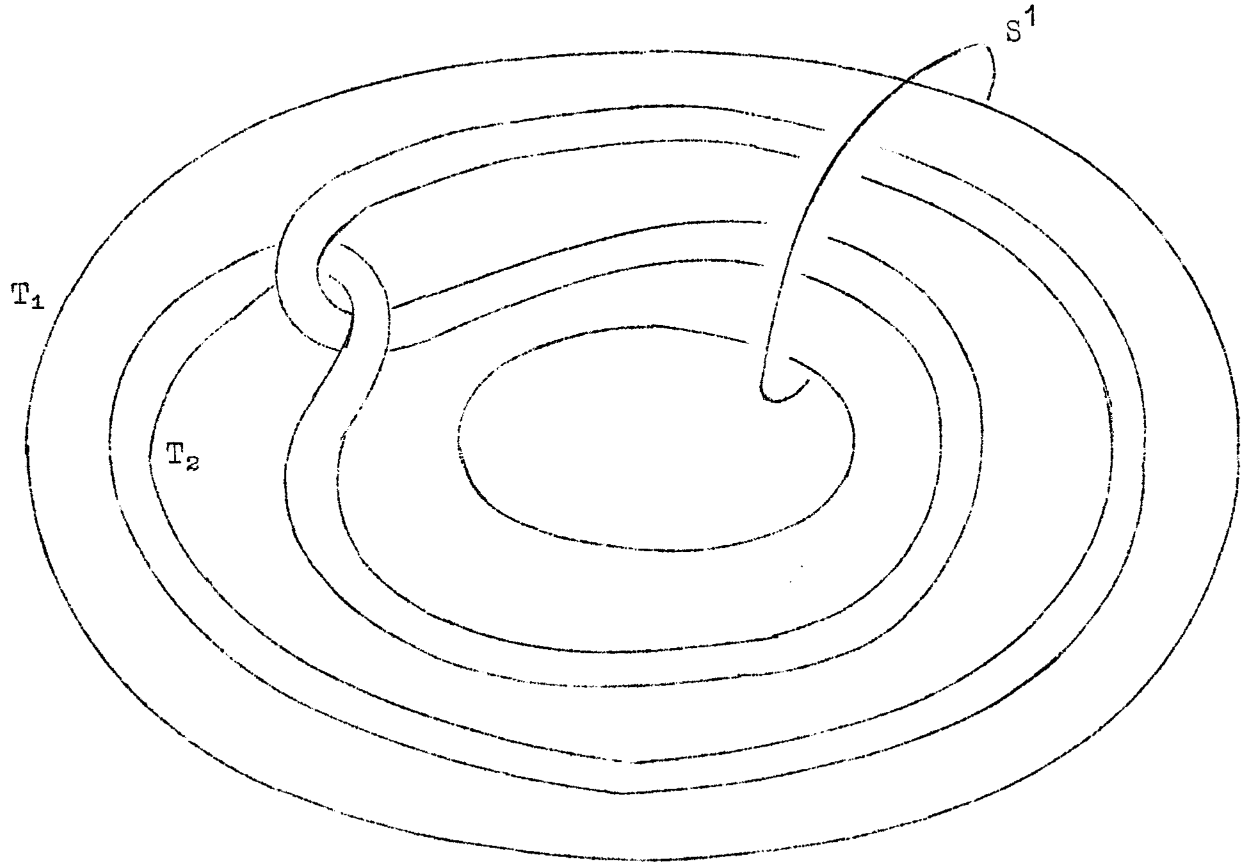
Therefore the condition $[m/r] \leq k$ is equivalent to saying that the set $\{0, 1, \ldots, n\}$ can be partitioned into r disjoint subsets G_i, $i = 1, \ldots, r$, each containing $\leq k+1$ integers.

Choose a triangulation of M, and let M' denote the barycentric derived complex. Divide the vertices of M' into r disjoint subsets J_i, by putting the barycentre of a q-simplex of M into J_i if $q \in G_i$. Let K_i be the subcomplex of M' consisting of all simplexes, all of whose vertices lie in J_i. The K_i's will then play the role that the complementary skeletons played in the proof of the preceding corollary. Let N_i be the simplicial neighbourhood of K_i in M'', the second derived. Then $M = \bigcup N_i$. By construction, for each i, $\dim K_i \leq k$, and so K_i lies in a ball B_i by Corollary 1. Ambient isotope B_i onto a ball A_i containing N_i. Then $M = \bigcup A_i$, as desired.

We now turn to the question of showing by examples that the hypotheses $x \leq m-3$ and $2x \leq m+k-2$ in Theorem 19 are the best possible. First suppose $x = m-2$.

Example 1. Whitehead's Example.

In 1937 Whitehead produced the following example of a contactible open 3-manifold M^3 (open means non-compact without boundary The manifold is remarkable in that it contains a curve S^1 that is inessential (since M^3 is contractible) but is not contained in a ball in M^3. The manifold is constructed as follows. Inside a solid torus T_1 in S^3 draw a smaller solid torus T_2, linked as shown; then inside T_2 draw T_3 similarly linked, and so on.



Define $M^3 = S^3 - \bigcap_1^\infty T_i$. If S^1 links T_1, then S^1 is not contained in a ball in M^3. We omit the proof, because the proof of the next example is simpler.

<u>Example 2.</u> <u>Mazur's Example.</u>

Poenaru (1960) and Mazur (1961) produced examples of a compact bounded contractible 4-manifold with non simply-connected boundary. For a description of Mazur's example M^4 see Chapter 3, page 10.

In particular M^4 has as spine the dunce hat D^2. Then D^2 is inessential, but not contained in a ball for the following reason.

Suppose D^2 were contained in a ball B. By replacing B by a regular neighbourhood if necessary we may assume D^2 lies in the interior of B. Let M_1 be a regular neighbourhood of D^2 in $\mathring{B}$. There is a homeomorphism $M \to M_1$ keeping D^2 fixed (Chapter 3, Theorem 8). Let B_1, M_2 be the images of B, M, under this homeomorphism. Therefore we have

$$M \supset B \supset M_1 \supset B_1 \supset M_2 \supset D^2.$$

By the regular neighbourhood annulus theorem (Theorem 8, Corollaries 2 and 3), we have

$$B - \mathring{B}_1 \cong S^3 \times I$$
$$M - \mathring{M}_1 \cong M - \mathring{M}_2 \cong \dot{M} \times I$$

Therefore in the commutative triangle induced by inclusions

$$\begin{tikzcd}
\pi_1(\dot{M}_1) \arrow[rr, "\cong"'] \arrow[dr] & & \pi_1(M - \mathring{M}_2) \\
& \pi_1(B - \mathring{B}_1) \arrow[ur] &
\end{tikzcd}$$

the top arrow is an isomorphism, and the bottom group zero, contradicting $\pi_1(\dot{M}) \neq 0$. Therefore D^2 is not contained in a ball.

<u>Remark.</u> It is significant that in the two examples above one of the manifolds is open, and the other is bounded. It is

conjectured that no similar example exists for <u>closed</u> manifolds. More precisely:

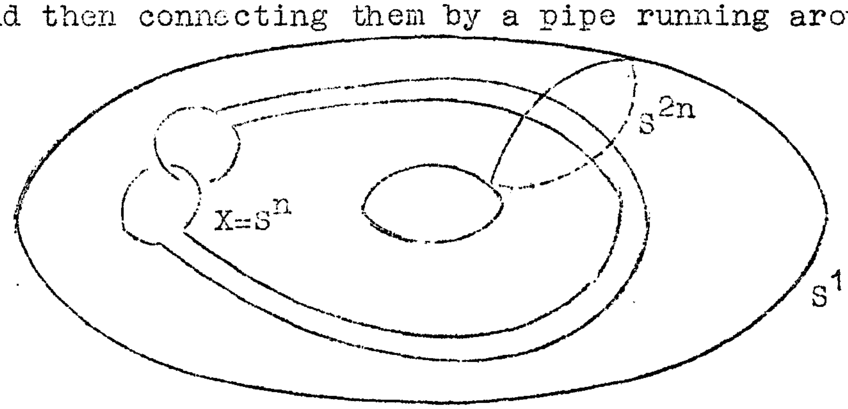
<u>Conjecture. Corollary 1 is true for $k = m-2$.</u>

Observe that this conjecture is true for $m \geqslant 5$, because the Poincaré Conjecture is true for $m \geqslant 5$. In the missing dimensions $m = 3,4$ the conjecture is equivalent to the Poincaré Conjecture, which is still unsolved. For, if the Poincaré Conjecture is true, then an $(m-2)$-connected m-manifold, $m \geqslant 3$, is a sphere, and so any proper subpolyhedron is contained in a ball. Conversely if the above conjecture is true, then the proof of Corollary 2 works for the missing dimensions $m = 3,4$, because there are complementary skeletons of codimension $\geqslant 2$.

Bing has shown that in dimension 3 a more delicate result will suffice: he has proved that if M^3 is closed manifold in which every simple closed curve lies in a ball, then $M^3 = S^3$.

<u>Example 3. Irwin's Example.</u>

We next give an example to show that the hypothesis $2x \leqslant m+k-2$ is necessary in Theorem 19. Let $M = S^1 \times S^m$, $n \geqslant 2$, and let $X = S^n$, embedded in M by first linking two little n-spheres locally, and then connecting them by a pipe running around the S'.



Notice that $m = 2n+1$, $x = n$, $k = 0$, and so the hypothesis fails by one dimension

$$2x \not\leq m + k - 2.$$

Next observe that we can homotope S^n to a point by pulling one end across the other and back around the S^1.

Therefore S^n is inessential. On the otherhand S^n cannot be contained in a ball, for otherwise we could unknot it in this ball (by Theorem 9 since $n \leq m-3$) and span it with an $(n+1)$-disk. In the universal cover $R \times S^{2n}$ of $S^1 \times S^{2n}$ the disk would lift to a countable set of disjoint disks, none of whose boundaries could therefore link. But by construction S^n lifts to a countable set of spheres, any one of which links its two neighbours. This contradiction shows that S^n cannot be contained in a ball.

Definition of a core.

There is a 1-dimensional obstruction in the last example, which suggests that if we cannot embed X in a ball, we might try to engulf it in some sort of 1-dimensional "core" of M.

More precisely define a closed subspace C to be a k-core of M if the pair (M, C) is k-connected; that is to say the relative homotopy groups $\pi_i(M,C)$ vanish for $i \leq k$. This condition is equivalent to saying that the inclusion $C \subset M$ induces isomorphisms $\pi_i(C) \overset{\cong}{\to} \pi_i(M)$, $i < k$, and an epimorphism $\pi_k(C) \to \pi_k(M)$.

Example 1. If M is k-connected, then a point, or a ball, or any collapsible set in M is a k-core.

Example 2. The k-skeleton of a triangulation of M is a k-core.

Example 3. The k-skeleton of a triangulation of a k-core is another k-core.

Example 4. A regular neighbourhood of a k-core is another k-core.

Example 5. If $D \searrow C$ then C is a k-core if and only if D is a k-core.

Example 6. If $p \leq q$ then $S^p \times$ point is a $(q-1)$-core of $S^p \times S^q$.

Definition of engulfing.

Let X, C be compact subspaces of M. We say that we can engulf X by pushing out a feeler from the core C, if there exists D, such that

$$X \subset D \searrow C$$
$$\dim (D-C) \leq x+1.$$

More briefly we describe this by saying engulf X, or engulf X from C, or engulf X in D. The feeler is D-C, and it is important for applications that it be of dimension only one more than X (and in special cases of the same dimension as X). For example in the next chapter we shall engulf singularities of maps, and the feeler itself may introduce new singularities, but these will be of lower dimension than the ones we started with, and so can be absorbed by successive engulfing. Rewriting Theorem 19 from this point of view, the core C would be a point, and X would be engulfed in a collapsible set.

The proof that this statement is equivalent to Theorem 19 is given by the following lemma.

Lemma 40. *Let C, X be compact subspaces of M. Then X can be engulfed from C if and only if X is contained in a regular neighbourhood of C.*

Proof. If X can be engulfed in D, then it is contained in regular neighbourhood N of D, which is also a regular neighbourhood of C, because $N \searrow D \searrow C$. Conversely given $X \subset N \searrow C$, triangulate N so that X, C are subcomplexes, and subdivide if necessary so that N collapses simplicially to C. Order the elementary simplicial collapses in order of decreasing dimension, by Lemma 11. Perform all those elementary collapses of dimension $\geqslant x+2$, leaving D, say. Then $\dim (D-C) \leqslant x+1$, and $D \supset X$ because we have only removed simplexes of dimension $\geqslant x+1$. Performing the rest of the elementary collapses gives $D \searrow C$.

Non-compact collapsing and excision.

We shall always assume X compact, but it is sometimes useful to have the core C non-compact, as for example in the proof of Theorem 22 below. So far collapsing has only been defined for compact spaces, and we extend the definition to non-compact spaces as follows. Define

$$D \searrow C \quad \text{if} \quad \begin{cases} \overline{D-C} \text{ compact, and} \\ \overline{D-C} \searrow \overline{D-C} \cap C \end{cases}$$

where the right hand side is compact collapsing. If C, D are non-compact the definition is new; if they are compact then the definition agrees with compact collapsing, because, given the right hand side, we can triangulate and perform the same sequence of elementary simplicial collapses on D, since C does not meet the free face of any elementary collapse. An immediate consequence of the definition is the <u>excision</u> property

$$A \searrow A \cap B \iff A \cup B \searrow B$$

because the condition for both sides is $\overline{A-B} \searrow \overline{A-B} \cap B$. Whenever we use this property in either direction we shall say <u>by excision</u>. Given $D \searrow C$, when we say <u>triangulate</u> the collapse we mean choose a triangulation of $\overline{D-C}$ such that $\overline{D-C}$ collapses simplicially to $\overline{D-C} \cap C$, and choose a particular sequence of elementary simplicial collapses.

The definition of engulfing from a non-compact core C remains the same, with the new interpretation given to the symbol $\searrow$.

<u>Remark</u>.

Stallings introduced a different point of view of engulfing. He envisaged an open set of M moving, amoeba like, until it had swallowed up X. Rewriting Theorem 19 from this point of view, the open set would be a small open m-cell, and we could isotope this onto the interior of the ball containing X. The following lemma illustrates the connection between our definition of engulfing and Stallings' point of view.

<u>Lemma 41.</u> <u>Let M be a manifold without boundary, and X a compact subspace. Let C be a closed subspace (not necessarily compact),</u>

<u>and U any open set containing C. If X can be engulfed from C, then there is a (piecewise linear) homeomorphism $h : M \to M$, isotopic to the identity by an isotopy keeping C fixed and supported by a compact set, such that $hU \supset X$.</u>

<u>Proof.</u> If C is compact the proof is easy, for choose one regular neighbourhood of C in U, and another containing X, by Lemma 40, and ambient isotop one onto the other keeping C fixed.

If C is non-compact, confine attention to a regular neighbourhood M_0 of $\overline{D-C}$ in M, which will be compact. Let $C_0 = C \cap M_0$, $D_0 = D \cap M_0$. Let N_C, N_D be second derived neighbourhoods of C_0, D_0 in some triangulation of M_0. Then $\overline{N_D - N_C}$ is contained in the interior of M_0, and so we can ambient isotop N_C into N_D keeping $C_0 \cup \dot{M}_0$ fixed, by (the proof of) Theorem 8(3). Extend the isotopy to M by keeping the rest of M fixed. If the triangulation was sufficiently fine, then $U \supset N_C$ and so U will be isotoped over X.

<u>Remark.</u>

In general $hU \not\supset U$ in Lemma 41. For example consider the case when $X \cup U = M$. Therefore the amoeba approach is no good for successive engulfings, because each new engulfing may mess up what has already been engulfed. The advantage of the feeler approach is that the core C stays fixed while successive feelers are added. The price that we have to pay for this advantage is that the core must satisfy a certain collapsibility condition (see the definition of q-collapsibility below). A further advantage of the feeler approach is that it can handle boundary problems which present certain difficulties. Before handling the general case, however, we consider the special case of a collapsible core. As in the case of Theorem 19, we shall be able to deduce the following Theorem 20 from the general Theorem 21, but again it is worth giving a short proof separately.

<u>Theorem 20.</u> <u>Let M^m be a k-connected manifold, $k \leq m-3$. Let C be a collapsible subspace and X^x a compact subspace, both in the interior of M. If $x \leq k$, then we can engulf X in a collapsible subspace D in the interior of M; that is to say $X \subset D \searrow C$ and $\dim (D-C) \leq x+1$.</u>

<u>Proof</u>. Triangulate $C \cup X$, and subdivide if necessary so that C is simplicially collapsible. Order the elementary simplicial collapses of C in order of decreasing dimension. We claim that it is possible to perform all those of dimension $> x$ on the complex $C \cup X$, collapsing it to X_0 say, $\dim X_0 = x$, as follows. There is no trouble during collapses of dimension $> x+1$, because X cannot get in the way. There looks as though there might be trouble with those of dimension $x+1$, for consider a collapse across A^{x+1} from the face F^x. It is possible that $F \subset X$, but since F is principal in X, F is still a free face of A, and so the collapse is valid.

Now X_0 is contained in a ball in $\mathring{M}$ by Theorem 19 Corollary 1. Therefore by Lemma 37, $C \cup X$ is also contained in a ball, B say, in $\mathring{M}$. We may assume $C \cup X \subset \mathring{B}$ by taking a regular neighbourhood if necessary.

Now a ball is a regular neighbourhood of any collapsible set in its interior, by Theorem 8 Corollary 1. Therefore X lies in the regular neighbourhood B of C. By Lemma 40 we can engulf X. This completes the proof of Theorem 20. We now show how the dimension of the feeler can be improved by one in special cases.

Definition of furling.

Let $C, W^w \subset \mathring{M}$. If there exists X^x such that $W \searrow X$, $x < w$, and $X \cap C = W \cap C$, then we say W can be furled to X relative to C, or, more briefly, W can be furled. The term comes from sailing, where C is a ship, and the 2-dimensional sails W can be furled to the 1-dimensional masts X.

Corollary to Theorem 20.

Let M be k-connected, $k \leq m-3$, C collapsible in $\mathring{M}$, and W^w compact in $\mathring{M}$. If W can be furled to X, $x \leq k$, then we can engulf $W \subset D \searrow C$ in $\mathring{M}$, such that $\dim (D-C) \leq w$.

Notice that there is no restriction on the dimension of W, and that the feeler has the same dimension as W. To prove the corollary we need a lemma, which is a more delicate version of Lemma 37. We take the opportunity while proving this lemma, to prove a sharpened version, sharper than is needed here, which will be useful later for boundary problems. For this we need some definitions.

Interior, boundary and admissible collapsing.

Let $X \searrow Y$ in the manifold M. Write

$$X \overset{o}{\searrow} Y \quad \text{if} \quad X-Y \subset \overset{o}{M}$$
$$X \overset{\beta}{\searrow} Y \quad \text{if} \quad X-Y \subset \dot{M}$$
$$X \overset{\alpha}{\searrow} Y \quad \text{if} \quad X \overset{o}{\searrow} (X \cap \dot{M}) \cup Y \overset{\beta}{\searrow} Y.$$

The ambient manifold M is not included in the notation but is always understood. We call $\overset{o}{\searrow}$ an <u>interior collapse</u>, $\overset{\beta}{\searrow}$ a <u>boundary collapse</u> and $\overset{\alpha}{\searrow}$ an <u>admissible collapse</u>. At the end of this chapter we shall define <u>inwards collapsing</u> $\overset{\gamma}{\searrow}$ which is in a sense opposite to admissible. Admissible collapsing was introduced by Irwin, and inwards collapsing by Hirsch, both for engulfing spaces that meet the boundary.

Notice that all three relations are transitive. The transitivity of o, β is obvious, and that of α depends upon the fact that two elementary simplicial collapses $K_1 \overset{\beta}{\searrow} K_2 \overset{o}{\searrow} K_3$ can be interchanged, because the free face of the second remains free in K_1, since it lies in $\overset{o}{M}$. Therefore given $X \overset{\alpha}{\searrow} Y \overset{\alpha}{\searrow} Z$, triangulate and push all the interior collapses to the front, leaving all the boundary collapses at the end, $X \overset{\alpha}{\searrow} Z$.

<u>Example 1.</u> If N is a derived neighbourhood of X in M then $N \overset{\alpha}{\searrow} X$.

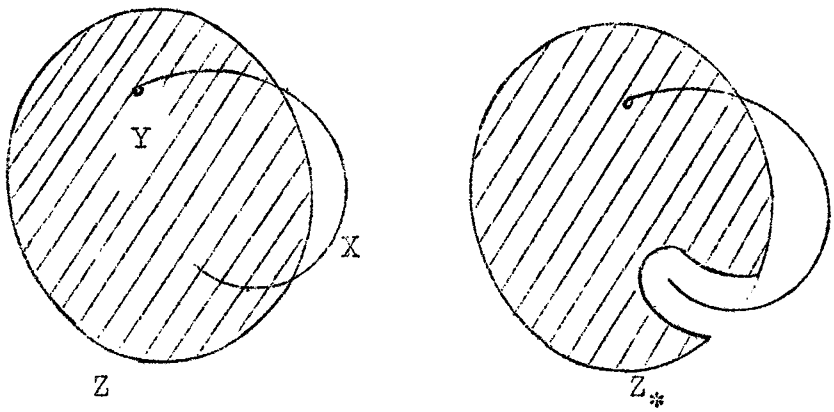
<u>Example 2.</u> If a ball B in M meets in $\dot{M}$ in a face, then B is admissibly collapsible, $B \overset{\alpha}{\searrow} 0$.

<u>Example 3.</u> If $X \overset{\alpha}{\searrow} 0$, then a derived neighbourhood of X in M is a ball meeting $\dot{M}$ in a face.

<u>Example 4.</u> A ball properly embedded in a manifold is not admissibly collapsible.

Lemma 42.

<u>If $X \overset{\alpha}{\searrow} Y \subset Z$ in M, then Z is ambient isotopic to Z_* keeping Y fixed such that $X \cup Z_* \overset{\alpha}{\searrow} Z_*$.</u>



Remarks.

1. The spaces are not necessarily compact.
2. $X-Z_* \subset X-Y$.
3. $\dim(X-Z_*) \leqslant \dim(X-Y)$, by 2.
4. The lemma is true if α is replaced by 0 or β, again by 2.

Proof.

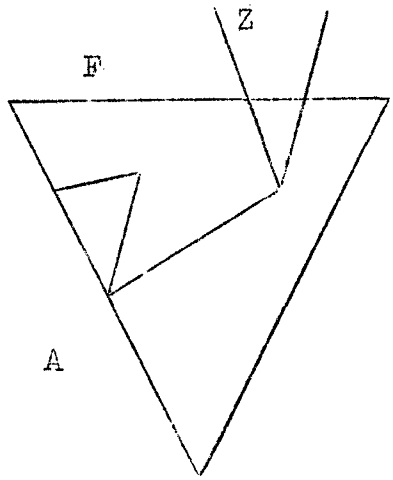
Triangulate a neighbourhood of $\overline{X-Y}$ in M, and by subdividing if necessary, triangulate the collapses

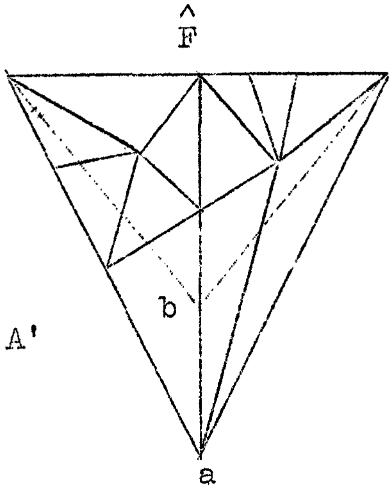
$$\overline{X-Y} \overset{\circ}{\searrow} (\overline{X-Y} \cap \dot{M}) \cup (\overline{X-Y} \cap Y) \overset{\beta}{\searrow} \overline{X-Y} \cap Y$$

As in the proof of Lemma 37, by induction on the number of elementary simplicial collapses, it suffices to consider the case when $X \searrow Y$ is an elementary simplicial collapse, across the simplex A from the face F, say. There are two cases according as to whether

$X \overset{\circ}{\searrow} Y$ or $X \overset{\beta}{\searrow} Y$. In the first case $A, F \not\subset \dot{M}$, and in the second case $A, F \subset \dot{M}$. (The purpose of the admissibility in the hypothesis was to exclude the third possibility $A \not\subset \dot{M}$, $F \subset \dot{M}$).

<u>First case: $A, F \not\subset \mathring{M}$.</u> As in Lemma 37 we do not assume Z to be a subcomplex of the triangulation, because Z is going to be isotoped around during the induction, until it reaches the position Z_*. Therefore we may expect $A \cap Z$ to be an arbitrary subpolyhedron of A.





Let $\hat{F}$ be the barycentre of F, and let A' be a subdivision of A containing $a\hat{F}$ and $A \cap Z$ as subcomplexes. Choose a point b in the interior of the segment $a\hat{F}$, and sufficiently close to the point a for there to be no vertices of A' in $ab\dot{F}$ other than those in $a\dot{F}$. We claim that

$$ab\dot{F} \overset{\circ}{\searrow} ab\dot{F} \cap Z.$$

To prove this, recall that $Z \supset Y \supset a\dot{F}$, and so the idea is to collapse $ab\dot{F}$ onto $a\dot{F}$, but leave sticking up those bits lying in Z. More precisely, let P be a simplex of A' meeting $b\dot{F}$ and not contained in Z. Let $P_1 = P \cap ab\dot{F}$, $P_2 = P \cap b\dot{F}$. Then P_1 is a convex linear cell with face P_2, and so we can collapse P_1 from P_2. Moreover this collapse is interior, because $\mathring{P}_1 \cup \mathring{P}_2 \subset \mathring{A} \subset \mathring{M}$. Perform collapses for all such P, in order of decreasing dimension, and this gives the required collapse. By excision

$$ab\dot{F} \cup Z \overset{\circ}{\searrow} Z.$$

Since $F \not\subset \dot{M}$, the link of F is a sphere, and so we can define the homeomorphism $g: M \to M$ described in the proof of Lemma 37, throwing $ab\dot{F}$ onto A. Let $Z_* = gZ$; then Z is ambient isotopic to Z_* keeping Y fixed. The image under g of the collapse $ab\dot{F} \cup Z \overset{\circ}{\searrow} Z$ is a collapse $A \cup Z_* \ {}^{\circ}\ Z_*$. But $A \cup Z_* = X \cup Z_*$ because $X = A \cup Y$ and $Z_* \supset Y$. Therefore we have the required collapse $X \cup Z_* \overset{\circ}{\searrow} Z_*$.

<u>Second case : $A, F \subset \dot{M}$.</u>

In this case the construction of $ab\dot{F}$ is as in the first case, but the definition of g is different because the link $L = \mathrm{lk}(F,M)$ is no longer a sphere but a ball. Let Δ be a simplex of dimension $1 + \dim L$, and let Γ be a top dimensional face of Δ, with barycentre $\hat{\Gamma}$. Let h be a homeomorphism given by mapping $\hat{F} \to \hat{\Gamma}$, $L \to \dot{\Delta} - \mathring{\Gamma}$, and joining linearly. Then $hb \in \mathring{\Gamma}$, because $A \subset \dot{M}$. There is a homeomorphism of Δ determined by mapping $hb \to \hat{\Gamma}$, keeping $\dot{\Delta} - \mathring{\Gamma}$ fixed, and joining linearly. As before this homeomorphism determines a homeomorphism of M, isotopic to the identity, throwing $ab\dot{F}$ onto A and keeping Y fixed. The rest of the proof is the same as the first case. The collapses this time are all boundary collapses.

The proof of Lemma 42 is complete. Notice that the reason for avoiding the third case $A \not\subset \dot{M}$, $F \subset \dot{M}$ is that otherwise the required isotopy would have had to push stuff off the boundary of M, which is impossible.

<u>Proof of Corollary to Theorem 20.</u>

We have to show that if W can be furled, then it can be engulfed with a feeler of the same dimension. Recall that furling means there exists X^x, $x < w$, such that $W \searrow X$ and $X \cap C = W \cap C$. This implies that $W \cup C \searrow X \cup C$.

Since $x \leqslant k$, we can engulf $X \subset E \searrow C$, such that $\dim(E - C) \leqslant x + 1 \leqslant w$, by Theorem 20. Apply Lemma 42 to the situation

$$W \cup C \overset{\circ}{\searrow} X \cup C \subset E$$

(the collapse being interior because all subspaces are interior to M), we can ambient isotop E to E_* keeping $X \cup C$ fixed, such that $(W \cup C) \cup E_* \searrow E_*$. Let $D = (W \cup C) \cup E_*$. Then $W \subset D \searrow E_*$, and $E_* \searrow C$ because the pair (E_*, C) is homeomorphic to (E, C). Therefore $W \subset D \searrow C$. Finally we have to check the dimension of the feeler : by the Remark after Lemma 42,

$$\begin{aligned} \dim(D - E_*) &\leqslant \dim[(W \cup C) - (X \cup C)] \\ &= \dim(W - X) \\ &\leqslant w. \end{aligned}$$

$$\dim(E_* - C) = \dim(E - C)$$
$$\leqslant x + 1$$
$$\leqslant w.$$

Therefore $\dim(D - C) \leqslant w$, and the proof of the Corollary is complete.

We have now completed the proofs of the engulfing theorems that we shall need in the ensuing chapters, apart from the piping lemma (Lemma 48). For the rest of this chapter we shall go on to prove the generalisation that allows for more complicated cores, and permits both C and X to meet the boundary of M. To state the generalisation we need two definitions.

<u>Definition of q-collapsibility</u>.

We describe the collapsibility condition on the core, that was mentioned in the Remark after Lemma 41. Let C be a closed subspace of M, not necessarily compact. Define C to be <u>q-collapsible in M</u> if there is a subspace Q such that $C \overset{\circ}{\searrow} Q$ and $\dim(Q \cap \mathring{M}) \leqslant q$.

<u>Example 1.</u> If $\dim C \leqslant q$ then C is q-collapsible.

<u>Example 2.</u> A collapsible subspace of $\mathring{M}$ is 0-collapsible.

<u>Example 3.</u> Any closed subspace of $\dot{M}$ is q-collapsible for all q.

<u>Example 4.</u> If C is compact and q-collapsible, then any regular neighbourhood of C, that meets $\dot{M}$ in a regular neighbourhood in $\dot{M}$ of $C \cap \dot{M}$, is also q-collapsible.

<u>Example 5.</u> An arc properly embedded in a 3-ball is <u>not</u> 0-collapsible.

<u>Definition of C-inessential</u>

Let C, X be subspaces of M. We call X <u>C-inessential in M</u> if the inclusion map $X \subset M$ is homotopic in M, keeping $X \cap C$ fixed, to a map $X \to C$.

Example 1. If C is a point the definition reduces to X inessential in M. Therefore the concept is a generalisation.

Example 2. If C is a k-core and $\dim X \leq k$ then X is C-inessential; for if X is triangulated so that $X \cap C$ is a subcomplex, then there is no obstruction to deforming into C each simplex of $X - C$, keeping its boundary fixed, in order of increasing dimension.

Example 3. If C is the northern hemisphere and X the southern hemisphere of S^n, then X is not C-inessential.

Theorem 21. Let C be a q-collapsible k-core of the manifold M^m, $q \leq m-3$. Let X^x be compact and C-inessential, and suppose X satisfies (1) or (2):

(1) $\dim (X \cap \dot{M}) < x$ and
$$x \leq m-3$$
$$2x \leq m+k-2$$
$$q+x \leq m+k-2$$

(2) $X \subset \mathring{M}$ and
$$x \leq m-4$$
$$2x \leq m+k-3$$
$$q+x \leq m+k-2.$$

Then we can engulf X; that is to say there exists D such that
$$X \subset D \searrow C,$$
$$\dim (D-C) \leq x+1,$$
$$D \cap \dot{M} = (X \cup C) \cap \dot{M}.$$

Notice that the converse is trivial: if we can engulf X then X is C-inessential, because the collapse $D \searrow C$ gives a deformation retraction of D onto C, which homotops X into C, keeping $X \cap C$ fixed.

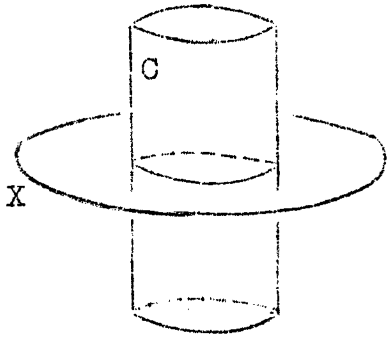
Before proving Theorem 21 we give some examples and corollaries.

Example 1. Let M be k-connected, let C be a point in $\mathring{M}$, and let X be inessential in $\mathring{M}$. Then we can engulf X in a collapsible set D. A regular neighbourhood of D is a ball, and so we deduce Theorem 19.

Example 2. Let M be x-connected. If C is a collapsible set in $\mathring{M}$, then C is an x-core. If $X^x \subset \mathring{M}$, there is no obstruction to deforming X into C, keeping $X \cap C$ fixed, and so X is C-inessential. Therefore we can engulf X, and so deduce Theorem 20.

Example 3. Consider Irwin's example, which was described above in Example 3 after Theorem 19. We have $M = S^1 \times S^{2n}$, and $X = S^n$ embedded in M by being self-linked around the S^1. Choose $C = S^1 \times$point in $S^1 \times S^{2n}$. Then C is a 1-collapsible (2n-1)-core. Theorem 21 tells us that X can be engulfed; by Lemma 39 this is equivalent to saying X is contained in a regular neighbourhood of S^1. Of course this can be easily seen by elementary methods - but the purpose was to illustrate how Theorem 21 can be applied in situations where Theorem 19 fails.

Example 4. We give an example to show that the hypothesis $q+x \leqslant m+k-2$ in Theorem 21 is the best possible. Let $M = E^{2n-1} = E^n \times E^{n-1}$, $n \geqslant 3$, $C = S^1 \times E^{n-1}$, $X = S^{n-1} \times 0$, where S^{n-1} contains S^1 in its interior.



Then C is an n-collapsible 1-core of M (it is not a 2-core because $\pi_2(M,C) \cong \pi_1(C) = Z$).

Putting $x = n - 1$, $q = n$, $m = 2n + 1$, $k = 1$ we see that $q + x \not\leq m + k - 2$ by one, but all the other hypotheses are satisfied. X is trivially C-inessential because it can be shrunk to a point of C. Suppose we could engulf X.

Let A be a regular neighbourhood of S^1 in E^n. Then $\mathring{A} \times E^{n-1}$ is an open set containing C, and so by Lemma 41 there is a homeomorphism h moving this open set over X. Let $N = h(A \times E^{n-1})$. Since $\pi_n(N) = 0$, X lies in a ball in N by Theorem 19. Span X with a disk D^n in this ball by Theorem 9. By Theorem 15, isotope D^n in N into general position with respect to $C \cap N$ keeping $X = \partial D^n$ fixed (which is possible since X does not meet C). Then $C \cap D^n$ is 1-dimensional, oriented by orientations of C, X, and therefore possesses a fundamental class in $H_1(C \cap D^n)$. Let $\xi \in H_1(C)$ be the image of this class under the inclusion homomorphism $C \cap D^n \subset C$. Then ξ is independent of D^n because it is the linking class of C, X in E^{2n-1}. We verify that $\xi \neq 0$ by spanning X with the unique disk in $E^n \times 0$. But $\xi = 0$ from the commutative diagram of inclusion homomorphisms

$$\begin{array}{ccc} H_1(C \cap D^n) & \longrightarrow & H_1(D^n) = 0 \\ \downarrow & & \downarrow \\ H_1(C) & \xrightarrow{\cong} & H_1(N) = Z. \end{array}$$

The contradiction shows that X cannot be engulfed.

Example 5. We give an example to show that the hypothesis $2x \leqslant m + k - 3$ is the best possible in the case that $X \subset \dot{M}$. Let $M = S^1 \times B^{2n+1}$, $n \geqslant 2$, and in the boundary $\dot{M} = S^1 \times S^{2n}$ let $X = S^n$ be as in Irwin's example. Let C be a point, which is a 0-collapsible 0-core. Then all the hypotheses are satisfied, except that

$2x \not\leq m+k-3$ by one dimension. We cannot engulf X, for if we were able to, then X would be contained in ball by Lemma 39, and so would span a disk in M. This disk would lift to a countable set of disjoint disks in the universal cover $R \times B^{2n+1}$ of M. The boundaries of these disks would therefore be homologically unlinked in $R \times S^{2n}$, but by construction we know that adjacent boundary spheres are linked. The contradiction shows that X cannot be engulfed, and that the hypothesis $2x \leq m+k-3$ is best possible.

<u>Example 6.</u> I do not know whether $x \leq m-4$ is best possible in the case that $X \subset \dot{M}$. It would be the best possible if the following conjecture were true.

<u>Conjecture.</u> <u>Let M^4 be a compact contractible manifold, and let S^1 be essential in $\dot{M}^4$. Then S^1 does not bound a disk in M^4.</u> Observe that S^1 does bound a <u>singular</u> disk because M^4 is contractible. A good candidate for this conjecture is Mazur's manifold M^4 (see Chapter 3, page 10), and for the curve choose $S^1 \times \text{point} \subset (S^1 \times B^3)^{\cdot}$, drawn so as to avoid the attached 2-handle. This curve bounds a disk with one self intersection, which seems impossible to remove.

Before proving Theorem 21, which will require several lemmas, we state and prove two corollaries.

<u>Corollary 1 to Theorem 21.</u>

<u>Let C, X be as in Theorem 21. If W can be admissibly furled to X, then we can engulf W with a feeler of the same dimension.</u>
The proof is the same as that of the Corollary to Theorem 20, with the

proviso that it is necessary to verify admissibility at each stage.

Corollary 2 to Theorem 21 (Irwin).

Let M^m be a k-connected manifold, $k \leqslant m-3$, and let Q^{m-1} be an h-connected submanifold of $\dot{M}$, $h \leqslant m-4$. Let $C \overset{\alpha}{\searrow} 0$ in M, and $C \cap \dot{M} \subset \mathring{Q}$. Let X be compact in M, $x \leqslant k$, and let $X \cap \dot{M} \subset \mathring{Q}$ with $\dim(X \cap \dot{M}) \leqslant h$. Then we can engulf $X \subset D \overset{\alpha}{\searrow} C$, such that $\dim(D-C) \leqslant x+1$ and $D \cap \dot{M} = (X \cup C) \cap \dot{M}$. Consequently X is contained in a ball that meets $\dot{M}$ in a face in $\mathring{Q}$.

Proof. Apply Theorem 21 to $X \cap \dot{M}$, $C \cap \dot{M}$ in $\mathring{Q}$ and engulf $X \cap \dot{M} \subset E \searrow C \cap \dot{M}$, where $\dim(E-C) \leqslant \dim(X \cap \dot{M}) +1$ and $E \subset \mathring{Q}$. Then $E \cup C \overset{\circ}{\searrow} E \overset{\beta}{\searrow} 0$. Now apply Theorem 21 to X, $E \cup C$ in M to engulf $X \subset D \searrow E \cup C$, where $\dim(D-(E \cup C)) \leqslant x+1$ and $D \cap \dot{M} = (X \cup E \cup C) \cap \dot{M} = E$. Because of the last remark, the collapse $D \searrow E \cup C$ is interior. Therefore $D \overset{\alpha}{\searrow} C$.

Therefore $D \overset{\alpha}{\searrow} 0$, and so a derived neighbourhood of D is a ball meeting $\dot{M}$ in a face in $\mathring{Q}$.

We now proceed to the proof of Theorem 21. The last statement of the thesis is taken care of by the following lemma.

Lemma 43. Given X, C if we can engulf $X \subset D \searrow C$, then we can choose D so that $D \cap \dot{M} = (X \cup C) \cap \dot{M}$.

Proof. The idea is to isotop D inwards a little, keeping $X \cup C$ fixed. Let $Y = X \cup C$. We can assume M compact by confining attention to a regular neighbourhood of $\overline{D-Y}$. Let $c : \dot{M} \times I \to M$, $c(x,0) = x$, $x \in \dot{M}$ be a collar given by Lemma 24 Corollary. Choose a cylindrical triangulation of $\dot{M} \times I$ (that is one such that the projection onto $\dot{M}$

is simplicial) such that $c^{-1}Y$ is a subcomplex. Define an isotopy of $\dot{M} \times I$ in itself as follows. For each vertex $v \in \dot{M} \cap Y$ keep $v \times I$ fixed. For each vertex $v \in \dot{M} - Y$ isotop $v \times 0$ along $v \times I$ and stop before hitting $(v \times 1) \cup c^{-1}Y$; extend to an isotopy of $v \times I$ in itself keeping $v \times 1$ and the intersection with $c^{-1}Y$ fixed. Now extend the isotopy cylinderwise to each prism $A \times I$, $A \in \dot{M}$ in order of increasing dimension, keeping $A \times 1$ and the intersection with $c^{-1}Y$ fixed. The image under c extends to an isotopy of M keeping Y fixed and moving $\dot{M} - Y$ into the interior. The restriction to D gives what we want.

<u>Definition of trails</u>

We introduce a notion due to Moe Hirsch, which will be useful in the proof of Theorem 21. Let s denote the simplicial collapse

$$K = K_0 \searrow K_1 \searrow \dots \searrow K_{n-1} \searrow K_n = L.$$

The symbol s includes the given ordering of elementary simplicial collapses. Let W be a subcomplex of K. We define the <u>trail of W under s</u> as follows, by induction on n. If $n = 0$, define $\text{trail}_s W = W$. Suppose inductively $\text{trail}_t W$ has been defined for the collapse t:

$$K_0 \searrow K_1 \searrow \dots \searrow K_{n-1},$$

and suppose $K_{n-1} \searrow K_n$ is across the simplex A from the face F. Define

$$\text{trail}_s W = \begin{cases} \text{trail}_t W, & F \notin \text{trail}_t W \\ A \cup \text{trail}_t W, & F \in \text{trail}_t W. \end{cases}$$

When there is no confusion we shall drop the suffix s.

Geometrically the trail is the track left by W during the deformation retraction $K \to L$ associated with the collapse $K \searrow L$. We leave the reader to verify the elementary properties:

(i) trail $K = K$, trail $L = L$.

(ii) trail $(W_1 \cup W_2)$ = trail $W_1 \cup$ trail W_2.

(iii) trail (trail W) = trail W.

Therefore "trail" is a closure operator on the set of all subcomplexes of K, and the trails form a sort of combinatorial fibering of the collapse.

Remark

Notice that the trail depends upon the triangulation and the order of the elementary collapses. If the same elementary collapses are re-ordered to give a different simplicial collapse $K \searrow L$ then the trails turn out to be the same, but if different elementary collapses are used to define a simplicial collapse $K \searrow L$ then the trails are different. Therefore the trail is not a piecewise linear invariant.

However trails can be related to piecewise linear invariants. For example admissibility is a piecewise linear invariant. If $X \searrow Y$ is an admissible collapse (in a manifold now) then we can triangulate so that the trail of anything in the boundary remains in the boundary. Therefore admissible collapsing is equivalent to "boundary preserving" in terms of trails. For Lemma 53 below we shall introduce another piecewise linear invariant called inwards collapsing, which is equivalent to "interior preserving" in terms of trails.

We now prove two important properties of trails.

Lemma 44. If $K \searrow L$ is a simplicial collapse and $W^w \subset K$, then

$$\dim (\text{trail } W) \leqslant w + 1$$

$$\dim (L \cap \text{trail } W) \leqslant w.$$

Proof Let s denote the collapse $K = K_0 \searrow K_1 \searrow \ldots \searrow K_n = L$. The proof is by induction on n, starting trivially with $n = 0$. Let t be the collapse $K_0 \searrow K_{n-1}$ of length $n - 1$, and let $T = \text{trail}_t W$. By induction $\dim T \leqslant w + 1$, and $\dim (K_{n-1} \cap T) \leqslant w$. Suppose $K_{n-1} \searrow K_n$ is across A from F. If $F \notin T$ then $\text{trail}_s W = T$, and $T \cap K_n = T \cap K_{n-1}$, and so both results hold for s. If $F \in T$ then $F \in K_{n-1} \cap T$, and so $\dim F \leqslant w$. Therefore $\dim A \leqslant w + 1$, and so $\dim (\text{trail}_s W) = \dim (A \cup T) \leqslant w + 1$. Also

$$\begin{aligned} L \cap \text{trail}_s W &= L \cap (A \cup T) \\ &\subset \dot{A} \cup (K_{n-1} \cap T), \end{aligned}$$

which is of dimension $\leqslant w$.

Lemma 45. If $K \searrow L$ is a simplicial collapse and $W \subset K$, then $K \searrow L \cup \text{trail } W \searrow L$.

Proof For each elementary collapse, across A from F, say, either both A, F are in the trail, or neither are. Perform, in order, all those elementary collapses for which neither A, F are in the trail, giving $K \searrow L \cup \text{trail } W$. These elementary collapses are valid, because if A principal in K_i, and $A \notin$ trail W, then A principal in $K_i \cup$ trail W; also if F a free face of A in K_i, and $F \notin$ trail W, then F is a free face of A in $K_i \cup$ trail W. Now perform, in order, the rest of the elementary collapses, giving $L \cup$ trail $W \searrow L$.

Corollary to Lemma 45. If $K \searrow L$ is a simplicial collapse and $W_2 \subset W_1 \subset K$, then $L \cup$ trail $W_1 \searrow L \cup$ trail W_2.

Proof Let s denote the collapse $K \searrow L$. Let t denote the induced collapse $L \cup \text{trail}_s W_1 \searrow L$ given by the lemma. Then $\text{trail}_t W_2 = \text{trail}_s W_2$ because $W_2 \subset W_1$. Now apply the lemma to t to obtain

$$L \cup \text{trail}_s W_1 \searrow L \cup \text{trail}_t W_2.$$

Substituting s for t in the right hand side gives what we want.

The relative mapping cylinder

Corresponding to the generalisation in the theorems from inessentiality to C-inessentiality, it is necessary in the proofs to generalise from the cone on X to the mapping cylinder of $X \to C$. It was for a similar purpose that Whitehead introduced the mapping cylinder in 1939. We need a relative version of the mapping cylinder here because $X \cap C$ is kept fixed. As it was pointed out in Chapter 2, the mapping cylinder is a simplicial rather than a piecewise linear construction: it is a tool rather than an end product.

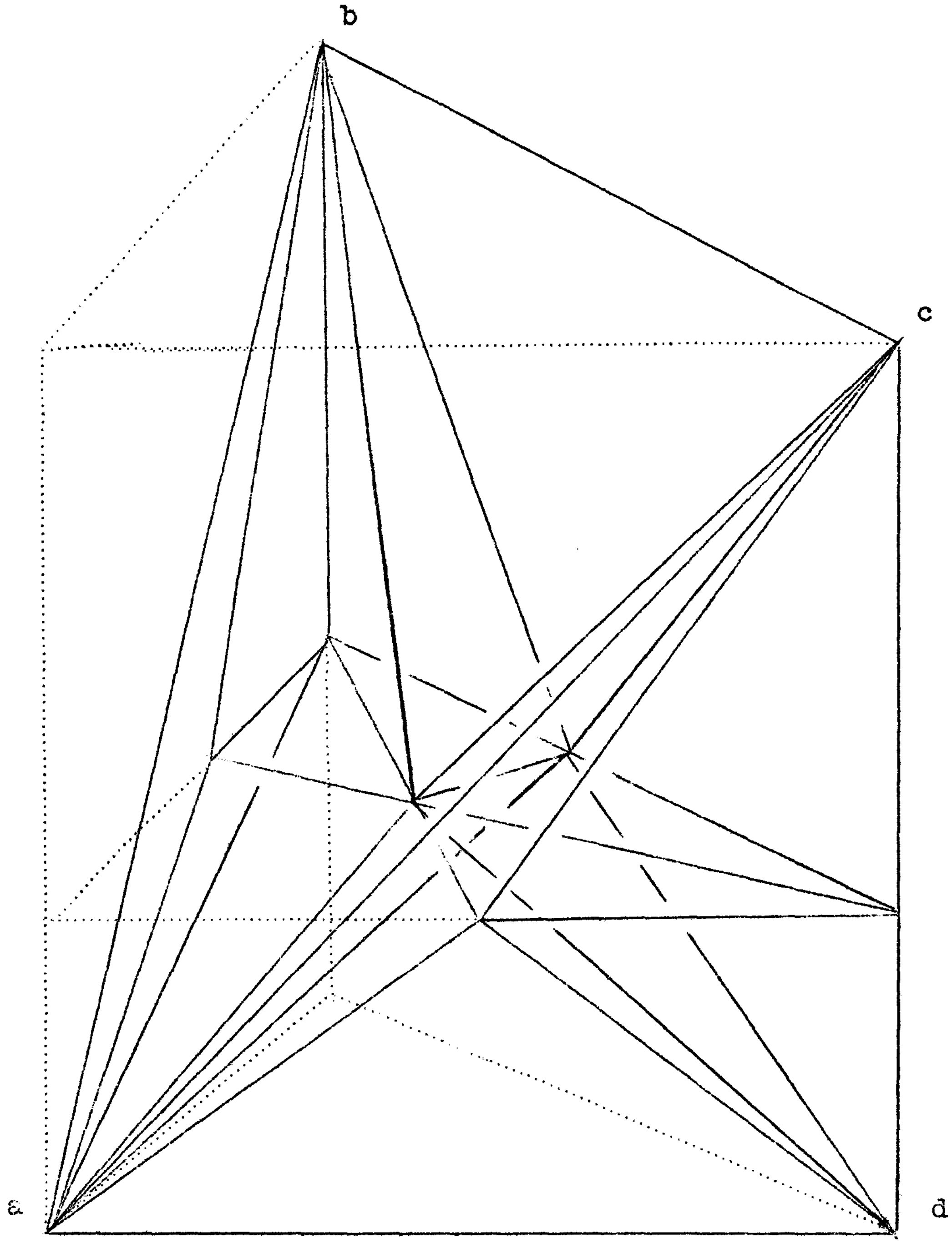
Let K, L be complexes meeting in the subcomplex $K \cap L$, and let $f:K \to L$ be a simplicial map such that $f|K \cap L = 1$. The <u>relative mapping cylinder</u> of f is a complex $\mu K \cup L$ defined as follows. For each simplex $A \in K - L$ choose a new vertex, a say. (Since our complexes always lie in some Euclidean space, choose one of sufficiently high dimension so that the new vertices are linearly independent of $K \cup L$ and each other). If $A \in K \cap L$ define $\mu A = \emptyset$. If $A \in K - L$, define

$$\mu A = a(\mu\dot{A} \cup A \cup fA)$$

inductively in order of increasing dimension, where $\mu\dot{A} = \bigcup\{\mu B;\ B \in \dot{A}\}$. Define

$$\mu K = \bigcup\{\mu A;\ A \in K\}.$$

The relative mapping cylinder is $\mu K \cup L$. In particular it contains $K \cup L$ as a subcomplex.



Let K be the 2-simplex, abc, and L be the 1-simplex, ad. The diagram shows the relative mapping cylinder of the simplicial map $f:K \to L$ given by $fa = fb = a$, $fc = d$.

<u>Lemma 46.</u> (<u>Newman</u>) <u>If $A^n \in K - L$ then μA is an $(n + 1)$-ball with face A.</u>

<u>Proof</u> For this proof only we shall use chains modulo 2 in the complex μK. The symbol μA will stand ambiguously for a subcomplex and an $(n + 1)$-chain. To emphasise the chain point of view we replace the statements $\mu A = \emptyset$, $f|A$ is degenerate by the formulae $\mu A = 0$, $fA = 0$, respectively. We replace $\cup$ by $+$, and write ∂ for boundary. Therefore $\mu A = a(\mu\partial A + A + fA)$. We deduce

$$\partial\mu = \mu\partial + 1 + f$$

by verifying inductively on simplexes, and extending additively to chains. If B is a ball, then ∂B is the chain of the complex $\dot{B}$, and so by the ambiguity of our notation, $\partial B = \dot{B}$. (Of course for a general chain C, ∂C is defined but $\dot{C}$ is not).

The proof is by a double induction. Let 1, 2 denote the following statements:

1(n): if $A^n \in K - L$ then μA is an $(n + 1)$-ball.

2(n): if $AB \in K$, $A \in K - L$, $\dim AB = n$, $0 \leqslant \dim A < n$, then $\mu(A\partial B)$ is an n-ball.

Notice that the lemma follows from 1(n) because A occurs with non-zero coefficient in $\partial\mu A$. Both 1, 2 are trivial when $n = 0$, and A is a vertex ($A \neq fA$ because $A \notin L$). We shall assume 1, 2 for dimensions $< n$, and prove first 2(n) and then 1(n).

The proof of 2(n) is by induction on dim B. Let 2(n, q) denote the statement of 2(n) when $\dim B = q$. The q-induction begins with 2(n, 0) trivially implied by 1(n - 1). We shall prove

$2(n, q - 1) \Rightarrow 2(n, q)$, $0 < q < n$.

Given $A^{n-q-1}B^q \in K$, write $B = xC^{q-1}$. Then

$$\begin{aligned}\mu A\partial B &= \mu A\partial(xC)\\ &= \mu AC + \mu Ax\partial C\\ &= X + Y, \text{ say}\end{aligned}$$

where X, Y are n-balls by $2(n, 0)$, $2(n, q - 1)$, respectively. Then $X + Y$ is an n-ball provided that X, Y meet in a common face. Now $X \cap Y = \partial X \cap Y$, because $X = a\partial X$, $a \notin Y$

$$\begin{aligned}&= (\mu(\partial A)C + \mu A\partial C + AC + fAC) \cap \mu Ax\partial C\\ &= \mu A\partial C + (fAC \cap fAx\partial C)\\ &\subset \partial Y.\end{aligned}$$

Therefore $X \cap Y = \partial X \cap \partial Y$. There are four cases.

(i) $fAC = 0$.

(ii) $fAxC \neq 0$. Therefore $fAC \cap fAx\partial C = fA\partial C$.

(iii) $fAC \neq 0$, $fx \in fA$. Therefore $fAx\partial C = 0$.

(iv) $fAC \neq 0$, $fx \in fC$. Therefore $fAx\partial C = fAC$.

In each of the first three cases $X \cap Y = \mu A\partial C$, which is an $(n - 1)$ ball by $2(n - 1)$. In the last case $X \cap Y = \mu A\partial C + fAC$, which is the union of two $(n - 1)$ balls meeting in the common face $fA\partial C$ (because $fAC \neq 0$), and consequently $X \cap Y$ is a ball. Therefore X, Y meet in a common face, so that $X + Y$ is an n-ball, and $2(n, q)$ is proved.

We now have to prove $2(n) \Rightarrow 1(n)$. Given $A^n \in K - L$, write $A = xB^{n-1}$, $x \in K - L$. Again there are four cases.

(i) $B \in K$.

(ii) $B \notin K$, $fA \neq 0$.

(iii) $B \notin K$, $fA = 0$, $fB \neq 0$.

(iv) $B \notin K$, $fA = fB = 0$.

In the first case we prove, by a separate induction on n, that $\mu A \cong (xyB)'$, where $y = fx$, and the dash means take a first derived complex modulo $xB + yB$. For if this is true for dimensions $< n$, then

$$\begin{aligned} \mu A &= \mu xB \\ &= a(\mu x\partial B + xB + yB), \text{ since } \mu\partial B = 0 \\ &\cong a((xy\partial B)' + xB + yB), \text{ by induction} \\ &= a(\partial(xyB))' \\ &\cong (xyB)'. \end{aligned}$$

In case (ii) μB is an n-ball by 1(n - 1). Therefore $\mu B + fA$ is an n-ball, because $\mu B \cap fA = fB$, which is a common face. Also $\mu x\partial B + (\mu B + fA)$ is an n-ball, because $\mu x\partial B$ is an n-ball by 2(n, n-1) and

$$\begin{aligned} \mu x\partial B \cap (\mu B + fA) &= \mu\partial B + fx\partial B \\ &\cong \mu\partial B + fB, \text{ by a homeomorphism,} \\ &= \partial\mu B + B, \end{aligned}$$

which is (n - 1) ball, because it is the complementary face to B of the n-ball μB, by 1(n - 1).

We have shown that

$$\mu\partial A + fA = \mu x\partial B + (\mu B + fA)$$

is an n-ball. But $\partial(\mu\partial A + fA) = \partial A$. Therefore $\mu\partial A + A + fA$ is an n-sphere, and joining to the point a gives μA an $(n + 1)$-ball.

Case (iii) is simpler than case (ii) because $fA = 0$. Therefore $\mu\partial A = \mu B + \mu x\partial B$, which is an n-ball because μB, $\mu x\partial B$ are n-balls meeting in the common face

$$\mu B \cap \mu x\partial B = \mu\partial B + fB, \quad \text{since } fx\partial B = fB,$$
$$= (n - 1)\text{-ball as above.}$$

Then $\mu\partial A + A$ is an n-sphere, and μA an $(n + 1)$-ball, as before.

Case (iv) is yet simpler because this time

$$\mu B \cap \mu x\partial B = \mu\partial B,$$
$$= \partial\mu B + B, \text{ since } fB = 0,$$
$$= (n - 1)\text{-ball as before.}$$

The proof of 1(n), and Lemma 46 is complete.

<u>Corollary to Lemma 46.</u> <u>$\mu K \cup L \searrow L$.</u>

<u>Proof</u> Collapse across μA from A, for each simplex $A \in K - L$, in order of decreasing dimension.

<u>Lemma 47.</u> <u>Topologically the relative mapping cylinder $\mu K \cup L$ of $f:K \to L$ can be obtained from $K \cup K \times I \cup L$ by identifying</u>

$$\underline{x = x \times 0, \qquad x \in K}$$
$$\underline{fx = x \times 1, \qquad x \in K}$$
$$\underline{x = fx = x \times t, \qquad x \in K \cap L,\ t \in I.}$$

<u>Proof</u> We construct a continuous map

$$\varphi:K \cup K \times I \cup L \longrightarrow \mu K \cup L$$

onto the mapping cylinder, that realises the identifications. We emphasise that the proof (of this lemma only) is topological and not piecewise linear. Define $\varphi | K \cup L$ to be the inclusion, and for $A \in K - L$ construct $\varphi | A \times I : A \times I \to \mu A$ by induction in order of increasing dimension, as follows.

For the inductive step, let $S^n = (A \times I)^{\bullet}$. We shall show that there is a pseudo-isotopy $h_t : S^n \to S^n$ such that $h_0 = 1$, h_t is a homeomorphism for $0 \leqslant t < 1$, and h_1 realises the identification. Assume for the moment that this pseudo-isotopy exists. By induction $\varphi | S^n$ has already been constructed so as to realise the identification, and so this map can be factored

$$\begin{tikzcd} S^n \arrow[rr, "\varphi"] \arrow[dr, "h_1"'] & & (\mu A)^{\bullet} \\ & S^n \arrow[ur, "\psi"'] & \end{tikzcd}$$

where ψ is a homeomorphism. Using Lemma 46 that μA is a ball, the pseudo-isotopy enables φ to be extended to a map of a collar of $A \times I$ onto a collar of μA. By filling in the complementary balls, we obtain a map of $A \times I$ onto μA, such that the interiors are mapped homeomorphically. This is the required map $\varphi | A \times I$, because under the identification no point of $(A \times I)^{\circ}$ is identified with any other point.

We now construct the pseudo-isotopy. Let B be the simplex spanning the vertices of $A \cap L$ (possibly $B = \emptyset$). Then $fb = b$ for each vertex of B, and so

$$fB = B = A \cap L \ (\neq A, \text{ because } A \notin L).$$

We first construct the pseudo-isotopy on $B \times I$ by defining

$$h_t : B \times I \to B \times [t, 1]$$

to be given by mapping the segment $x \times I$ linearly onto $x \times [t, 1]$, each $x \in B$. Therefore h_t keeps $B \times 1$ fixed, $h_0 = 1$, h_t is a homeomorphism for $0 \leqslant t < 1$, and h_1 is the required identification $B \times I \to B \times 1$. Next we construct the pseudo-isotopy on $A \times 1$, as follows.

Lift $A \to fA$; that is to say choose a simplicial embedding $g : fA \to A$ such that $fg = 1$, and $B \subset gfA$. Given a vertex $a \in A$, and $t \in I$, define $a_t = (1 - t)a + t(gfa)$. Let A_t be the simplex with vertices $\{a_t ; a \in A\}$, and define the simplicial map

$$h_t : A \times 1 \to A_t \times 1.$$

Therefore h_t keeps $gfA \times 1$ fixed, $h_0 = 1$, h_t is a homeomorphism for $0 \leqslant t < 1$, and h_1 is the required identification $A \times 1 \to gfA \times 1$. Moreover h_t is compatible with pseudo-isotopy already defined on $B \times I$, because both keep fixed the intersection $B \times I \cap A \times 1 = B \times 1$. We can show by elementary geometry, that given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that given $0 \leqslant s < t < 1$ such that $t - s < \delta$, then the isotopy from h_s to h_t of $B \times I \cup A \times 1$ can be extended to an ε-isotopy of S^n, keeping fixed outside any chosen neighbourhood of $h_s(B \times I \cup A \times 1)$.

Choose strictly monotonic sequences $\varepsilon_i \to 0$ and $t_i \to 1$ such that $t_{i+1} - t_i < \delta(\varepsilon_i)$. Choose a sequence of neighbourhoods V_i such that $\bigcap V_i = B \times I \cup A \times 1$. Suppose inductively that we have chosen the isotopy h_t of S^n for $0 \leqslant t \leqslant t_i$. Now extend the isotopy h_t of $B \times I \cup A \times 1$ for $t_i \leqslant t \leqslant t_{i+1}$ to an ε_i-isotopy of S^n keeping fixed outside $h_{t_i}(V_i)$. Therefore $\{h_{t_i}\}$ is a Cauchy sequence of homeomorphisms of S^n, and consequently the limit map h_1 exists, making $\{h_t;\ 0 \leqslant t \leqslant 1\}$ a pseudo-isotopy. Any point of S^n outside $B \times I \cup A \times 1$ has a neighbourhood outside V_i, for some i, which is kept fixed for $t_i \leqslant t \leqslant 1$. Therefore the point is not identified with any other point under h_1. By construction h_1 makes the required identification on $B \times I \cup A \times 1$, therefore on S^n. Therefore the construction of the pseudo-isotopy, and the proof of Lemma 47, are complete.

<u>Definition of cylinderlike</u>

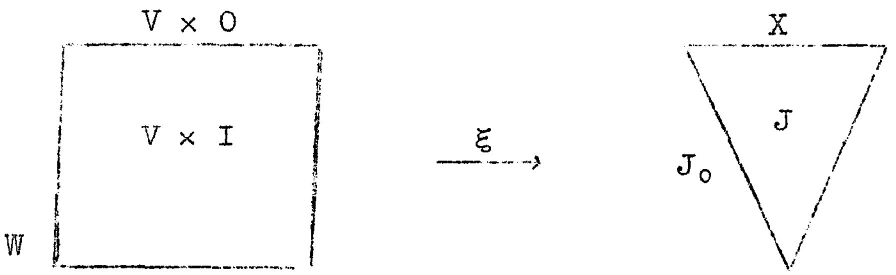
Let V be a manifold, and $V \times I$ the cylinder on V. Let $W = (\dot{V} \times I) \cup (V \times 1)$, the walls and base of the cylinder (perversely we regard $V \times 0$ as the top and $V \times 1$ as the base of the cylinder).

We call a triad $X^x, J_0^x \subset J^{x+1}$ of compact spaces <u>cylinderlike</u> if there exists a manifold V and a map $\xi : V \times I \to J$ such that

$$\xi(V \times 0) \subset X$$

$$\xi W \subset J_0$$

and ξ maps $(V \times I)-W$ homeomorphically onto $J - J_0$.



Example 1.

Let X^x be a complex, J^{x+1} the cone on X, and J_o^x the subcone on the (x - 1)-skeleton of X. Then X, $J_o \subset J$ is cylinderlike. For let V be the disjoint union of a set of x-simplexes $\{A_i\}$ in 1-1 correspondence with the x-simplexes $\{B_i\}$ of X. Define ξ by mapping $A_i \times 0$ isomorphically onto B_i, and extending to a homeomorphism of $A_i \times I$ onto the subcone on B_i, for each i. We have already used this example in the proof of Lemma 39 above.

Example 2.

Let $g:X^x \to C^c$ be a simplicial map between two complexes, and suppose $c \leqslant x$. Let J^{x+1} be the relative mapping cylinder of g. Let g_o be the restriction of g to the (x - 1)-skeleton of X, and let J_o^x be the submapping cylinder of g_o. Then X, $J_o \subset J$ is cylinderlike. For choose V to be the disjoint union of a set of x-simplexes $\{A_i\}$ in 1-1 correspondence with the x-simplexes $\{B_i\}$ of X - C. Define ξ by mapping the pair $A_i \times I$, $A_i \times 0$ homeomorphically onto μB_i, B_i (by Lemma 46) for each i.

In the special case that C is a point, not in X, example 2 reduces to example 1.

Lemma 48 (The piping lemma)

<u>Let M^m be a manifold, and let X^x, $J_o^x \subset J^{x+1}$ be cylinderlike, $x \leqslant m - 3$. Let $f:J \to M$ be a map in general position for the pair J, $X \cup J_o$ and such that $f(J - J_o) \subset \mathring{M}$. Then there exists a map $f_1:J \to M$, homotopic to f keeping $X \cup J_o$ fixed, and a subspace $J_1 \subset J$,</u>

<u>such that</u>

$$f_1(J - J_0) \subset \mathring{M}$$

$$S(f_1) \subset J_1$$

$$\dim J_1 \leqslant 2x - m + 2$$

$$\dim (J_0 \cap J_1) \leqslant 2x - m + 1$$

$$J \searrow J_0 \cup J_1 \searrow J_0$$

<u>Remark</u>

The meat of the lemma is the combination of the collapsing condition $J \searrow J_0 \cup J_1$ together with the dimension $\dim J_1 \leqslant 2x - m + 2$. In order to achieve this the homotopy $f \to f_1$ has to be global, rather than local like the homotopies of simplicial approximation and general position.

Before proving the lemma we introduce notation. The proof will then follow, and involve three sublemmas.

<u>Cylindrical triangulations.</u>

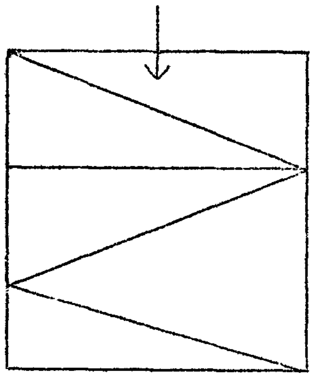
Let V be compact. We call a triangulation of $V \times I$ <u>cylindrical</u> if the subcylinder through each simplex is a subcomplex. Given any triangulation, we can find a cylindrical subdivision by Theorem 1, by merely making the projection $\pi : V \times I \to V$ simplicial. Given a cylindrical triangulation we can choose a cylindrical derived complex by Lemma 5.

In a cylindrical triangulation there are two types of simplex: call A <u>horizontal</u> if $\pi|A : A \to \pi A$ is a homeomorphism, and call A <u>vertical</u> if $\pi A = \pi \dot{A}$. In an arbitrary triangulation of $V \times I$

it is possible to have simplexes that are neither horizontal nor vertical, but in a cylindrical triangulation, or any subdivision thereof, every simplex is either horizontal or vertical.

Cylinderwise collapsing.

Let K be a cylindrical triangulation of $V \times I$. For each simplex $A^a \in \pi K$, the subcylinder $A \times I$ consists of horizontal a-simplexes and vertical $(a+1)$-simplexes, arranged alternately. Removing the interiors of the top horizontal and the top vertical simplex is an elementary simplicial collapse. Proceeding in this way we obtain a collapse $A \times I \searrow (\dot{A} \times I) \cup (A \times 1)$.



Do this for all simplexes $A \in \pi K$, in some order of decreasing dimension, and we have a simplicial collapse $V \times I \searrow V \times 1$. We call this collapsing *cylinderwise.*

Let P be a subcomplex of a cylindrical triangulation of $V \times I$. Call P *solid* if P contains every simplex beneath a simplex of P. Equivalently P is solid if P = trail P under a cylinderwise collapse.

Example 1. A subcylinder is solid.

Example 2. The intersection and union of solids are solid.

Example 3. If V is a manifold and W the walls and base of $V \times I$, then W is solid.

Corollary 2 to Lemma 45. *If P, Q are solid subcomplexes such that $P \supset Q \supset V \times 1$, then $P \searrow Q$ cylinderwise.*

The result follows immediately from Corollary 1 because $P = \text{trail } P$, $Q = \text{trail } Q$ under a cylinderwise collapse.

Proof of Lemma 48

Without loss of generality we can assume M compact, for otherwise replace M by a regular neighbourhood of fJ in M. Let $z = 2x - m + 2$. Let $\xi : V \times I \to J$ be the map given by the cylinderlike hypothesis, and let $\varphi = f\xi$.

$$\begin{tikzcd} V \times I \arrow[rr, "\varphi"] \arrow[dr, "\xi"'] & & M \\ & J \arrow[ur, "f"'] & \end{tikzcd}$$

Let $Z = \xi^{-1}S(f)$. Then Z has the properties

(1) $\dim Z \leqslant z$.

(2) $\dim [Z \cap (V \times I)^{\cdot}] \leqslant z - 1$.

These properties follow from three facts: firstly f is in general position, implying $\dim S(f) \leqslant z$, and $\dim (S(f) \cap (X \cup J_0)) \leqslant z - 1$; secondly ξ is non-degenerate because $\xi|(V \times I)^{\circ}$ is a homeomorphism; and thirdly $\xi^{-1}(X \cup J_0) = (V \times I)^{\cdot}$.

If we could now find a subspace $Q \supset Z$ such that

$$\dim(Q \cap W) < \dim Q \leqslant z$$

$$V \times I \searrow W \cup Q \searrow W$$

then the proof would be finished by defining $f_1 = f$ and $J_1 = \xi Q$. In particular the collapses $J \searrow J_0 \cup J_1 \searrow J_0$ would follow from

Lemma 38 because $W \supset S(\xi)$. However in general no such Q exists. What we have to do is to homotop f to f_1, and replace Z by $Z_1 = \xi^{-1}S(f_1)$, so that there does exist a Q containing Z_1 and with the above properties.

Digression. We digress for a moment to explain the obstruction to finding a Q containing Z, and to describe the intuitive idea behind the proof. Let $\pi: V \times I \to V$ be the projection. Then $\pi Z \times I$ is the subcylinder through Z and we can collapse cylinderwise

$$V \times I \searrow W \cup (\pi Z \times I).$$

It is no good putting $Q = \pi Z \times I$ because this is one dimension too high. And the trouble is that if we start collapsing $\pi Z \times I$ cylinderwise to try and reduce the dimension by one, then the horizontal z-simplexes of Z form an obstruction to collapsing away the $(z + 1)$-dimensional stuff underneath them. Therefore the idea is to punch holes in these simplexes in order to release the stuff underneath. Now the only way to "punch holes" in the singular set of a map (which is essentially what Z is) is to alter the map. Roughly speaking we alter f to f_1 and Z to Z_1 so that Z_1 equals Z minus the punch-holes. More precisely we shall describe a homotopy from φ to φ_1 keeping $(V \times I)^{\bullet}$ fixed, which will determine a homotopy from f to f_1 keeping $X \cup J_0$ fixed, because $\xi|(V \times I)^{\circ}$ is a homeomorphism.

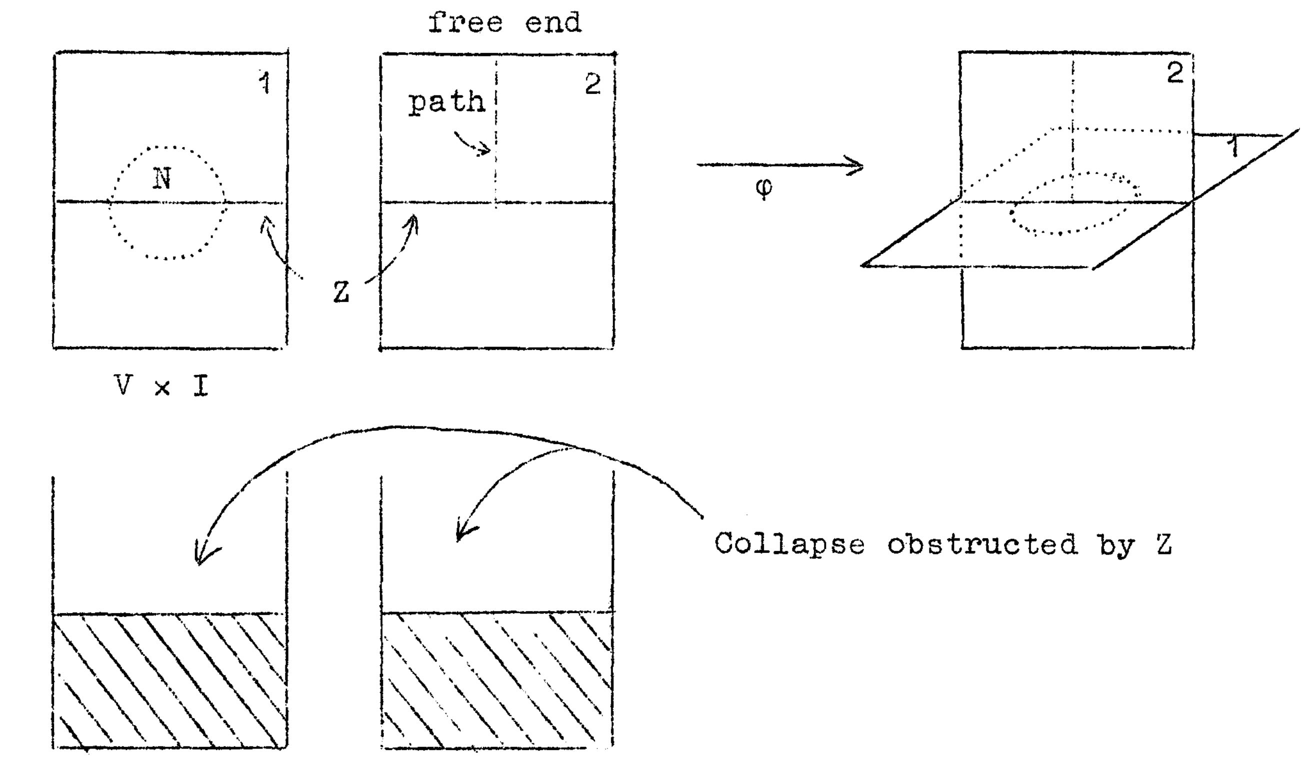
The way to punch holes in a simplex $A^z \in Z$ is as follows.

Since A is a top-dimensional simplex of a singular set, it arises from where two sheets of $\varphi(V \times I)$ cross one another. Choose an interior point of A, and a neighbourhood N of this point in the sheet containing A. Then pipe N over the free end of the other sheet. The "free end" means $\varphi(V \times 0)$. The "piping" is done by dividing N into a central disk N' surrounded by an annulus N", and replacing φN by

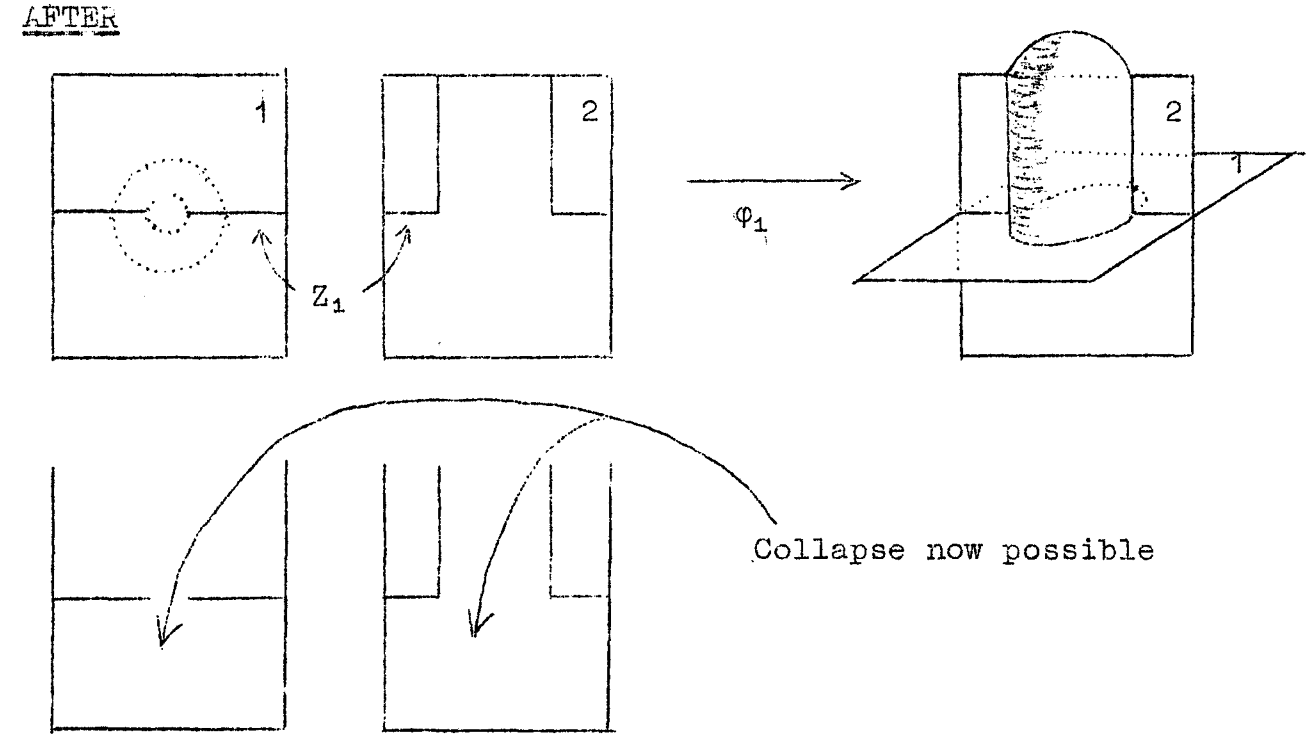
$$\varphi_1 N = \varphi_1 N' \cup \varphi_1 N''.$$

Where $\varphi_1 N''$ is a long pipe running along a path in the second sheet to the free end, and $\varphi_1 N'$ is a cap on the end of this pipe, whose interior does not meet the rest of $\varphi(V \times I)$. The reason that we had to have $\varphi(V \times I - W)$ in the interior of M was to make room for the pipe and the cap over the end. Define $\varphi_1 = \varphi$ on the rest of $V \times I$. The following pictures illustrate the idea when $z = 1$, and show how the piping enables us to perform the collapse that we want. Of course the pictures are inaccurate in that $x \not\leq m - 3$.

BEFORE



AFTER



Remark.

A technical difficulty in the proof is that in general it is impossible to find a cylindrical triangulation of $V \times I$ with respect to which the map $\varphi : V \times I \to M$ is simplicial. The reason is that we cannot make both φ, π simplicial, as is shown by Example 1 after Theorem 1. As the proof progresses we shall sometimes want φ simplicial, and other times π, and so it will be necessary to switch back and forth.

Continuing the proof of Lemma 48.

Let K, L triangulate the pair $V \times I$, Z. We shall construct successive subdivisions K_1, K_2, ... of K, and L_1, L_2, ... will denote the induced subdivisions of L. First let K_1 be a cylindrical subdivision of K. Next let K_2 be a subdivision of K_1 such that $\varphi : K_2 \to M$ is simplicial (for a suitable triangulation of M). Then L_2 has the properties:

(1) $\dim L_2 \leq z$.

(2) The z-simplexes of L_2 are interior to K_2.

(3) φ identifies the z-simplexes of L_2 in pairs, and identifies interiors of those simplexes with no other points.

The properties (1), (2) follow from the properties (1), (2) of Z above. Property (3) comes from the general position and Theorem 17.

Sublemma 1.

We can choose ε, K so that L_3 satisfies the further property:

(4) If A is a horizontal z-simplex of L_2 then

$$\pi A \cap \pi(Z - A) \subset \pi \dot{A}.$$

<u>Proof</u>. Although K_1 is cylindrical, L_1 does not in general satisfy (4) because two horizontal z-simplexes of L_1 may lie in the same subcylinder, and therefore have the same image under π. If they do, then we can move one of them, A say, sideways out of the way as follows. Let K' be a first derived of K_1 modulo the z-skeleton. Choose a point $v_A \in st(A, K')$ such that $\pi v_A \notin \pi A$, which is possible because $\dim A = z < x = \dim V$, since $x \leq m - 3$. (This is one of the places where codimension ≥ 3 is essential.) Since both v_A and A lie in a simplex of K', the linear join $v_A\dot{A}$ is well defined. By property (2) A lies in the interior of K', and so there is a homeomorphism ψ of $\overline{st}(A, K')$ throwing A onto $v_A\dot{A}$, and keeping the boundary fixed. Extend ψ to a homeomorphism

$$\psi : V \times I \to V \times I$$

fixed outside st(A, K'). Since ψ keeps $\overline{Z - A}$ fixed, and since $\pi v_A \notin$ (z-skeleton of $\pi K'$), we have

$$\pi\psi A \cap \pi\psi(Z - A) \subset \pi\psi\dot{A}.$$

Now choose a point v_A for each horizontal z-simplex $A \in L_1$, such that the images $\{\pi v_A\}$ are distinct. Since the stars $\{st(A, K')\}$ are disjoint, we can define the homeomorphism ψ so as to shift all the A's simultaneously. Let

$$\tilde{\xi} = \xi\psi^{-1} : V \times I \to J,$$

which is a valid alternative for the cylinderlike hypothesis, because ψ keeps $(V \times I)^{\bullet}$ fixed. Let

$$\tilde{Z} = \tilde{\xi}^{-1}S(f) = \psi Z.$$

Let $\tilde{K} = \psi K_1$, $\tilde{L} = \psi L_1$. Then $\tilde{K}$, $\tilde{L}$ is a triangulation of the pair $V \times I$, $\tilde{Z}$. Moreover by construction $\tilde{L}$ satisfies property (4), and consequently any subdivision of $\tilde{L}$ also satisfies (4). Therefore if we replace ξ, K in our original construction by $\tilde{\xi}$, $\tilde{K}$ then L_2 will automatically satisfy (4). This completes the proof of Sublemma 1.

Construction of one pipe.

So far we have constructed a triangulation K_2, L_2 of $V \times I$, Z satisfying properties (1, 2, 3, 4). Let K_3 be the barycentric first derived of K_2. Let K_4 be a cylindrical subdivision of K_3. Let K_5 be a cylindrical second derived of K_4.

Let A, A_* be two z-simplexes of L_2 identified by φ, by property (3). Label them so that one of the following cases occurs:

(i) both vertical

(ii) both horizontal

(iii) A horizontal and A_* vertical.

One of these cases always occurs because every simplex of K_2 is either horizontal or vertical, because K_2 is a subdivision of the cylindrical K_1 (this was why we bothered with K_1). In case (i) there is no need to do any piping, because neither A nor A_* will have any $(z + 1)$-dimensional stuff underneath them that we want to get rid of. Therefore we can assume A is horizontal.

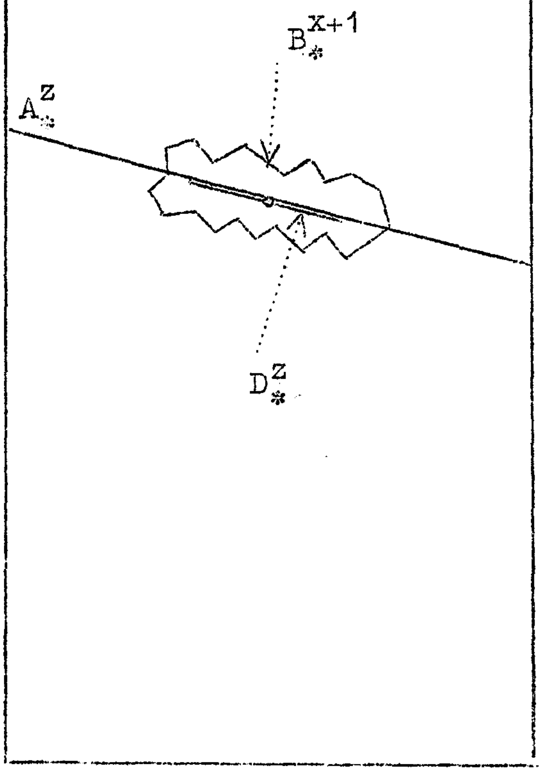
Let $\hat{A}$ be the barycentre of A, and let P (P stands for path) be the vertical interval above $\hat{A}$ joining $\hat{A}$ to $\pi\hat{A} \times 0$. By construction $\hat{A}$ is a vertex of K_3 and P is a subcomplex of K_4, and so the second derived neighbourhood

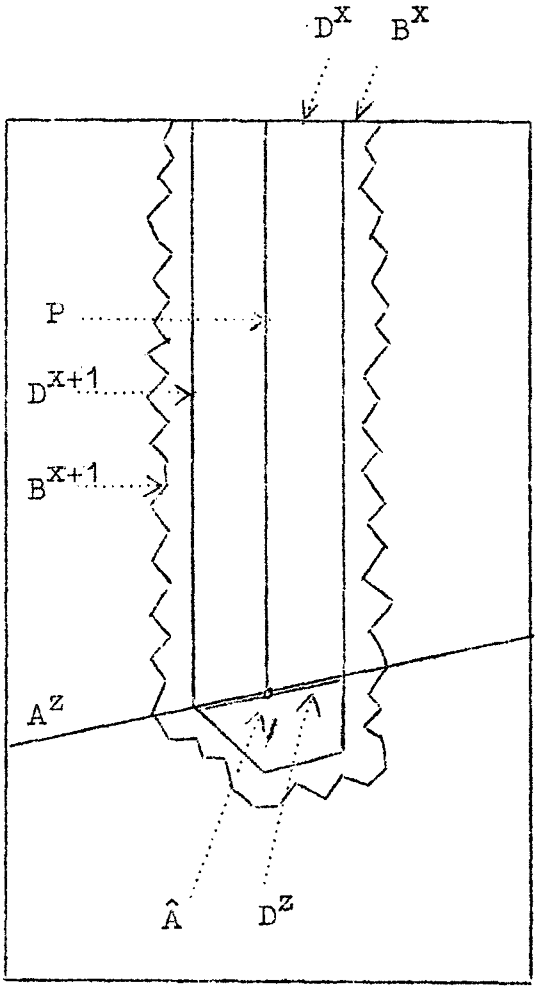
$$D^{x+1} = N(P, K_5)$$

is an $(x + 1)$-ball meeting $V \times 0$ in a face, D^x say. Let

$$D^z = A \cap D^{x+1},$$

which is a z-ball, being the closed star of $\hat{A}$ in A_5 (A_5 being the subdivision of A induced by K_5).





Let

$$D^z_* = A_* \cap \varphi^{-1}\varphi D^z$$

which is a z-ball by property (3) above, but is not in general a subcomplex of K_5, because φ is not in general simplicial on K_5.

Both P and $\pi(\overline{Z - A}) \times I$ are subcomplexes of K_4, and are disjoint by property (4). Their second derived neighbourhoods are disjoint in K_5. Therefore $D^{x+1} \cap Z = D^z$. Therefore $\varphi|D^{x+1}$ is an embedding, and

$$(5) \qquad \varphi^{-1}\varphi D^{x+1} = D^{x+1} \cup D^z_*.$$

Let K_6 be a subdivision of K_5, and M_6 a triangulation of M such that $\varphi : K_6 \to M_6$ is simplicial. Consequently D^z_* becomes a subcomplex of K_6. Let K_7, M_7 be barycentric second deriveds of K_6, M_6. Then $\varphi : K_7 \to M_7$ is simplicial because φ is non-degenerate. Let

$$B^m = N(\varphi D^{x+1}, M_7)$$
$$B^{x+1} = N(D^{x+1}, K_7)$$
$$B^{x+1}_* = N(D^z_*, K_7).$$

Notice that these are three balls, because they are second derived neighbourhoods of balls. Also B^{x+1} meets $V \times 0$ in a face, B^x say (because D^{x+1} did); B^{x+1}_* lies in the interior of $V \times I$ (because D^z_* did); and B^m lies in the interior of M (because φD^{x+1} did). From (5) we deduce

$$(6) \qquad \varphi^{-1}(B^m) = B^{x+1} \cup B^{x+1}_*$$
$$\varphi^{-1}(\dot{B}^m) = (\dot{B}^{x+1} - \mathring{B}^x) \cup \dot{B}^{x+1}_*.$$

By Theorem 6, $V \times I$ is homeomorphic to closure $(V \times I - D^{x+1})$, and so we can choose an embedding

$$h: V \times I \to V \times I$$

that is the identity outside D^{x+1}, and maps $V \times I$ onto closure $(V \times I - D^{x+1})$. The two maps

$$\varphi | B^x, \quad \varphi h | B^x : B^x \to B^m$$

are both proper and agree on the boundary. Therefore they are ambient isotopic by Theorem 9, because $x \leqslant m - 3$, and so there exists a homeomorphism $k: B^m \to B^m$, keeping the boundary fixed, such that

$$\varphi | B^x = k\varphi h | B^x : B^x \to B^m.$$

Extend k by the identity to a homeomorphism of M. Define φ_1 to be the composition

$$\underbrace{V \times I \xrightarrow{h} V \times I \xrightarrow{\varphi} M \xrightarrow{k} M}_{\varphi_1} .$$

Notice that $\varphi = \varphi_1$ outside $B^{x+1} \cup B_*^{x+1}$, and the only difference between φ, φ_1 is to alter the embeddings of B^{x+1}, B_*^{x+1} in B^m. Therefore φ_1 is homotopic to φ. Also $\varphi_1 = \varphi$ on $(V \times I)^{\cdot}$, because the only place where they might not agree is B^x, and here they agree by choice of k. Therefore $\varphi_1 \simeq \varphi$ keeping $(V \times I)^{\cdot}$ fixed.

<u>Remark</u>.

We have completed the construction of one pipe. Notice that the neighbourhood N referred to in the digression above is B_*^{x+1}. The pipe was thrown up indirectly by the homeomorphism k, rather than by drilling directly along the path φP (which might be pretty rugged if $x = m - 3$, because the embedding $\varphi|D^{x+1}: D^{x+1} \to M^m$ could then be locally knotted along P).

Construction of f_1.

Recall the commutative diagram

$$\begin{tikzcd} V \times I \arrow[rr, "\varphi"] \arrow[dr, "\xi"'] & & M \\ & J \arrow[ur, "f"'] & \end{tikzcd}$$

Let $B = B^{x+1} \cup B_*^{x+1}$. Since $\varphi_1 = \varphi$ outside B, and since ξ maps B homeomorphically onto ξB, and maps no other points into ξB, we can define $f_1 = \varphi_1 \xi^{-1}$ unambiguously. Therefore

$$\begin{tikzcd} V \times I \arrow[rr, "\varphi_1"] \arrow[dr, "\xi"'] & & M \\ & J \arrow[ur, "f_1"'] & \end{tikzcd}$$

is commutative. Since $f = f_1$ outside ξB, and $f(J - J_0) \subset \mathring{M}$, and $f_1(\xi B) \subset B^m \subset \mathring{M}$, we have $f_1(J - J_0) \subset \mathring{M}$ as required. Let $Z_1 = \xi^{-1}S(f_1)$.

Sublemma 2. $hZ_1 = Z - (\mathring{D}^z \cup \mathring{D}_*^z)$.

In other words we have punched holes in the simplexes A, A_*.

Proof. Let $B = B^{x+1} \cup B_*^{x+1}$. Then f, f_1 agree outside ξB. Therefore

$$Z_1 = (Z - B) \cup \xi^{-1}S(f_1|\xi B).$$

Since h keeps $Z - B$ fixed,

$$hZ_1 = (Z - B) \cup h\xi^{-1}S(f_1|\xi B).$$

Now

$$h\xi^{-1}S(f_1|\xi B) = h\xi^{-1}S(k\varphi h\xi^{-1}|\xi B),\ \text{by definitions of } \varphi_1,\ f_1,$$
$$= S(k\varphi|hB),\ \text{because } h\xi^{-1}|\xi B:\xi B \to hB \text{ is a homeomorphism},$$
$$= S(\varphi|hB),\ \text{because } k \text{ is a homeomorphism},$$
$$= (B^z - \overset{\circ}{D}{}^z) \cup (B_*^z - \overset{\circ}{D}{}_*^z).$$

Therefore $hZ_1 = Z - (\overset{\circ}{D}{}^z \cup \overset{\circ}{D}{}_*^z)$, as required.

Simultaneous construction of all the pipes.

Let $\{(A_i, A_{*i})\}$ be the set of pairs of z-simplexes of L_2 of type (ii) or (iii), where i runs over some indexing set. For each i we construct a pipe as above, using the same subdivisions $K_3, .., K_7$ of K_2. We now verify that the constructions are mutually disjoint.

Firstly the paths $\{P_i\}$ are mutually disjoint, and disjoint from $\bigcup A_{*i}$ by property (4) above. Also these are subcomplexes of K_4 and so their second derived neighbourhoods $\{D_i^{x+1}\}$ are mutually disjoint, and disjoint from $\bigcup A_{*i}$. Therefore, using (5) above, the images $\{\varphi D_i^{x+1}\}$ are mutually disjoint subcomplexes of M_6. Therefore their second derived neighbourhoods $\{B_i^m\}$ are mutually disjoint. Therefore we can define h, k so as to construct a pipe inside each B_i^m simultaneously. As before define $\varphi_1 = k\varphi h$, $f_1 = \varphi_1\xi^{-1}$.

Sublemma 3.

Let T be the subcylinder through the (z - 1)-skeleton of L_2. Then

$$V \times I \searrow W \cup T \cup Z_1 \searrow W.$$

<u>Proof</u>. Since h is an embedding it suffices by Lemma 38 to show that

$$h(V \times I) \searrow h(W \cup T \cup Z_1) \searrow hW.$$

By property (4) each path P_i is disjoint from $W \cup T$, and therefore the second derived neighbourhoods D_i^{x+1}, B_i^{x+1} are also disjoint from $W \cup T$. Therefore h keeps $W \cup T$ fixed, because h only moves inside the $\{B_i^{x+1}\}$. Therefore it suffices to show

$$h(V \times I) \searrow W \cup T \cup hZ_1 \searrow W.$$

Now $h(V \times I)$ is the subcomplex of the cylindrical triangulation K_5 obtained by removing the open simplicial neighbourhoods of the $\{P_i\}$. Therefore $h(V \times I)$ is solid because if a simplex does not meet $\bigcup P_i$, neither does any simplex beneath it.

For each horizontal z-simplex $A \in L_2$, let A' denote the subcomplex of K_5 beneath A, and let $A'' = A' \cap h(V \times I)$. Let $E = \bigcup A''$, the union for all such A. Then A' is solid, and so is A'' being the intersection of solids, and so is E being the union of solids. Therefore $W \cup T \cup E$ is solid. We have

$$h(V \times I) \supset W \cup T \cup E \supset V \times 1 \text{ and so}$$

$$h(V \times I) \searrow W \cup T \cup E$$

cylinderwise by Corollary 2 to Lemma 45. Next we have to show

$$W \cup T \cup E \searrow W \cup T \cup hZ_1.$$

We do this by examining each A'' separately. There are two cases, according to whether A is the first or second member of a pair.

<u>Case (i):</u> A is the first member of a pair. Now A' is a

$(z+1)$-ball, because A is interior to the prism $\pi A \times I$ by property (2), and so A' triangulates the convex linear $(z+1)$-cell beneath A. The subball $\overline{st}(\hat{A}, A')$ meets $\dot{A}'$ in the common face D^z, and so the complement

$$A'' = A' - st(\hat{A}, A')$$

is also a $(z+1)$-ball with face $F = lk(\hat{A}, A')$. Therefore we can collapse across A'' from F.

<u>Case (ii).</u> A_* is the second member of a pair. In this case $A''_* = A'_*$ and so we can collapse A''_* from D^z_*. What is left after all these collapses is $W \cup T \cup hZ_1$ by Sublemma 2.

Next collapse $W \cup T \cup hZ_1 \searrow W \cup T$ by collapsing

$$A - \mathring{D}^z \searrow \dot{A}$$
$$A_* - \mathring{D}^z_* \searrow \dot{A}_*$$

for all horizontal z-simplexes A or A_* in L_2. Finally collapse $W \cup T \searrow W$ cylinderwise. We have shown

$$h(V \times I) \searrow W \cup T \cup hZ_1 \searrow W$$

as required.

<u>Completion of the proof of Lemma 48.</u>

Define $J_1 = S(f_1) \cup \xi(\overline{T - W})$. We must verify that J_1 satisfies the three conditions:

(i) $\dim J_1 \leqslant z$

(ii) $\dim (J_0 \cap J_1) \leqslant z - 1$

(iii) $J \searrow J_0 \cup J_1 \searrow J_0$.

To prove (i) observe that $\dim S(f_1) \leqslant z$ and $\dim T \leqslant z$.

To prove (ii); $J_0 \cap J_1 = (J_0 \cap S(f_1)) \cup (J_0 \cap \xi(\overline{T - W}))$

$$= (J_0 \cap S(f)) \cup (\xi W \cap \xi(\overline{T - W}))$$

$$= (J_0 \cap S(f)) \cup \xi(W \cap \overline{T - W}).$$

$\dim (J_0 \cap S(f)) < z$, by general position of f.

$\dim (W \cap \overline{T - W}) < \dim T \leqslant z$.

Therefore $\dim (J_0 \cap J_1) < z$.

To prove (iii); $J = \xi(V \times I)$

$$J_0 \cup J_1 = \xi W \cup \xi(\overline{T - W}) \cup \xi Z_1$$

$$= \xi(W \cup T \cup Z_1)$$

$$J_0 = \xi W.$$

Therefore the collapse $J \searrow J_0 \cup J_1 \searrow J_0$ follows from Sublemma 3 by Lemma 37 because $W \supset S(\xi)$. The proof of the piping lemma is complete.

Lemma 49 (c.f. Lemma 39)

Let C be a closed subspace of M^m, such that $\dim (C \cap \mathring{M}) \leqslant q$. Let X^x be compact, C-inessential and $X \cap \dot{M} \subset C$. Then these are compact spaces $Y \supset Z$, such that

$$X \cup C \subset Y \cup C \overset{\circ}{\searrow} Z \cup C$$

$$Y \cap \dot{M} \subset C$$

$$\dim Y \leqslant x + 1$$

$$\dim Z \leqslant \max (q, x) + x - m + 2.$$

Proof. Notice that the interiorness of the collapse follows trivially from the other results, because $(X \cup C) \cap \dot{M} = (Y \cup C) \cap \dot{M} = (Z \cup C) \cap \dot{M} = C \cap \dot{M}$.

The lemma is trivial if max $(q, x) > m - 3$, because then choose $X = Y = Z$. Therefore assume $q, x \leqslant m - 3$. Let $z = \max(q, x) + x - m + 2$. There are two cases according as to whether $x < q$ or $x \geqslant q$.

Case 1 $x < q$.

Therefore $z = q + x - m + 2$. Without loss of generality we can assume $X = \overline{X - C}$, because if we prove the result for $\overline{X - C}$, then trivially it follows for X. The C-inessentiality means that the inclusion $X \subset M$ is homotopic to a map $f: X \to C$ by a homotopy $h: X \times I \to M$ keeping $X \cap C$ fixed. Both f, h are continuous-maps, not necessarily piecewise linear, but before we make h piecewise linear we first want to factor it through a mapping cylinder.

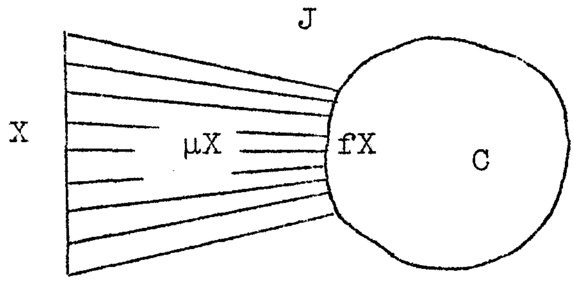
Without loss of generality we can assume M to be compact. For if not, replace M by a compact submanifold M_* containing a neighbourhood of $h(X \times I)$. (Construct M_* by covering $h(X \times I)$ with a finite number of balls and taking a regular neighbourhood of their union). Replace C by $C \cap M_*$. If the result holds for M_* then using the same Y, Z it also holds for M, by excision.

Therefore assume that M is compact, and consequently the pair M, C is triangulable. By the relative simplicial approximation theorem, we can homotop $f: X \to C$ to a piecewise linear map keeping $X \cap C$ fixed. Triangulate X, C so that f is simplicial, and let $J = \mu X \cup C$ be the relative mapping cylinder. Since h realises the identifications of the topological relative mapping cylinder, we can factor h through J by Lemma 47,

$$\begin{tikzcd} X \times I \arrow[rr, "h"] \arrow[dr] & & M \\ & J \arrow[ur, "g"'] & \end{tikzcd}$$

where $g|X \cup C$ is the identity.

Again using the relative simplicial approximation theorem, make $g: J \to M$ piecewise linear keeping $X \cup C$ fixed.



Let $Y = g(\mu X)$, which is one of the spaces to be found. Then $Y \supset X$ and $\dim Y \leq x + 1$. Let $Y_0 = X \cup fX = g(X \cup fX)$. Then $Y_0 \subset Y$. In particular $g|Y_0$ is in general position, because it is the identity, and so by Theorem 18 we can homotop $g|\mu X$ into general position keeping Y_0 fixed. At the same time we can ensure that $g(\mu X - Y_0) \subset \mathring{M}$. Therefore $Y - Y_0 \subset \mathring{M}$. Therefore $Y \cap \dot{M} = Y_0 \cap \dot{M} \subset C$, by the hypothesis $X \cap \dot{M} \subset C$. The homotopy extends trivially to a homotopy of g keeping $X \cup C$ fixed, because $\mu X \cap C = fX$. Since dim $(\mu X) \leqslant x + 1$, the general position implies

$$\begin{aligned}\dim S(g|\mu X) &\leqslant 2(x+1) - m\\ &\leqslant z - 1,\end{aligned}$$

because $x < q$. What we want is $\dim S(g) \leqslant z - 1$, but as yet $Y = g(\mu X)$ may intersect C in too high a dimension, and so another general position move is necessary.

Let C_0 = closure $(C \cap \mathring{M})$. Then $\dim C_0 \leqslant q$ by hypothesis. Since $Y - Y_0 \subset \mathring{M}$, by Theorem 15 we can ambient isotop Y keeping Y_0 fixed, until $Y - Y_0$ is in general position with respect to C_0. Therefore

$$\begin{aligned}\dim ((Y - Y_0) \cap C_0) &\leqslant (x+1) + q - m\\ &= z - 1.\end{aligned}$$

The isotopy of Y keeping Y_0 fixed determines a homotopy of g keeping $X \cup C$ fixed, that does not alter $S(g|\mu X)$ but does reduce $S(g)$, because

$$\begin{aligned}g(\mu X) \cap g(J - \mu X) &= Y \cap (C - fX)\\ &= (Y - fX) \cap C\\ &= (Y - fX - X) \cap C, \text{ since } X \cap C \subset fX\\ &= (Y - Y_0) \cap C\\ &= (Y - Y_0) \cap C_0, \text{ since } Y - Y_0 \subset \mathring{M}.\end{aligned}$$

Therefore, since g is non-degenerate,

$$\dim g^{-1}(g(\mu X) \cap g(J - \mu X)) \leqslant z - 1.$$

Writing $J = \mu X \cup (J - \mu X)$, we see that

$S(g) = S(g|\mu X) \cup S(g|J - \mu X) \cup$ closure $g^{-1}(g(\mu X) \cap g(J - \mu X))$, of which the second term is empty, and the other two terms we have made of dimension $\leqslant z - 1$. Therefore

$$\dim S(g) \leqslant z - 1.$$

Now triangulate the collapse $J \searrow C$ of the mapping cylinder (given by the Corollary to Lemma 46), and let $T = \text{trail } S(g)$. Then $J \searrow T \cup C$ by Lemma 45. Therefore $gJ \searrow g(T \cup C)$ by Lemma 38, because $T \supset S(g)$.

Let $Z = gT \cap Y$, which is the other space that we had to find. Then

$$\begin{aligned} \dim Z &\leqslant \dim T \\ &\leqslant 1 + \dim S(g), \text{ by Lemma 44} \\ &\leqslant z. \end{aligned}$$

Now $Y \cup C = gJ$, and
$$\begin{aligned} Z \cup C &= (gT \cap Y) \cup C \\ &= (gT \cup C) \cap (Y \cup C) \\ &= g(T \cup C) \cap gJ \\ &= g(T \cup C). \end{aligned}$$

Therefore

$$X \cup C \subset Y \cup C \searrow Z \cup C,$$

which completes the proof of case (1).

<u>Case (2), $x \geqslant q$.</u>

Therefore $z = 2x - m + 2$. This time we shall need to use the piping lemma. As before assume M, C compact, make $f: X \to C$ simplicial, and let $J = \mu X \cup C$ be the relative simplicial mapping cylinder of f. Without loss of generality we can assume $\dim C \leqslant x$; for otherwise let $C^{(x)}$ denote the x-skeleton of C.

Then $fX \subset C^{(x)}$, and if we prove the lemma for $C^{(x)}$:

$$X \cup C^{(x)} \subset Y \cup C^{(x)} \searrow^{\circ} Z \cup C^{(x)},$$

then it follows for C by excision, because $C - C^{(x)} \subset \dot{M}$ by the hypothesis $\dim (C \cap \mathring{M}) \leqslant q \leqslant x$.

Therefore assume $\dim C \leqslant x$. Hence $\dim J \leqslant x + 1$. Let X_0 be the $(x - 1)$-skeleton of X, and $J_0 = \mu X_0 \cup C$ the submapping cylinder of $f|X_0$. As before construct from the homotopy h a piecewise linear map $g:J \to M$ such that $g|X \cup C$ is the identity. In particular $g|X \cup C$ is in general position, and $J \supset X \cup J_0 \supset X \cup C$, and so by Theorem 18 Corollary 2 we can homotop g into general position for the pair $J, X \cup J_0$ keeping $X \cup C$ fixed. At the same time we can ensure that $g(J - X - C) \subset \mathring{M}$. Therefore $g(J - C) \subset \mathring{M}$, because $X - C \subset \mathring{M}$ by the hypothesis $X \cap \dot{M} \subset C$.

This time let $Y = gJ$. Then $Y \supset X$, $\dim Y \leqslant x + 1$, and $Y \cap \dot{M} \subset C$ because $Y - C = gJ - gC \subset g(J - C) \subset \mathring{M}$. Now the triple $X^x, J_0^x \subset J^{x+1}$ is cylinderlike, $g(J - J_0) \subset \mathring{M}$ and $x \leqslant m - 3$. Therefore by the piping lemma (Lemma 48) we can homotop g keeping $X \cup J_0$ fixed and $g(J - J_0) \subset \mathring{M}$, and choose $J_1 \supset S(g)$ such that

$$\dim J_1 \leqslant z$$
$$\dim (J_0 \cap J_1) \leqslant z - 1$$
$$J \searrow J_0 \cup J_1 \searrow J_0.$$

Triangulate the mapping cylinder collapse $J_0 \searrow C$ given by Lemma 46 Corollary, and let $T = \text{trail } (J_0 \cap J_1)$. Then by Lemma 44

$$\dim T \leqslant 1 + \dim (J_0 \cap J_1)$$
$$\leqslant z.$$

Also $J_0 \searrow T \cup C$ by Lemma 45. Therefore $J_0 \cup J_1 \searrow T \cup C \cup J_1$ because $J_0 \cap J_1 \subset T$. Therefore

$$J \searrow J_0 \cup J_1 \searrow T \cup J_1 \cup C.$$

Therefore by Lemma 38, $gJ \searrow g(T \cup J_1 \cup C)$, because $J_1 \supset S(g)$.

Now let $Z = g(T \cup J_1)$; then $\dim Z \leqslant z$ because both $\dim T$, $\dim J_1 \leqslant z$. We have $gJ = Y = Y \cup C$, and $g(T \cup J_1 \cup C) = Z \cup C$, and so

$$X \cup C \subset Y \cup C \searrow Z \cup C,$$

which completes the proof of Lemma 48.

Admissible regular neighbourhoods.

Definition: a regular neighbourhood N of X in M is called admissible if the collapse $N \searrow X$ is admissible.

Lemma 50. Let N be an admissible regular neighbourhood of X in M. Let F be the frontier of N in M. If $Y \subset N - F$, and $X \overset{a}{\searrow} Y$ or $Y \overset{a}{\searrow} X$, then N is also an admissible regular neighbourhood of Y in M.

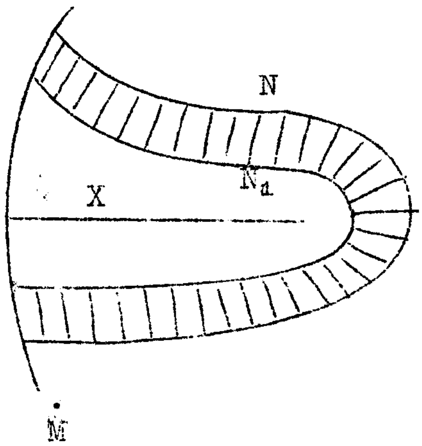
Proof. If $X \overset{a}{\searrow} Y$ the result is trivial because then $N \overset{a}{\searrow} X \overset{a}{\searrow} Y$. Therefore assume $Y \overset{a}{\searrow} X$. There are two cases: the absolute case when $X \subset \overset{\circ}{M}$ and the relative case when X meets $\dot{M}$. In the absolute case both N, Y must also lie in $\overset{\circ}{M}$, and so $F = \dot{N}$; therefore the result follows from Theorem 8 Corollary 4.

The relative case is similar, and we indicate the steps of the proof, leaving the details to the reader.

(i) A derived neighbourhood is admissible (c.f. Corollary to Lemma 14).

(ii) Any two admissible regular neighbourhoods are ambient isotopic (c.f. Theorem 8 (2)).

(iii) If N_1 is another admissible regular neighbourhood of X in M, and $N_1 \subset N - F$, then there is a homeomorphism $\varphi : F \times I \to \overline{N - N_1}$ such that $\varphi(x, 0) = 0$, $x \in F$ and $\varphi(\dot{F} \times I) = \dot{M} \cap (\overline{N - N_1})$, (c.f. Theorem 8 Corollary 2).



Now, given the situation in the lemma, let N_1 be an admissible regular neighbourhood of Y in $N - F$. Then N_1 is also an admissible regular neighbourhood of X in M because $N_1 \searrow^{a} Y \searrow^{a} X$. Hence $N \searrow^{a} N_1$ cylinderwise by (iii). Therefore N is an admissible regular neighbourhood of Y because $N \searrow^{a} N_1 \searrow^{a} Y$. The proof of Lemma 50 is complete.

We now prove Theorem 21 in the special case that $X \cap \dot{M} \subset C$. In particular this covers the case when $X \subset \mathring{M}$.

Lemma 51. Let C be a q-collapsible k-core of M^m, and let X^x be compact, C-inessential, $X \cap \dot{M} \subset C$. Let

$$x, q \leqslant m - 3$$

$$2x, q + x \leqslant m + k - 2$$

Then we can engulf $X \subset D \overset{\circ}{\searrow} C$, such that $\dim (D - C) \leqslant x + 1$.

Proof: Let $c = \dim(C \cap \overset{\circ}{M})$. The q-collapsibility means that $C \overset{\circ}{\searrow} Q$, where $\dim (Q \cap \overset{\circ}{M}) \leqslant q$. Of course $c > q$ in general. We consider separately the three cases: (1) $c \leqslant q$ (2) $c > q > x$ (3) $c > q \leqslant x$.

Case (1): $c \leqslant q$ (c.f. the proof of Theorem 19).

The proof is by induction on x, starting trivially with $x = -1$. Assume the result true for dimensions $< x$. By Lemma 49 choose Y, Z such that $X \subset Y \cup C \overset{\circ}{\searrow} Z \cup C$, $Z \cap \dot{M} \subset C$, and

$$z = \dim Z \leqslant \max (q, x) + x - m + 2.$$

Therefore $z \leqslant k$ by the hypothesis $x + \max (q, x) \leqslant m + k - 2$. Therefore Z is C-inessential. Also $z < x$ by the hypothesis $x \leqslant m - 3$. Therefore by induction we can engulf $Z \subset E \overset{\circ}{\searrow} C$, with $\dim (E - C) \leqslant z + 1 \leqslant x$, and $E \cap \dot{M} = C \cap \dot{M}$. Apply Lemma 42 to the situation

$$Y \cup C \overset{\circ}{\searrow} Z \cup C \subset E,$$

and ambient isotop E to E_* keeping $Z \cup C$ fixed so that $(Y \cup C) \cup E_* \overset{\circ}{\searrow} E_*$. Since ambient isotopy preserves interior collapsibility we have $E_* \overset{\circ}{\searrow} C$. Therefore putting $D = Y \cup C \cup E_*$, we have $D \overset{\circ}{\searrow} E_* \overset{\circ}{\searrow} C$, and so

$$X \subset D \overset{\circ}{\searrow} C.$$

Also $\dim (D - C) \leqslant x + 1$ because $D - C = (Y - E_*) \cup (E_* - C)$, dim $Y \leqslant x + 1$, and dim $(E_* - C) = \dim E - C \leqslant x$.

Case (2): $c > q > x$

Let $C \overset{\circ}{\searrow} Q$ be given by q-collapsibility. We can choose Q so that $X \cap C = X \cap Q$; for if not let T = trail $(X \cap C)$ under some triangulation of the collapse $C \overset{\circ}{\searrow} Q$. Then dim $T \leqslant x + 1 \leqslant q$ by Lemma 44, and $C \overset{\circ}{\searrow} Q \cup T$ by Lemma 45. Therefore $Q \cup T$ is as good a candidate as Q in the definition of q-collapsibility. In fact it is better because $X \cap C = X \cap (Q \cup T)$.

Therefore we can suppose $X \cap C = X \cap Q$. Therefore $X \cup C \overset{\circ}{\searrow} X \cup Q$ by excision, and $X \cap \dot{M} \subset Q$. The deformation retraction $C \to Q$ ensures that if X is C-inessential then X is Q-inessential also. Therefore by case (1) we can engulf $X \subset E \overset{\circ}{\searrow} Q$. Apply Lemma 42 to the situation

$$X \cup C \overset{\circ}{\searrow} X \cup Q \subset E$$

and ambient isotop E to E_* keeping $X \cup Q$ fixed, so that $(X \cup C) \cup E_* \overset{\circ}{\searrow} E_*$. Let $F = X \cup C \cup E_*$. Then $F \overset{\circ}{\searrow} E_* \overset{\circ}{\searrow} Q$, because $E \overset{\circ}{\searrow} Q$. Therefore $F \supset C \supset Q$, $F \overset{\circ}{\searrow} Q$ and $C \overset{\circ}{\searrow} Q$.

If we could deduce that $F \overset{\circ}{\searrow} C$ we should be finished, but this cannot be deduced, as is shown by the example at the end of Chapter 3. Therefore we have to get round the difficulty by taking a regular neighbourhood of F; but it is necessary first to restrict attention to a compact subset in order that the regular neighbourhood should exist.

Let M_1 be a regular neighbourhood of $\overline{F - Q}$ in M. Let C_1, F_1, Q_1 denote the intersections of M_1 with C, F, Q respectively.

Then $F_1 \overset{\circ}{\searrow} Q_1$ and $C_1 \overset{\circ}{\searrow} Q_1$ in M_1 because none of the collapses meet the frontier of M_1. Let N be an admissible regular neighbourhood of F_1 in M_1. Then C_1 does not meet the frontier of N because $C_1 \subset F_1$. Since $F_1 \overset{\circ}{\searrow} Q_1 \overset{\circ}{\nearrow} C_1$ in M_1 we also have N an admissible regular neighbourhood of C_1 in M_1, by Lemma 50. Therefore $N \searrow C_1$. Adding $C - C_1$ to both sides, $N \cup C \searrow C$ by excision. Now $X \subset F \subset F_1 \cup C \subset N \cup C$. Let D = trail X under some triangulation of the collapse $N \cup C \searrow C$. Then $X \subset D \searrow C$ by Lemma 45, and $\dim (D - C) \leqslant x + 1$ by Lemma 44. By Lemma 43 isotop D keeping $X \cup C$ fixed so that $D \cap \dot{M} = (X \cup C) \cap \dot{M} = C \cap \dot{M}$. This completes the proof of case (2).

<u>Case (3): $c > q \leqslant x$</u>

Let $C \overset{\circ}{\searrow} Q$ be given by the q-collapsibility. Triangulate $X \cup \overline{C - Q}$ so that X is a subcomplex, and subdivide if necessary so that $C \overset{\circ}{\searrow} Q$ collapses simplicially. Let T^x be the trail of the $(x - 1)$-skeleton of $\overline{C - Q}$. Then $C \overset{\circ}{\searrow} T \cup Q$ by elementary simplicial collapses of dimension $\geqslant x + 1$. It is valid to perform these on $X \cup C$ because although an x-simplex of X may occur as the free face of some elementary collapse, it is nevertheless principal in X, and therefore remains a free face when X is added. Therefore

$$X \cup C \overset{\circ}{\searrow} X_1 \cup Q,$$

where $X_1 = (X - (C - (T \cup Q))) \cup T$. We now want to engulf X_1 from Q.

Observe that $\dim X_1 \leqslant x$ because $\dim T \leqslant x$. Also

$X_1 \cap \dot{M} \subset (X \cup C) \cup \dot{M} = C \cap \dot{M} = Q \cap \dot{M} \subset Q$. Since X is C-inessential, so is $X \cup C$, and therefore so also is X_1. The deformation retraction $C \to Q$ ensures that X_1 is Q-inessential. Therefore by case (1) we can engulf $X_1 \subset E \overset{\circ}{\searrow} Q$. Apply Lemma 42 to the situation

$$X \cup C \overset{\circ}{\searrow} X_1 \cup Q \subset E,$$

and proceed as in case (2). The proof of Lemma 51 is complete.

The problem of proving the general case of Theorem 21 is that the deformation of X into C may involve some boundary-to-interior type collapses, and so the engulf involve the inverse process interior-to-boundary type expansions. But when we try to expand interior-to-boundary we hit an obstruction, because there is no room to push other stuff out of the way. We drew attention to this situation at the end of the proof of Lemma 42. Therefore we introduce a device of Moe Hirsch to cope with the difficulty.

<u>Inwards collapsing</u>.

<u>Definition</u>: Let $K \supset L$, J be complexes. We say that an (ordered) simplicial collapse $K \searrow L$ is <u>away from</u> J if $J \cap \text{trail } W = J \cap W$ for every subcomplex $W \subset J$.

<u>Example</u> Let K be a cylindrical triangulation of $X \times I$. Then any cylinderwise collapse $K \searrow$ base $X \times 1$ is away from the top $X \times 0$.

<u>Lemma 52</u>.

<u>Let $K \searrow L$ be a simplicial collapse away from J, and let $W \subset K$. Then the induced collapse $K \searrow L \cup \text{trail } W$ is also away from J.</u>

Proof. Let s denote the collapse $K \searrow L$, and t the induced collapse $K \searrow L \cup \text{trail } W$ given by Lemma 45. The elementary simplicial collapses of t are a subset of those of s, with the induced ordering. If $V \subset K$, $\text{trail}_s V$ is obtained by adding to V an ordered set of simplexes, while $\text{trail}_t V$ is obtained by adding a subset; therefore $\text{trail}_t V \subset \text{trail}_s V$. Therefore

$$J \cap V \subset J \cap \text{trail}_t V \subset J \cap \text{trail}_s V = J \cap V,$$

because s is away from J. Therefore t is also.

Definition. Let $X \searrow Y$ in the manifold M. We call the collapse inwards, and write $X \overset{\gamma}{\searrow} Y$ if, given any triangulation of $\overline{X - Y}$, there exists a subdivision and a simplicial collapse $\overline{X - Y} \searrow \overline{X - Y} \cap Y$ away from $\overline{X - Y} \cap \dot{M}$. It follows at once from the definition that γ is invariant under excision:

$$X \overset{\gamma}{\searrow} X \cap Y \iff X \cup Y \overset{\gamma}{\searrow} Y$$

Example.

Let M be compact, X a collar on M, and Y the inside boundary of the collar. Then $X \overset{\gamma}{\searrow} Y$, because given any triangulation of X there exists a cylindrical subdivision and a cylinderwise collapse away from $\dot{M}$.

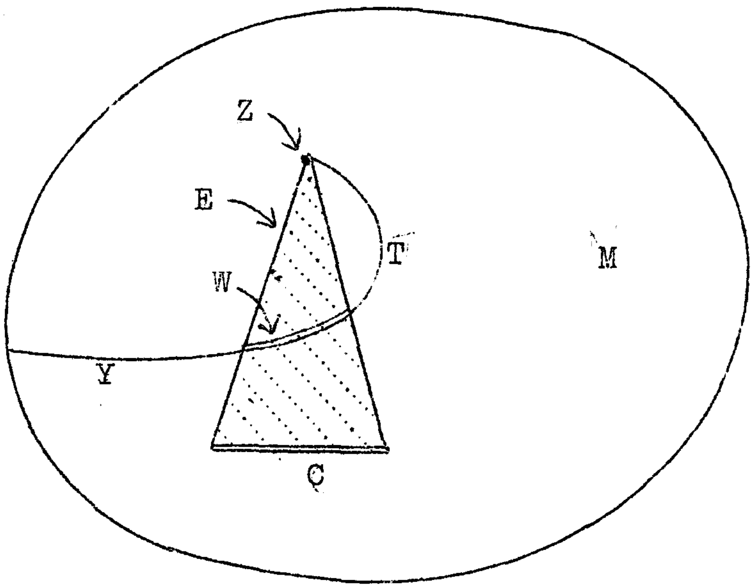
Lemma 53 (Hirsch)

Let C be a q-collapsible k-core of M. Suppose Y^y, Z^z are compact, $Y \supset Z$, $Y \cup C \overset{\gamma}{\searrow} Z \cup C$, Z is C-inessential, and $Z \cap \dot{M} \subset C$. Let

$$q, y \leqslant m - 3$$

$$q + z,\ y + z \leqslant m + k - 2.$$

Then we can engulf $Y \subset D \searrow C$ such that $\dim (D - C) \leqslant \max(y, z + 1)$.



Proof. The proof is by induction on z, starting trivially with $z = -1$, for then choose $D = Y \cup C$. Therefore assume the lemma true for dimensions less than z.

Since $z \leqslant y$, the hypothesis of Lemma 51 is satisfied for Z, and so we can engulf $Z \subset E \overset{\circ}{\searrow} C$, $\dim (E - C) \leqslant z + 1$. By Theorem 15 ambient isotop E, keeping $C \cup Z$ fixed, until $E - (C \cup Z)$ is in general position with respect to Y.

Let $W = \text{closure } (Y \cap [E - (C \cup Z)])$. Then

$$\dim W \leqslant y + (z + 1) - m.$$

Now $W \subset \overline{(Y \cup C) - (Z \cup C)}$. Therefore triangulate $\overline{(Y \cup C) - (Z \cup C)}$ so that W is a subcomplex. By the definition of γ there exists a subdivision and a simplicial collapse $Y \cup C \searrow Z \cup C$ such that if $T = \text{trail } W$, then $T \cap \dot{M} = W \cap \dot{M}$.

We claim that Y, T, E satisfy the hypotheses for Y, Z, C in the lemma. Once this claim has been established, we can appeal to induction, because $t = \dim T$

$$\begin{aligned} &\leqslant 1 + \dim W, \text{ by Lemma 44} \\ &\leqslant y + z + 2 - m \\ &< z, \text{ by the hypothesis } y \leqslant m - 3. \end{aligned}$$

Therefore by induction engulf $Y \subset D \searrow E$. Therefore $Y \subset D \searrow C$ because $E \overset{\circ}{\searrow} C$. Therefore $Y \subset D \searrow C$ because $D \searrow E \searrow C$. Finally

$\dim (D - C) \leqslant \max (y, z + 1)$, because

$\dim (D - E) \leqslant \max (y, t + 1)$, by induction, and

$\dim (E - C) \leqslant z + 1$, by choice of E.

There remains to establish the claim about Y, T, E satisfying the hypotheses.

First E is q-collapsible because $E \overset{\circ}{\searrow} C$. Next E is a k-core because $\pi_i(E, C) = 0$, all i. Next T is compact, because W is compact. Next $Y \supset T$ because $Y \supset W$ and so, taking trails under the collapse $Y \cup C \searrow Z \cup C$,

$$Y = \text{trail } Y \supset \text{trail } W = T.$$

Next $Y \cup E \overset{Y}{\searrow} T \cup E$ by excision, because $Y \cup C \overset{Y}{\searrow} T \cup Z \cup C$ by Lemma 52, and

$$\begin{aligned} (Y \cup C) \cup (T \cup E) &= Y \cup E \\ (Y \cup C) \cap (T \cup E) &= [Y \cap (T \cup E)] \cup C \\ &= T \cup (Y \cap E) \cup C \\ &= T \cup [W \cup Z \cup (Y \cap C)] \cup C \\ &= T \cup Z \cup C. \end{aligned}$$

Next T is E-inessential because E is a k-core, and

$$t = y + z + 2 - m$$
$$\leqslant k, \text{ by the hypothesis } y + z \leqslant m + k - 2.$$

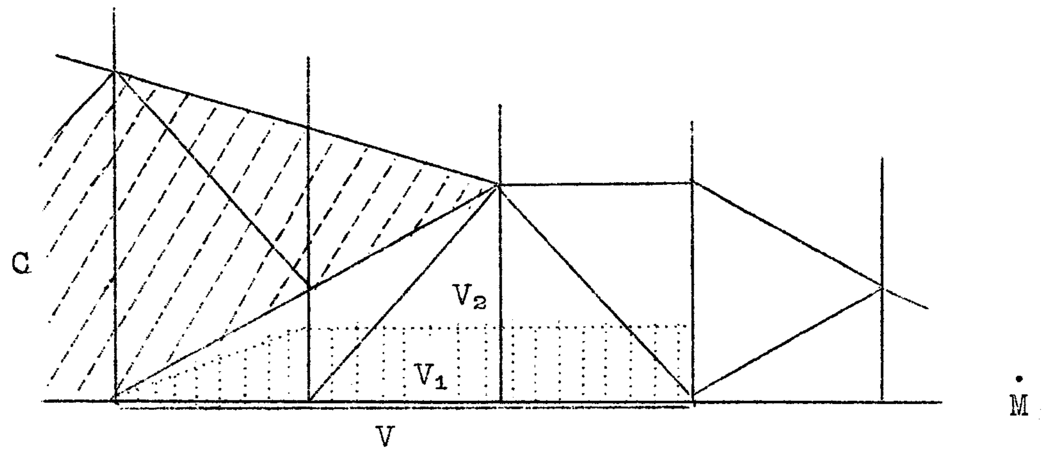
Finally the dimensional hypotheses are satisfied because the only change is to substitute t for z, and $t < z$. Therefore the proof of Lemma 53 is complete.

<u>Relative collars.</u>

Let M be compact, $C \subset M$, $V \subset \dot{M}$. We construct a collar on V mod C in M as follows. Let $c:\dot{M} \times I \to M$, $c(y, 0) = y$, $y \in \dot{M}$ be a collar on M. Choose a cylindrical triangulation of $\dot{M} \times I$, and a triangulation of M such that V, C are full subcomplexes and c simplicial. (This can be done as follows: first triangulate the triple M, V, C, next take a first derived, next subdivide to make c simplicial, then subdivide to make $\dot{M} \times I$ cylindrical, and finally extend the subdivision to M). Choose $\varepsilon > 0$ such that $\dot{M} \times (0, \varepsilon]$ contains no vertices. Let $f:V \to [0, \varepsilon]$ be the simplicial map determined by mapping vertices of $V \cap C$ to 0 and vertices of $V - C$ to ε. Define

$$V_1 = \{c(v, t);\ v \in V,\ 0 \leqslant t \leqslant fv\}$$
$$V_2 = \{c(v, fv);\ v \in V\}.$$

We call V_1 a <u>collar on V mod C in M</u>, and we call V_2 the <u>inside boundary</u> of the collar.



Suppose, further, that we are given $X^x \subset M$ such that $\dim (X \cap \dot{M}) < x$, and that we chose the triangulation of M so as to have X a subcomplex.

Lemma 54.

i) $V \cap C = V_1 \cap C = V_2 \cap C = V_2 \cap \dot{M}$.

ii) $V_1 \xrightarrow{\Upsilon} V_2$.

iii) $V_1 \xrightarrow{\circ} V \cup (X \cap V_1)$.

iv) $\dim (X \cap V_2) < x$.

Proof.

i) If $v \in V - C$, then $v \in \mathring{A}$, some simplex $A \notin C$. The fibre $v \times [0, fv] \subset \text{int } [st(A \times 0, \dot{M} \times I)]$ because the triangulation is cylindrical, and because of our choice of ε. Therefore the image $c(v \times [0, fv])$ does not meet C. Therefore $V_1 \cap C = V \cap C$. Next $V_2 \cap C = V \cap C$ because $V \cap C \subset V_2 \subset V_1$. Finally $V_2 \cap \dot{M} = f^{-1}0 = V \cap C$, because $V \cap C$ is full in V.

ii) $V_1 \searrow V_2$ because take a cylindrical subdivision such that V_1 is a subcomplex, and collapse cylinderwise away from V.

iii) Let $p:\dot{M} \times I \to I$ denote the projection. If $A^a \in M$ meets $V_1 - V$ then pA is 1-dimensional, and $A \cap V_2$ is a convex linear cell of dimension $a - 1$ separating A into two components, one of which is $A \cap V_1$. Collapse across $A \cap V_1$ from $A \cap V_2$ for all such simplexes A not in X, in order of decreasing dimension, and we have the collapse $V_1 \overset{\circ}{\searrow} V \cup (X \cap V_1)$.

iv) If $A \in X$, then either $A \cap V_2 \subset \dot{M}$, whence $\dim (A \cap V_2) \leqslant \dim (X \cap \dot{M}) < x$, by hypothesis, or else $A \cap V_2$ contains an interior point, whence $\dim (A \cap V_2) < \dim A \leqslant x$. Therefore $\dim (X \cap V_2) < x$.

<u>Proof of Theorem 21. Case (2).</u>

We are given $X \subset \dot{M}$ to engulf, where X is C-inessential, C is a q-collapsible k-core, and

$$q \leqslant m - 3$$

$$x \leqslant m - 4$$

$$q + x \leqslant m + k - 2$$

$$2x \leqslant m + k - 3.$$

Let Y be a collar on X mod C, with inside boundary Z. We now want to apply Lemma 53, and so let us check the hypotheses. First $Y \cup C \overset{Y}{\searrow} Z \cup C$ by excision, because $Y \overset{Y}{\searrow} Z$ and $Y \cap C = Z \cap C$ by Lemma 54 (i) and (iv). Next Z is C-inessential because the collar furnishes a homotopy from Z to X keeping $X \cap C = Z \cap C$ fixed, and because X is C-inessential by hypothesis.

Next $Z \cap C = Z \cap \dot{M} \subset \dot{M}$, by Lemma 54(i). Finally the dimensional hypotheses are satisfied because $y = z + 1$, $z = x$. Therefore by Lemma 53 we can engulf $Y \subset D \searrow C$. Therefore $X \subset D \searrow C$ because $X \subset Y$. Next

$$\dim (D - C) \leqslant \max (y, z + 1) = x + 1.$$

Finally we can choose D so that $D \cap \dot{M} = (Y \cap C) \cap \dot{M}$ by Lemma 43. The proof of Theorem 21 Case (2) is complete.

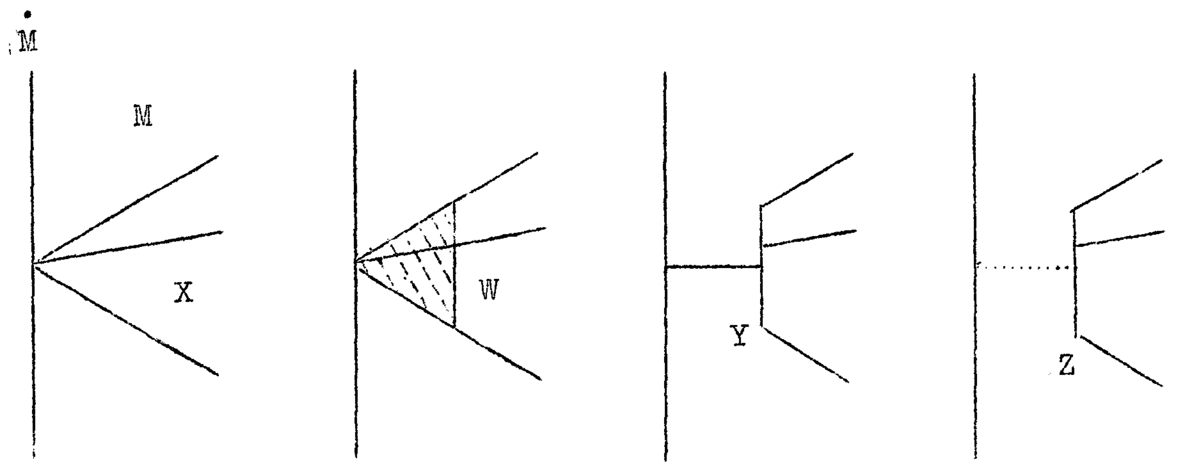
Lemma 55.

Let $X, C \subset M$, X compact and C-inessential, and $\dim (X \cap \dot{M}) < x$. Then there exist W^{x+1}, Y^x, Z^x all C-inessential, such that

$$X \subset W \cup C \overset{\circ}{\searrow} Y \cup C \overset{Y}{\searrow} Z \cup C$$

$$X \cap \dot{M} = W \cap \dot{M} = Y \cap \dot{M}$$

$$Z \cap \dot{M} \subset C.$$



<u>Remark</u>. The important part of the lemma is to get $Z \cap \dot{M} \subset C$ and the γ-collapse.

<u>Proof.</u> Without loss of generality we may assume M compact, for otherwise replace M by a regular neighbourhood of X in M, and perform all the constructions therein. Let

$$X_0 = \overline{(X \cap \dot{M}) - C}$$

$$C_0 = X_0 \cap C.$$

We assume $X_0 \neq \emptyset$, otherwise the lemma is trivial: $X = W = Y = Z$. Triangulate $\dot{M}$ such that X_0 and $\dot{M} \cap C$ are subcomplexes and take a first derived. Let V_0 be the closed simplicial neighbourhood of $X_0 - C_0$; in other words V_0 is the union of all closed simplexes of $\dot{M}$ meeting $X_0 - C_0$. (Notice that V_0 may not be a manifold at points of C_0, and is therefore not a regular neighbourhood in general). We deduce

(1) $V_0 \searrow X_0$, by Lemma 14 Corollary.

(2) $V_0 \cap C = C_0$, because of the first derived.

Let V_1 be a collar on V_0 mod C in M, with the proviso that we have X a subcomplex during the construction (so that Lemma 54 is applicable). Let X_1 be the subcollar on X_0, and let V_2, X_2 denote the inside boundaries of the collars. We claim:

(3) $V_1 \cap \overline{X - V_1} = X \cap V_2$

(4) $\dim (X \cap V_2) \leqslant x - 1$

(5) $V_1 \cup X \cup C \searrow X \cup C$.

(6) $V_1 \overset{Y}{\searrow} X_1 \cup V_2 \overset{\circ}{\searrow} X_1$

To prove (3), let F_0 = frontier of V_0 in $\dot{M}$, and let F_1 be the subcollar on F_0. Then

$$\begin{aligned} F_0 \cap X &= F_0 \cap X_0 \\ &= C_0, \text{ because } V_0 \text{ is a neighbourhood of } X_0 - C_0 \text{ in } \dot{M} \\ &= F_0 \cap C. \end{aligned}$$

Therefore by construction F is a collar mod X as well as mod C. Therefore $F_1 \cap X = C_0$ by Lemma 54 (i). Therefore

$$\begin{aligned} V_1 \cap \overline{X - V_1} &= X \cap (\text{frontier of } V_1 \text{ in } M) \\ &= X \cap (F_1 \cup V_2) \\ &= C_0 \cup (X \cap V_2) \\ &= X \cap V_2. \end{aligned}$$

(4) follows from Lemma 54 (iv). To prove (5) observe that $V_1 \searrow V_0 \cup (X \cap V_1)$ by Lemma 54 (iii). Next $V_0 \cup (X \cap V_1) \searrow X \cap V_1$

by excision from (1), because $V_0 \cap (X \cap V_1) = X \cap V_0 = X_0$. Therefore $V_1 \searrow X \cap V_1$ by composition, and so $V_1 \cup X \cup C \searrow X \cup C$ by excision, because

$$\begin{aligned} V_1 \cap (X \cup C) &= (V_1 \cap X) \cup (V_0 \cap C) \text{ by Lemma 54 (i)} \\ &= (V_1 \cap X) \cup C_0 \quad \text{by (2)} \\ &= V_1 \cap X. \end{aligned}$$

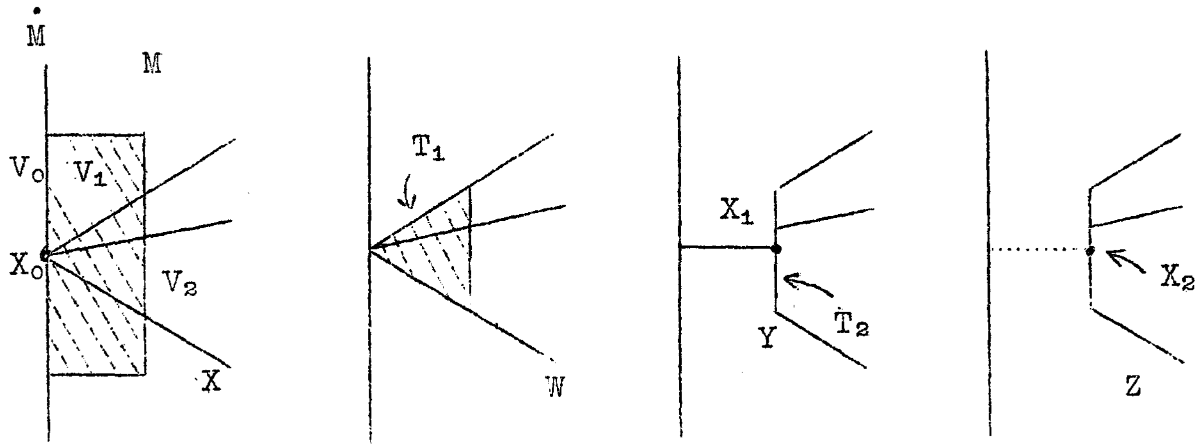
To prove (6) observe that $V_1 \overset{Y}{\searrow} X_1 \cup V_2$ cylinderwise away from V_0. Next $V_2 \searrow X_2$ by (1), because the pair V_2, X_2 is homeomorphic to V_0, X_0. Also

$$\begin{aligned} V_2 \cap \dot{M} &= V_0 \cap \dot{C} \text{ by Lemma 54 (i)} \\ &= C_0 \qquad \text{by (2)} \\ &= X_2 \cap \dot{M}, \text{ again by Lemma 54 (i).} \end{aligned}$$

Therefore $V_2 \overset{\circ}{\searrow} X_2$. Therefore $X_1 \cup V_2 \overset{\circ}{\searrow} X_1$ by excision because $X_1 \cap V_2 = X_2$. The proofs of (3, 4, 5, 6) are complete.

Let $T_1 = \text{trail}\ (X \cap V_1)$, $T_2 = \text{trail}\ (X \cap V_2)$ under some triangulation of the composite collapse (6). Let

$$\begin{aligned} W &= X_1 \cup T_1 \cup (X - V_1) \\ Y &= X_1 \cup T_2 \cup (X - V_1) \\ Z &= X_2 \cup T_2 \cup (X - V_1). \end{aligned}$$



We must now show that W, Y, Z satisfy the properties in the Lemma. First dim $W \leq x + 1$ and dim $Y, Z \leq x$ because

$$\begin{aligned}
\dim X_1 &\leq 1 + \dim X_0, \text{ because } X_1 \text{ is a collar on } X_0,\\
&\leq x, \text{ by the hypothesis } \dim (X \cap \dot{M}) < x.\\
\dim X_2 &\leq x - 1, \text{ because } X_2 \text{ is homeomorphic to } X_0.\\
\dim T_1 &\leq 1 + \dim (X \cap V_1), \text{ by Lemma 44}\\
&\leq 1 + x\\
\dim T_2 &\leq 1 + \dim (X \cap V_2)\\
&\leq x, \text{ by (4).}
\end{aligned}$$

Next $V_1 \cup X \cup C$ is $(X \cup C)$-inessential by (5), and therefore is C-inessential because X is C-inessential. Therefore W, Y, Z are also C-inessential because they are subspaces of $V_1 \cup X \cup C$. Next $X \subset W$ because

$$X = (X \cap V_1) \cup (X - V_1) \subset T_1 \cup (X - V_1) \subset W.$$

Next we have to show $W \cup C \overset{\circ}{\searrow} Y \cup C$. First observe that

$X_1 \cup T_1 \searrow X_1 \cup T_2$ by Lemma 45 Corollary; moreover this collapse is interior for the following reasons. If $T_1' = \text{trail}\ (X \cap V_1)$ under the first part of (6), then

$$\begin{aligned} T_1' \cap \dot{M} &= (X \cap V_1) \cap \dot{M}, \text{ by the property } \gamma, \\ &= X \cap V_0 \\ &= X_0. \end{aligned}$$

Also $T_1 - T_1' \subset X_1 \cup V_2$, because it comes from the second part of (6), and so $(T_1 - T_1') \cap \dot{M} \subset (X_1 \cup V_2) \cap \dot{M} = X_0$, by Lemma 54 (i). Therefore $(X_1 \cup T_1) \cap \dot{M} = X_0 = (X_1 \cup T_2) \cap \dot{M}$. Therefore $X_1 \cup T_1 \overset{\circ}{\searrow} X_1 \cup T_2$. We can add $C \cup \overline{X - V_1}$ to both sides because

$$\begin{aligned} T_1 \cap C &\subset V_1 \cap C \\ &= C_0, \text{ by Lemma 54 (i)} \\ &\subset T_2. \end{aligned}$$

$$\begin{aligned} T_1 \cap \overline{X - V_1} &\subset V_1 \cap \overline{X - V_1} \\ &= X \cap V_2 \text{ by (3)} \\ &\subset T_2. \end{aligned}$$

Therefore $W \cup C \overset{\circ}{\searrow} Y \cup C$ by excision.

Next $X_1 \overset{\Upsilon}{\searrow} X_2$ by Lemma 54 (ii). We can add $C \cup T_2 \cup \overline{X - V_1}$ to both sides because

$$X_1 \cap C = C_0 \subset X_2, \text{ by Lemma 54 (i)}$$

$$X_1 \cap T_2 \subset X_1 \cap V_2 = X_2.$$

$$X_1 \cap \overline{X - V_1} \subset X_1 \cap V_2 = X_2, \text{ by (3).}$$

Therefore $Y \cup C \overset{\Upsilon}{\searrow} Z \cup C$ by excision.

Next X, W, Y all meet $\dot{M}$ in the same set because $X \cap V_1$, $X_1 \cup T_1$, $X_1 \cup T_2$ all meet $\dot{M}$ in the same set, namely X_0. Finally

$Z \cap \dot{M} \subset C$ because

$$\begin{aligned}(X_2 \cup T_2) \cap \dot{M} &\subset V_2 \cap \dot{M} \\ &= V_0 \cap C \quad \text{by Lemma 54 (i)} \\ &\subset C, \text{ and} \\ (X - V_1) \cap \dot{M} &\subset (X - X_0) \cap \dot{M} \\ &\subset C, \text{ by definition of } X_0.\end{aligned}$$

The proof of Lemma 55 is complete.

<u>Proof of Theorem 21 Case 1.</u>

We are given $X \subset M$ to engulf, where $\dim (X \cap \dot{M}) < x$. X is C-inessential where C is a q-collapsible k-core, and

$$q, x \leqslant m - 3$$

$$q + x, 2x \leqslant m + k - 2.$$

By Lemma 55 choose W^{x+1}, Y^x, Z^x such that $X \subset W \cup C \overset{\circ}{\searrow} Y \cup C \overset{Y}{\searrow} Z \cup C$. Since Z is C-inessential and $Z \cap \dot{M} \subset C$ we can engulf $Y \subset D \searrow C$ by Lemma 53. Apply Lemma 42 to the situation $W \cup C \overset{\circ}{\searrow} Y \cup C \subset E$, and ambient isotope E to E_*, keeping $(Y \cup C) \cup \dot{M}$ fixed, so that $(W \cup C) \cup E_* \searrow E_*$. Let $D = W \cup E_* = (W \cup C) \cup E_*$. Then $X \subset D \searrow C$ because $D \searrow E \searrow C$. Also $\dim (D - C) \leqslant x + 1$, because

$$\begin{aligned}D - E_* &\subset W - Y, \\ \dim (D - E_*) &\leqslant \dim W = x + 1, \\ E_* - C &\cong E - C, \\ \dim (E_* - C) &= \dim (E - C) \\ &\leqslant \max (y, z + 1) \\ &= x + 1.\end{aligned}$$

Finally we can choose D such that $D \cap \dot{M} = (X \cup C) \cap \dot{M}$ by Lemma 43.

This completes the proof of our main engulfing theorem, Theorem 21.

<u>The uniqueness of piecewise linear structure of E^n.</u>

We conclude this chapter with an application of engulfing, a theorem of John Stallings which implies the uniqueness of structure of E^n. More precisely, given two piecewise linear structures (= polystructures) on E^n (there are obviously infinitely many) then they are piecewise linearly homeomorphic, provided that they are piecewise linear manifold structures, and $n \neq 4$. The case $n = 1$ is trivial, $n = 2$ is classical, and $n = 3$ is the Hauptvermutung Theorem of Moise. We shall prove the case $n \geq 5$.

<u>Question 1.</u> <u>Is the result true for $n = 4$?</u>

The proof below fails for the same reason that the Poincaré Conjecture proof fails.

<u>Question 2.</u> <u>Has E^n a non-piecewise linear manifold structure, $n \geq 4$?</u>

This is the Hauptvermutung for manifolds. The obvious case to look at is:

<u>Question 3.</u> <u>Is the double suspension of a Poincaré sphere topologically homeomorphic to S^5?</u>

By a Poincaré sphere we mean a closed 3-manifold M^3, which is a homology 3-sphere, but not simply-connected. The double suspension of M^3 is the same as the join $S^1 * M^3$. This cannot be a polyhedral sphere, because the link of a 1-simplex on the suspension ring S^1 is M^3, not S^3.

Call a manifold M <u>open</u> if it is non-compact without boundary, and its structure has a countable base. This is equivalent to saying there is a triangulation of M by an infinite complex, in which the links of vertices are (m - 1)-spheres.

The key idea of Stallings is the following definition. Let M be 2-connected. We call M <u>1-connected at infinity</u> if given compact $P \subset M$ there is a larger compact $Q \subset M$ (Q is not necessarily a subpolyhedron) such that M - Q is 1-connected. This is equivalent, by the exact homotopy sequence, to saying that the pair (M, M - Q) is 2-connected. The property is topological, independent of any structure on M.

<u>Example 1.</u> If $m \geqslant 3$, E^m is 1-connected at infinity.

<u>Example 2.</u> Whitehead's example M^3 (given in Example 1 after Theorem 19 Corollary 3) is a contractible open 3-manifold not 1-connected at infinity. In fact, if S^1 is the curve not contained in a ball, and Q is compact $\supset S^1$, then the fundamental group of M - Q is not finitely generated.

<u>Example 3.</u> The interior of Mazur's example M^4 (given after Whitehead's example) is a contractible open 4-manifold not 1-connected at infinity. In fact, if D^2 is the spine, and Q is compact $\supset D^2$, then the fundamental group of M - Q must contain $\pi_1(\dot{M}^4)$ as a subgroup. The dimension 4 is not significant in

Mazur's examples, because Curtis has given similar examples for dimensions ≥ 5.

It is no coincidence that we use the same examples to illustrate non-engulfability and non-connectedness at infinity; in fact the idea behind the proof of Stallings theorem is that connectedness at infinity implies a certain engulfability.

Theorem 22. (Stallings)

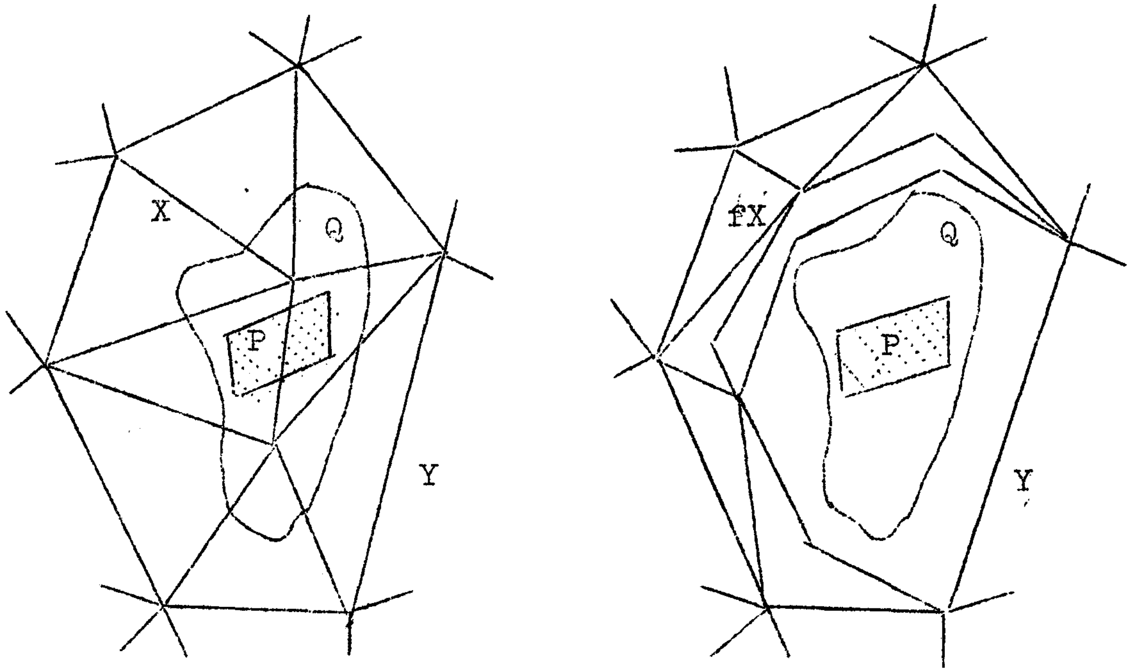
Let M^m be a contractible open manifold, 1-connected at infinity. If $m \geq 5$, then $M^m \cong E^m$.

Proof. Let P be a compact subspace of M. The main step of the proof is to show that P is contained in a ball. We cannot engulf P directly because it is not in general of codimension ≥ 3. Therefore we have to start indirectly by engulfing a 2-skeleton of M "away from P". So choose a triangulation of M by an infinite complex.

By hypothesis, choose compact $Q \supset P$ such that $M - Q$ is 1-connected. It is important to observe that Q is not a subpolyhedron in general (in order that the definition of 1-connectedness at infinity be a topological invariant). Forget P for the moment. Let

X = union of all 2-simplexes meeting Q

Y = union of all 2-simplexes not meeting Q.



In the diagram the 2-skeleton is represented by a 1-skeleton.

Since $(M, M - Q)$ is 2-connected the inclusion $X \subset M$ is homotopic in M, keeping $X \cap Y$ fixed, to a map $f:X \to M - Q$. We can assume f is piecewise linear, by using the relative simplicial approximation theorem in $M - Q$, keeping $X \cap Y$ fixed. (Since $M - Q$ is open in M it is also a piecewise linear manifold). By Theorem 15 ambient isotop the image fX in $M - Q$ keeping $X \cap Y$ fixed, so that $fX - (X \cap Y)$ is in general position with respect to $X \cap (M - Q)$. Since both are 2-dimensional in ≥ 5 dimensions, they are disjoint.

Therefore $fX \cap X = X \cap Y$. Let $C = fX \cup Y$. Then $C \subset M - Q$. C is connected because it is the image under $f \cup 1: X \cup Y \to fX \cup Y$ of $X \cup Y$, which is connected, being the 2-skeleton of connected M. Therefore $\pi_0(C) = 0$ and $\pi_1(M, C) = 0$. Therefore C is a 2-collapsible 1-core.

Now X is C-inessential in M because $X \cap C = X \cap Y$, and the inclusion $X \subset M$ is homotopic to $f: X \to C$ keeping $X \cap Y$ fixed. Putting $k = 1$, $q = x = 2$, $m \geqslant 5$ the hypotheses of Theorem 21 part (1) are satisfied. Therefore we can engulf X from C.

Let $U = M - P$. Then $U \supset M - Q \supset C$. Therefore by Lemma 41 there is a homeomorphism $h: M \to M$ isotopic to the identity keeping C fixed, such that $hU \supset X$. Therefore $hU \supset X \cup C \supset X \cup Y$. Therefore hP does not meet the 2-skeleton $X \cup Y$. We have achieved our first objective of pushing P off the 2-skeleton. This makes hP "effectively" of codim 3, and so we can now start engulfing hP in a ball.

More precisely, notice that since hP is compact it does not meet a neighbourhood of $X \cup Y$. Choose a second derived of M such that hP does not meet the second derived neighbourhood of $X \cup Y$. Therefore hP is contained in the complementary second derived neighbourhood of the dual $(m - 3)$-skeleton. Again using the compactness, hP is contained in the second derived neighbourhood N of some compact subspace Z of the dual $(m - 3)$-skeleton. N is now a regular neighbourhood of Z, because Z is compact, and so $N \searrow Z$.

By Theorem 19 Corollary 1 Z is contained in a ball.

Therefore N is contained in a ball, B say, by Lemma 37. Therefore $P \subset h^{-1}B$, because $hP \subset N \subset B$. We have completed the main step of the proof, which was to show that any compact subspace is contained in a ball.

We now use this result to cover M by an ascending sequence of balls $\{B_i\}$ as follows. Choose a triangulation of M. Since M is connected the triangulation is countable, and so order the simplexes $A_1, A_2, \ldots$. Define $B_1 = \overline{st}(A_1, M)$. For $i > 1$, define B_i inductively to be a regular neighbourhood of a ball containing $A_i \cup B_{i-1}$. Then $\{B_i\}$ is an ascending sequence of balls, each contained in the interior of its successor, such that $\bigcup B_i = \bigcup A_i = M$. The proof of Theorem 22 is completed by:

Lemma 56.

If M^m is the union of an ascending sequence of balls, each in the interior of its successor, then $M^m \cong E^m$.

Proof. Let $E^m = \bigcup \Delta_i$, the ascending sequence of m-simplexes, each in the interior of its successor. Choose a homeomorphism $f_1 : B_1 \to \Delta_1$, and inductively extend f_{i-1} to $f_i : B_i \to \Delta_i$ by the combinatorial annulus theorem (Theorem 8 Corollary 3) which says that $\overline{B_i - B_{i-1}} \cong \overline{\Delta_i - \Delta_{i-1}} \cong S^{m-1} \times I$.

The two corollaries to Theorem 22 are also due to Stallings.

Corollary 1. *The piecewise linear structure of M^m $m \geqslant 5$, is unique up to homeomorphism.*

<u>Corollary 2.</u> <u>Let M^m, Q^q be contractible open manifolds. If $m + q \geqslant 5$ then $M \times Q \cong E^{m+q}$.</u>

This result is interesting in view of the non-trivial examples above.

<u>Proof.</u> It suffices to show that $M \times Q$ is 1-connected at infinity, and this is done using algebraic topology. First, if $m \geqslant 2$ then M^m has one end because $H_f^1(M) \cong H_{n-1}(M) = 0$.

Therefore if both $m, q \geqslant 2$ we can find arbitrarily large compact subspaces $A \subset M$, $B \subset Q$ such that $M - A$, $Q - B$ are connected. Therefore

$$(M \times Q) - (A \times B) = M \times (Q - B) \cup (M - A) \times Q$$

is 1-connected by Van Kampen's Theorem, because in the free product with amalgamation both sides are killed by the amalgamation.

On the other hand if $m > q = 1$, then Q = the real line, and so choose B to be an arbitrarily large interval. Then $(M \times Q) - (A \times B)$ is homotopy equivalent to two copies of M sewn along $M - B$, which again is 1-connected by Van Kampen's Theorem.

While discussing E^m, we mention the analogous result to Theorem 10 for spheres.

<u>Lemma 57.</u>

<u>Any orientation preserving homeomorphism of E^m is ambient isotopic to the identity.</u>

<u>Proof.</u> Given a homeomorphism f, first ambient isotop f to g, where g keeps a ball B^m fixed, as in the proof of Theorem 10.

Now embed $E^m - \overset{\circ}{B}$ in a simplex Δ^m onto the complement of the barycentre $\hat{\Delta}$. The restriction $g|E^m - \overset{\circ}{B}$ extends to a continuous-homeomorphism $h:\Delta \to \Delta$ by mapping $\hat{\Delta} \to \hat{\Delta}$, that keeps $\dot{\Delta}$ fixed and is piecewise linear except at $\hat{\Delta}$. By Alexander's Theorem (Lemma 16) h is isotopic to 1 by an isotopy $H:\Delta \times I \to \Delta \times I$ that keeps $\dot{\Delta}$ and $\hat{\Delta}$ fixed, and is piecewise linear except on $\hat{\Delta} \times I$. Therefore the restriction of H to $\Delta - \hat{\Delta}$ determines a piecewise linear ambient isotopy of E^m moving g to 1.

<u>Remark.</u> Let P denote the group of piecewise linear homeomorphisms of E^m, and let L denote the subgroup of linear homeomorphisms, which deformation retracts onto the orthogonal group. Therefore both P, L have two components, corresponding to the two orientations of E^m. However this is a deceptive remark, because the Lie group topology on L is not the same as the topology induced from P. The higher homotopy groups of P are not known, but they are known to differ from those of L.

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1963 (Revised 1966)

Seminar on Combinatorial Topology

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Chapter 8 : EMBEDDING AND UNKNOTTING

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Chapter 8 : EMBEDDING AND UNKNOTTING

In this chapter we wish to classify embeddings of one manifold in another. "Classify" means sort into equivalence classes and then list the classes. The natural equivalence relation to use is ambient isotopy, because this has the same geometric quality as the embeddings. In Chapter 5 we saw that ambient isotopy was the same as isotopy. Listing is done by means of algebra, and the way to pass into algebraic topology is via homotopy theory. Geometrically the notion of homotopy is a horrible idea, because during a homotopy a nice embedding gets all mangled up. But the virtue of homotopy theory is that the homotopy classes of maps are often finite or finitely generated, and frequently computable, and so out of the mess we get something interesting. Therefore our classification technique will be to map the ambient isotopy classes of embeddings (geometry) into the homotopy classes of maps (algebra). If this map is an isomorphism then the algebra classifies the geometry; if not then we have a knot theory to play with.

Definition. Let M be a closed manifold and Q a manifold without boundary (open or closed). We say M unknots in Q if any two embeddings $M \subset Q$ are

$$\text{ambient isotopic} \Longleftrightarrow \text{homotopic.}$$

Otherwise we say M knots in Q.

Example (i) The classical example is S^1 knots in S^3. A knotted curve is homotopic, but not isotopic, to a circle. Similarly S^m knots in S^{m+2}, $m \geqslant 1$, and this kind of knotting is characteristic of codimension 2.

Example (ii) If $q - m \geqslant 3$ then S^m unknots in S^q, by Corollary 2 of Theorem 9 in Chapter 4.

It is the latter example that we want to generalise to arbitrary manifolds, and in Corollary 1 below we give sufficient conditions for M to unknot in Q. While proving an unknotting theorem it is natural to prove an embedding theorem in the same context, the relation between the two being explained as follows. Let

$\mathrm{Iso}(M \subset Q)$ = ambient isotopy classes of embeddings (geometry)

$[M, Q]$ = homotopy classes of maps (algebra)

$[M \subset Q]$ = homotopy classes of embeddings (hybrid).

There are natural maps

$$\mathrm{Iso}(M \subset Q) \xrightarrow[\text{surjective}]{\mu} [M \subset Q] \xrightarrow[\text{injective}]{\lambda} [M, Q].$$

To say that M unknots in Q is the same as saying that μ is bijective. The main results of this Chapter are Theorems 23 and 24 below, which give sufficient conditions for λ and μ to be bijective. In other words conditions for there to be a classification isomorphism

$$\mathrm{Iso}(M \subset Q) \xrightarrow{\cong} [M, Q].$$

Remark. We think of a "knot" mapwise, as an isotopy class of embeddings. Other authors, notably Fox, prefer to think of a "knot" setwise as an isotopy (or homeomorphism) class of subsets. Clearly the mapwise definition is finer than the setwise, because potentially it gives more knots. Therefore our mapwise unknotting theorems are stronger. However our preference for a mapwise rather than a setwise approach is dictated by our aim to classify knots in terms of homotopy.

Statement of main theorems. Let M^m, Q^q be manifolds (with or without boundary). We shall always suppose that M is compact. We shall state the theorems in relative form; the absolute form can be deduced by putting $\dot{M} = \emptyset$. Throughout this chapter let

$$d = 2m - q.$$

The letter d stands for double-point dimension because this would be the dimension of the double points were an arbitrary map $M \to Q$ put in general position.

Embedding Theorem 23. (Irwin) Let $f: M \to Q$ be a map such that $f|\dot{M}$ is an embedding of $\dot{M}$ in $\dot{Q}$. Then f is homotopic to a proper embedding keeping $\dot{M}$ fixed, provided

$$\begin{cases} m \leq q - 3 \\ M \text{ is } d\text{-connected} \\ Q \text{ is } (d+1)\text{-connected.} \end{cases}$$

Remark. As usual we always assume everything to be piecewise linear, unless we explicitly draw attention to the contrary. However Theorem 23 is an exception. Because of relative simplicial approximation, it is only necessary to assume that f is a continuous map such that $f|\dot{M}$ is a piecewise linear embedding; we can still deduce the existence of a piecewise linear embedding homotopic to f. In other words this is the strongest way round:

continuous hypothesis $\Longrightarrow$ piecewise linear thesis.

In the following theorem everything is piecewise linear.

Unknotting Theorem 24. Let $f, g: M \to Q$ be two proper embeddings such that $f|\dot{M} = g|\dot{M}$. If f, g are homotopic keeping $\dot{M}$ fixed then they are ambient isotopic keeping $\dot{Q}$ fixed, provided

$$\begin{cases} m \leqslant q - 3 \\ M \text{ is } (d+1)\text{-connected} \\ Q \text{ is } (d+2)\text{-connected.} \end{cases}$$

In the absolute case the two theorems can be combined to give.

Corollary 1. Let M be closed and Q without boundary. Then M unknots in Q, and $\mathrm{Iso}(M \subset Q) \cong [M, Q]$, provided

$$\begin{cases} m \leqslant q - 3 \\ M \text{ is } (d+1)\text{-connected} \\ Q \text{ is } (d+2)\text{-connected.} \end{cases}$$

The proofs of the theorems are a mixture of the ingredients of the last four chapters, namely unknotting balls, covering isotopy, general position and engulfing, and we give the proofs at the end of this chapter. But before we give them, we deduce some more corollaries, make some remarks about further developments, suggest some problems, and give counterexamples to show that the dimensional restrictions are the best possible. We also illustrate in Theorem 25 how the theorems can be used to classify certain links of spheres in spheres, and knots of spheres in solid tori. First the corollaries: they follow immediately from the statement of the theorems, and are obtained by specialising M or Q. In the first corollary we put Q equal to Euclidean space.

Corollary 2. Any closed k-connected manifold M^m, $k \leq m - 3$, can be embedded in E^{2m-k}, and unknots in one higher dimension.

In particular any homeomorphism $M \to M$ can be realised by an ambient isotopy of E^{2m-k+1}, in the same way that S^1 can be embedded in E^2, and any homeomorphism of S^1 can be realised by an ambient isotopy of E^3 (but not of E^2).

The next corollary is obtained by putting M equal to a sphere, and is a higher dimensional analogue of the Sphere Theorem.

Corollary 3. If Q^q is $(2m-q+1)$-connected, where $m \leq q - 3$, then any element of $\pi_m(Q)$ can be represented by an m-sphere embedded in $\mathring{Q}$. If Q is one higher connected then $\pi_m(Q)$ classifies the ambient isotopy classes of $S^m \subset \mathring{Q}$, provided $m > 1$.

By Theorem 24 the ambient isotopy classes are the same as the homotopy classes, and since $m > 1$ there is no base point trouble. If $m = 1$ then the isotopy classes are classified by the free homotopy classes, in other words the conjugacy classes of $\pi_1(Q)$. A special case of Corollary 3 is:

Corollary 4. *If Q^q is m-connected, $m \leq q - 3$, then any two embeddings $S^m \subset \mathring{Q}$ are ambient isotopic.*

The next corollary is obtained by putting M equal to a disk, and is in some ways a higher dimensional analogue of Dehn's Lemma and the Loop Theorem.

Corollary 5. *Let $S^{m-1} \subset \dot{Q}$ and suppose S^{m-1} is inessential in Q. If Q is (2m-q+1)-connected, where $m \leq q - 3$, then S^{m-1} can be spanned by a properly embedded disk $D^m \subset Q$. If Q is one higher connected then $\pi_m(Q)$ classifies the ambient isotopy classes of such disks, keeping $\dot{Q}$ fixed.*

The correspondence between isotopy classes of disks and elements of $\pi_m(Q)$ is not natural as in Corollary 2, but is obtained by choosing a base disk, D^m_* say, and associating with any other disk D^m the difference element of $\pi_m(Q)$ given by $D^m \cup D^m_*$. This time the case m = 1 is not exceptional because we can choose a fixed base point on S^0.

We now make some remarks about the two main theorems.

Remark 1: Hudson's improvements.

John Hudson has improved both theorems by weakening the hypotheses: instead of requiring the manifolds to be connected he requires only the maps to be connected. We say that the map $f:M \to Q$ is k-connected if the pair (F, M) is k-connected where F is the mapping cylinder of f. This is equivalent to saying that f induces isomorphisms $\pi_i(M) \xrightarrow{\cong} \pi_i(Q)$ for $i < k$, and an epimorphism $\pi_k(M) \to \pi_k(Q)$. As before let

$$d = 2m - q = \text{double-point dimension}$$

$$t = 3m - 2q = \text{triple-point dimension.}$$

Hudson's improvement in the Embedding Theorem 23 is to replace

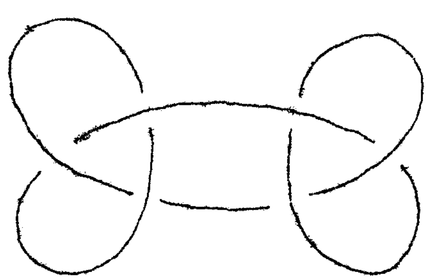
$$\left.\begin{array}{l} M \text{ d-connected} \\ Q \text{ (d+1)-connected} \end{array}\right\} \text{ by } \left\{\begin{array}{l} f \text{ (d+1)-connected} \\ M \text{ (t+1)-connected.} \end{array}\right.$$

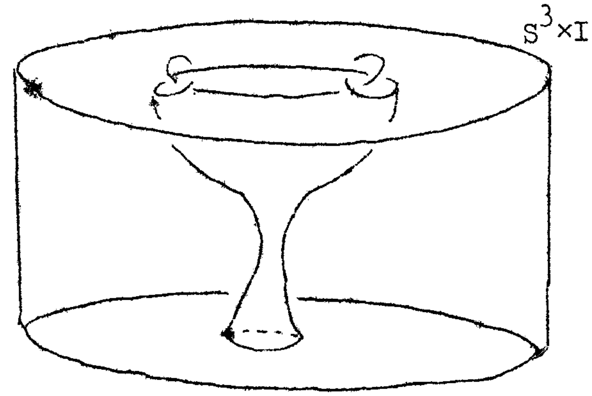
and in the Unknotting Theorem 24 to replace

$$\left.\begin{array}{l} M \text{ (d+1)-connected} \\ Q \text{ (d+2)-connected} \end{array}\right\} \text{ by } \left\{\begin{array}{l} f \text{ (d+2)-connected} \\ M \text{ (t+3)-connected.} \end{array}\right.$$

In both cases the connectivity of M implies the same for Q because $t \leqslant d - 3$ and so f induces isomorphisms of homotopy groups in the range concerned.

Hudson's proofs are too long to give here, and so we content ourselves with proving the theorems as stated. His main idea is combine the techniques given here with those developed by Haefliger for the smooth case. Another basic idea is to use concordance. Two embeddings f, $g:M \to Q$ are called concordant if there is a proper embedding $F:M \times I \to Q \times I$ that agrees with f at the top and g at the bottom; there is no requirement that F should be level-preserving in between, as there is in isotopy. In codimension 2 concordance is strictly weaker than isotopy: for example the reef knot





is concordant to a circle by a locally flat concordance, because it bounds a locally flat disk in the 4-ball, but the reef knot is not isotopic to a circle by a locally flat isotopy. However in codimension $\geqslant 3$ Hudson has shown that two embeddings are

$$\text{concordant} \iff \text{isotopic},$$

and so the unknotting theorem becomes a corollary of the embedding theorem. However we shall prove the two separately.

Remark 2. Codimension 1.

Our results are essentially unknotting results in codimension ≥ 3. The situation in codimension 2 is fundamentally different, because knotting occurs and is detected by the fundamental group. In codimension 1 the situation is again different, because of orientation. S^n knots in E^{n+1} because two embeddings with opposite orientation are homotopic but not isotopic, and therefore $\mathrm{Iso}(S^n \subset E^{n+1})$ contains at least two elements. We do not know whether there are more than two elements because the piecewise linear Schönflies Conjecture is still unsolved for $n \geq 3$. In fact the Schönflies Conjecture is equivalent to:

Conjecture: $\mathrm{Iso}(S^n \subset E^{n+1})$ has two elements.

Remark 3. Codimension 0.

Again the situation in codimension zero is quite different, and there are many more unsolved problems. If M is a closed manifold then $\mathrm{Iso}(M \subset M)$ is a group, namely the quotient of the group of all homeomorphisms of M by the component of the identity. It is called the homeotopy group of M. Only three examples of homeotopy groups are known.

Example (i). The homeotopy group of S^n is Z_2 by Theorem 10 of Chapter 4. The isotopy class of a homeomorphism is determined by the degree ± 1. In other words

$$\mathrm{Iso}(S^n \subset S^n) \cong [S^n \subset S^n] \cong Z_2.$$

Example (ii). Gluck has shown that the homeotopy group of $S^1 \times S^2$ is $Z_2 \times Z_2 \times Z_2$. The first two factors correspond to orientation reversals of S^1 and S^2, and the third factor Z_2 is generated by the homeomorphism $h: S^1 \times S^2 \to S^1 \times S^2$ given by $h(\theta, x) = (\theta, \rho_\theta x)$, where ρ_θ is rotation of S^2 through angle θ about the poles. Recently Browder has proved (unpublished) the same result for $S^1 \times S^n$, $n \geqslant 3$.

Example (iii). It follows from a theorem of Baer that the homeotopy group of a 2-manifold is isomorphic to the automorphism group of $\pi_1(M)$ modulo inner automorphisms. In each of these three cases the manifold unknots in itself, but the following example shows that this is not true in general.

Example (iv). Browder has shown that $S^3 \times S^5$ knots in itself, although the homeotopy group of $S^3 \times S^5$ is not yet known. He gives a homeomorphism h of $S^3 \times S^5$ onto itself that is homotopic but not isotopic to 1. We sketch the proof. Choose an element $\alpha \in \pi_3(SO(6))$, $\cong Z$, choose a smooth representative $f \in \alpha$, and use f to twist the fibres of the product bundle $S^3 \times S^5 \to S^3$. The result is a smooth fibre-homeomorphism h of $S^3 \times S^5$ onto itself. We claim that if α is a multiple of 24 then h is fibre-homotopic to 1, because if F_5 denotes the space of maps of S^5 to itself of degree 1, then $\pi_3(F_5) \cong \pi_8(S^5) \cong Z_{24}$, and so α is killed by the homomorphism $\pi_3(SO(6)) \to \pi_3(F_5)$. To place outselves in the piecewise-linear category choose a piecewise linear homeomorphism h^1 concordant to h. Browder then shows that h is not topologically concordant to 1, and therefore h^1 is not piecewise linearly isotopic to 1. To prove the non-concordance let T_h denote the mapping torus, obtained from $S^3 \times S^5 \times I$ by identifying $(x, 0) = (hx, 1)$ for all $x \in S^3 \times S^5$. If h were concordant to 1 then T_h would be topologically homeomorphic to $T_1 = S^3 \times S^5 \times S^1$. But it transpires that $\alpha \in Z$ classifies the Pontrjagin class $p_1(T_h)$, and so if $\alpha \neq 0$ then the rational Pontrjagin class of T_h is non-zero. But the rational Pontrjagin class of $S^3 \times S^5 \times S^1$ is zero, and is a topological invariant, and so we have a contradiction.

Example (v).

In smooth theory it is well known that a manifold can knot in itself. For example the piecewise linear homeotopy group of S^6 is Z_2, but the smooth homeotopy group of S^6 is the dihedral group D_{28}. The orientation preserving subgroup Z_{28} corresponds to exotic 7-spheres, by using the homeomorphisms of S^6 to glue two 7-balls together. In piecewise linear theory, on the other hand, there are no exotic spheres (at any rate in dimension ≥ 5) by the Poincaré Conjecture, which we shall prove in the next chapter.

Remark 4. Higher homotopy groups $\pi_i(M \subset Q)$.

Our so called classification of embeddings of M in Q has only touched the surface of the problem. More generally we can study the space $(M \subset Q)$ of all embeddings of M in Q, regarded either as a piecewise linear space (as in Chapter 2) or as a semi-simplicial complex. In particular we can study the higher homotopy groups $\pi_i(M \subset Q)$. So far in Theorem 24 we have only said something about the zero homotopy group

$$\pi_0(M \subset Q) = \mathrm{Iso}(M \subset Q).$$

For example we might generalise Theorem 9 the unknotting of spheres by:

Conjecture. $\pi_i(S^m \subset S^q) = 0$, provided $i + m \leq q - 3$.

Remark 5. An obstruction theory.

In the critical dimension, when the map is just not sufficiently connected for unknotting, Hudson has developed an obstruction for homotopic maps to be isotopic, with the obstruction in a quotient of the first non-vanishing homology group of the map (with certain coefficients). One would like to develop a more general obstruction theory, and fit it into an exact sequence, perhaps including the terms $\pi_i(M \subset Q) \to \pi_i[M, Q]$, $i \geq 0$. In Corollary 2 to Theorem 25 below we give a non-trivial example that looks as though it ought to fit into an exact sequence.

We now discuss counterexamples to show that the dimensional restrictions in the two main theorems are the best possible. In each of the six cases we relax a single hypothesis by one dimension, and show that the theorem then becomes false.

Embedding theorem	1 Codimension 2	2 M only (d+1)-connected	3 Q only d-connected
Unknotting theorem	4 Codimension 2	5 M only d-connected	6 Q only (d+1)-connected.

Counterexample 1. This is the only one of the six cases where the counterexample is conjectured rather than proved. Let D^2 be a disk, and Q^4 a contractible 4-manifold with non-simply connected boundary. Let

$$f: D^2 \to Q^4$$

be a map such that f embeds $\dot{D}^2$ onto an essential curve in $\dot{Q}^4$. Such a map exists because the curve is inessential in Q^4. By the Conjecture in Example 6 after Theorem 21 in Chapter 7, the curve does not bound a non-singular disk in Q^4 and so f cannot be homotopic to an embedding keeping the boundary fixed. Notice that D^2 and Q^4 satisfy the connectivity conditions because they are both contractible.

Counterexample 2. Let m be a power of 2 and $m \geq 4$. Let $M = P^m$, real projective space, and let $Q = E^{2m-1}$. Then $d = 1$ and P just fails to be 1-connected. Meanwhile Q is 2-connected and the codimension is ≥ 3. Since P^m cannot be embedded in E^{2m-1}, no map $P^m \to E^{2m-1}$ can be homotopic to an embedding.

Counterexample 3. (Irwin) Let $m \geq 3$, and let $f: S^m \to S^{2m}$ be a map with exactly one double point, where the two sheets of fS^m cross transversally. We shall show that such a map exists in a moment. If m were allowed to equal 1 then the figure 8 would give a correct picture. Let Q^{2m} be a regular neighbourhood of fS^m in S^{2m}. We claim that $f:S^m \to Q^{2m}$ cannot be homotopic to an embedding. Notice that $d = 0$ and S^m is 0-connected, but Q just fails to be 1-connected. In fact $\pi_1(Q) = Z$, generated by a loop starting from the double point along one sheet and back along the other. Notice also that the codimension is ≥ 3.

Proof that f exists: Write $S^m = D_0^m \cup S^{m-1} \times I \cup D_1^m$. Embed the two disks transversally as $D_0^m \times 0$ and $0 \times D_1^m$ in a little ball $D_0^m \times D_1^m \subset S^{2m}$. Let B^{2m} be the complementary ball, and extend the embedding of the boundaries of the disks $S^{m-1} \times \dot{I} \to \dot{B}^{2m}$ to a map $S^{m-1} \times I \to B^{2m}$. Now use Theorem 23 to homotop this map into a proper embedding, keeping the boundary fixed. The result gives what we want.

Proof that $f \not\simeq$ embedding. Suppose on the contrary that f was homotopic to an embedding $g:S^m \to Q^{2m}$. Let P^{2m} denote the universal cover of Q, which consists of a countable number of copies of $S^m \times D^m$, plumbed together in sequence. We can lift f, g to a countable number of maps f_i, $g_i:S^m \to P$, $i \in Z$. By construction each f_i is an embedding, and, for each i, f_iS^m cuts $f_{i+1}S^m$ transversally once. Meanwhile g_iS^m is disjoint from $g_{i+1}S^m$ because g was an embedding. Here we have a contradiction, because the intersection of the f's is homological, and must be algebraically the same as the g's because $f_i \simeq g_i$, each i.

More precisely, if ξ is a generator of $H_m(S^m)$, and

$$D : H_m(P) \xrightarrow{\cong} H_c^m(P, \dot{P})$$

is Poincaré Duality (where H_c stands for compact cohomology) then in $H_c^{2m}(P, \dot{P})$ we have the contradiction

$$0 = Dg_i\xi \cup Dg_{i+1}\xi = Df_i\xi \cup Df_{i+1}\xi \neq 0.$$

Counterexample 4. S^m knots in S^{m+2}, $m \geq 1$.

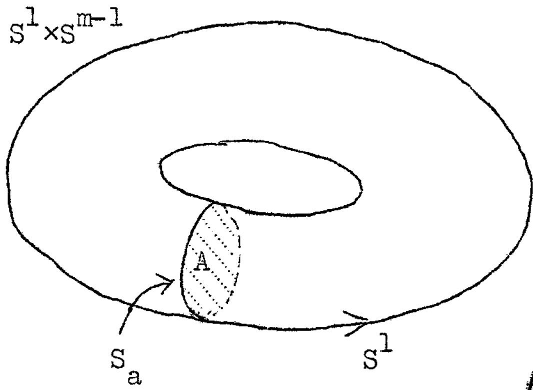
Counterexample 5. (Hudson) $S^1 \times S^{m-1}$ knots in S^{2m}, $m \geq 3$.

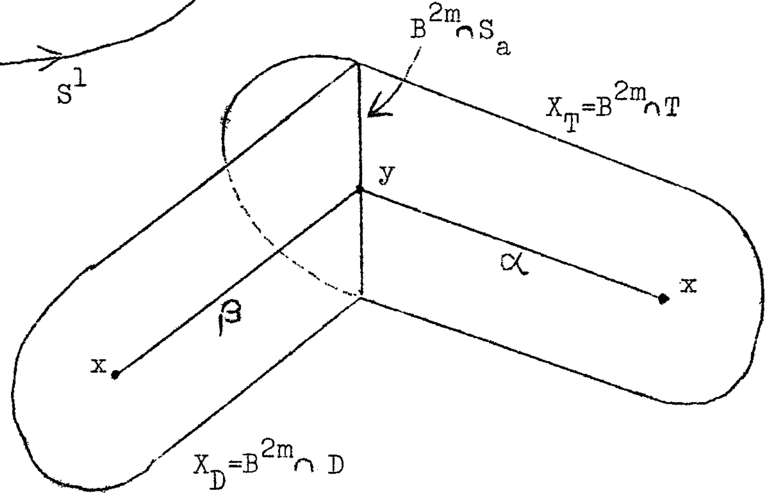
Notice that $d = 0$ and $S^1 \times S^{m-1}$ just fails to be $(d+1)$-connected.

Proof of the knotting. Given an embedding $f : S^1 \times S^{m-1} \to S^{2m}$ we shall define a knotting number $k(f) \in Z_2$, and prove that it is an invariant of the ambient isotopy class of f. We shall then describe two embeddings f_0, f_1 with knotting numbers 0, 1 respectively.

Let $T = f(S^1 \times S^{m-1})$, the embedded torus. Given $a \in S^1$, let $S_a = f(a \times S^{m-1})$, an embedded $(m-1)$-sphere.

Lemma 58. There is an m-ball A in S^{2m}, spanning S_a and not meeting T again.





Proof. Since $m \geqslant 3$, S_a is unknotted in S^{2m}, and so can be spanned by an m-ball, D say. By Theorem 15 ambient isotop D, keeping $\dot{D} = S_a$ fixed, until $\mathring{D}$ is in general position with respect to T. Therefore $\mathring{D}$ meets T in a finite number of points, which we can remove one by one as follows. Let x be one of these points. Choose $y \in S_a$, choose an arc $\alpha \subset T$ joining xy, and an arc β in D joining xy. We can choose the arcs so as to avoid the other points of $T \cap \mathring{D}$ and so as to meet S_a only in y. Let Δ^2 be a 2-disk in S^{2m} spanning $\alpha \cup \beta$, and not meeting $T \cup D$ again (this is possible by general position since $m \geqslant 3$). Triangulate everything and take a second derived neighbourhood B^{2m} of Δ^2 in S^{2m}, which is a ball since Δ^2 is collapsible. Consider the set

$$X = B^{2m} \cap (T \cup D).$$

Now X consists of three m-balls glued together along the common face $B^{2m} \cap S_a$, and embedded in B^{2m} with one self intersection at the interior point x. Let

$$\dot{X} = \dot{B}^{2m} \cap (T \cup D)$$

which consists of the three complementary faces glued together along the common (m-2)-sphere $\dot{B}^{2m} \cap S_a$. Let Y be a cone on $\dot{X}$ in B^{2m}. If we replace X by Y, behold we have removed the intersection point x; but we have moved the torus meanwhile, and so we must now move it back. We can write

$$X = X_T \cup X_D,$$

where X_T, X_D are the m-balls $B^{2m} \cap T$, $B^{2m} \cap D$. Similarly

$$Y = Y_T \cup Y_D$$

where Y_T, Y_D are the cones on the (m-1)-sphere $\dot{B}^{2m} \cap T$ and the (m-1)-ball $\dot{B}^{2m} \cap D$. Since X_T, Y_T are two m-balls in B^{2m} with the same boundary, and since $m \geqslant 3$, by Theorem 9 Corollary 1 we can ambient isotop Y_T onto X_T keeping $\dot{B}^{2m}$ fixed. This moves the torus back into position.

Meanwhile the isotopy carries Y_D to Y'_D say. Then replacing D by

$$(D - X_D) \cup Y'_D$$

has the effect of reducing the intersections of $T \cap \mathring{D}$ by one. After a finite number of steps we obtain A as required. This completes the proof of Lemma 58, and we return to the construction of the knotting number k(f).

Choose three points a, b, $c \in S^1$. By the lemma choose three m-balls A, B, C spanning S_a, S_b, S_c respectively, and not meeting the torus again. We can choose the balls in general position relative to one another, and so each pair cuts transversally in a finite number of points. Let AB denote the number of intersections of A and B, modulo 2. Define

$$k(f) = AB + BC + CA.$$

We have to show that k is independent of the choices made. First we show k is independent of A. Let [bc] denote the interval of S^1 not containing a, and let S_{BC} denote the immersed m-sphere

$$S_{BC} = B \cup f([bc] \times S^{m-1}) \cup C.$$

Then the homological linking number mod 2 of S_a^{m-1} and S_{BC}^m in S^{2m} is given by

$$L(S_a, S_{BC}) = AB + AC,$$

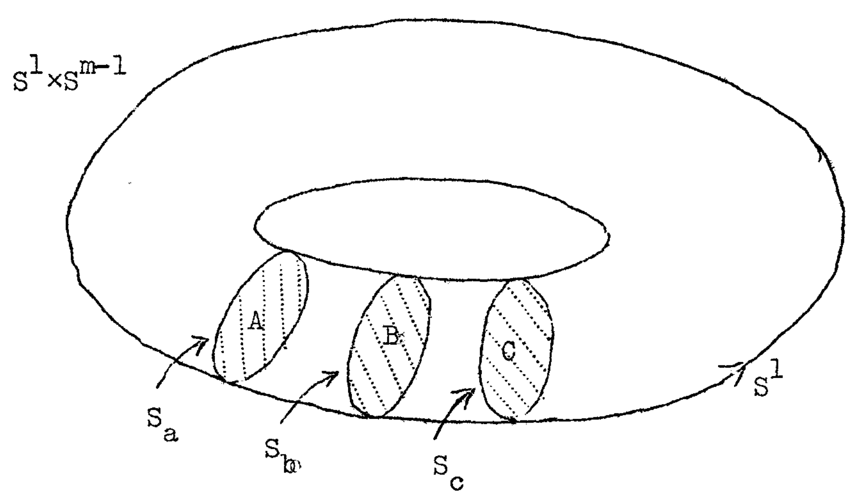
because A does not meet $f([bc] \times S^{m-1})$. Therefore

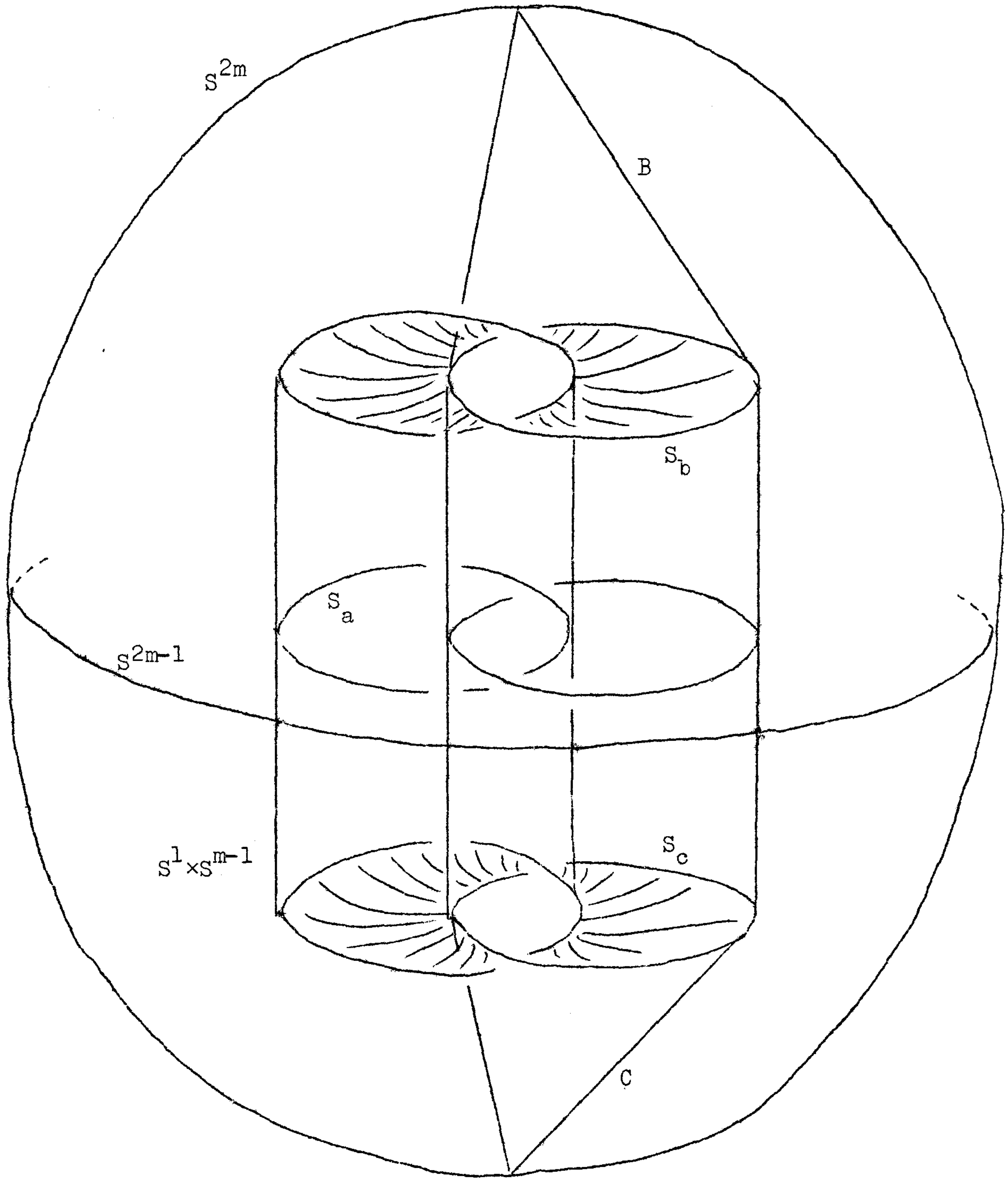
$$k(f) = L(S_a, S_{BC}) + BC,$$

which is independent of A. Also k is independent of a, because if we move a (without meeting [bc]), then the resulting isotopy of S_a does not alter the linking number $L(S_a, S_{BC})$. Similarly k is independent of B, C, b, c. Therefore k is well-defined. Clearly k is an ambient isotopy invariant, because any ambient isotopy carries with it the whole construction of A, B, C.

Finally we have to produce embeddings f_0, f_1 with different knotting numbers. Define the embedding $f_0 : S^1 \times S^{m-1} \to S^{2m}$ to be the obvious one given by the boundary of an embedded $S^1 \times D^m$. Then we can draw A, B, C disjoint as in the picture below. Therefore $k(f_0) = 0$.

Construct the embedding $f_1 : S^1 \times S^{m-1} \to S^{2m}$ as follows.




S^{2m}
B
S_b
S_a
S^{2m-1}
S_c
$S^1 \times S^{m-1}$
C

First link two $(m-1)$-spheres in the equator S^{2m-1} with linking number 1. Then collar each of these into each hemisphere. Finally connect the tops of the two collars by a cylinder $I \times S^{m-1}$ in the tropic of Cancer, and connect the bottoms of the two collars by similar cylinder in the tropic of Capricorn. The only moment of doubt occurs as to whether two linking spheres can be connected by a cylinder, but this doubt is resolved by glancing at the image of the diagonal $\times$ I under the identification map

$$S^{m-1} \times S^{m-1} \times I \to S^{m-1} * S^{m-1} = S^{2m-1}.$$

To compute $k(f_1)$, choose S_a to be one of the $(m-1)$-spheres in the equator, and choose S_b, S_c to be the top and bottom of the other collar. Form B by joining S_b to the north pole, and C by joining S_c to the south pole. Then $L(S_a, S_{BC}) = 1$, because we can compute it by spanning S_a with an m-ball in the equator, that meets the other $(m-1)$-sphere and hence also S_{BC}, in exactly one point. Meanwhile $BC = 0$ because B, C are disjoint. Therefore

$$k(f_1) = L(S_a, S_{BC}) + BC = 1.$$

This completes the proof of Counterexample 5.

Remark on knotted tori.

Hudson has shown that if the codimension is even, then his knotting number described above is in fact sufficient to classify the knots, but if the codimension is odd then an anologous knotting number in the integers Z is required. More generally he has proved that

$$\mathrm{Iso}(S^p \times S^{m-p} \subset S^{2m-p+1}) \cong \begin{cases} Z_2, & \text{codimension even} \\ Z, & \text{codimension odd} \end{cases}$$

provided $m \geq 3$ and $1 \leq p < m - p$. This is the critical dimension for knotting tori, because they unknot in all higher dimensions, by Corollary 1.

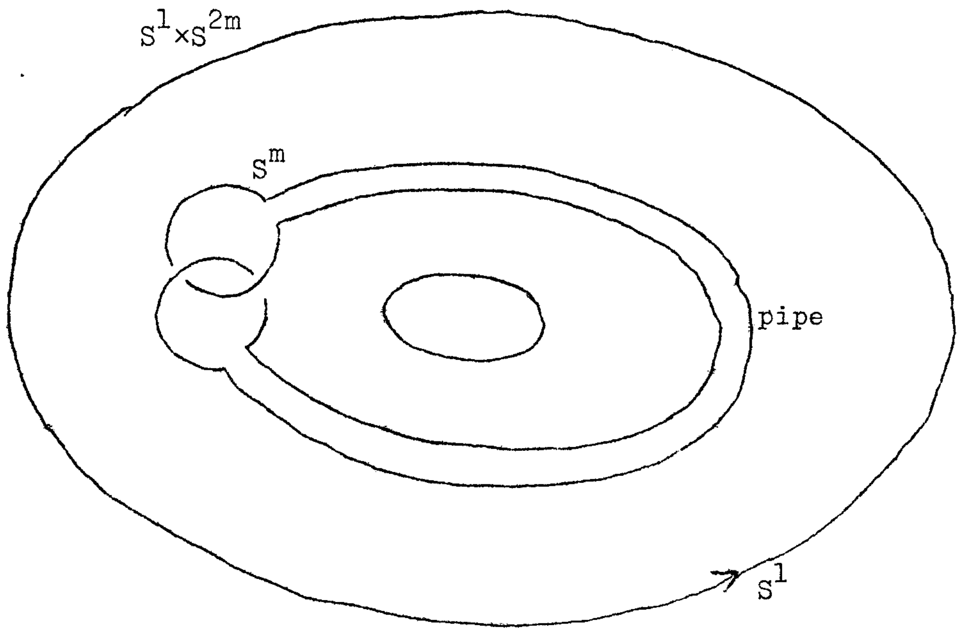
The case $p = 0$ turns out to be exceptional. Here the torus $S^0 \times S^m$ consists of two spheres, and the ambient isotopy classes of links of spheres in the critical dimension are classified by their linking number, as we shall show below:

$$\mathrm{Iso}(S^0 \times S^m \subset S^{2m+1}) \cong Z, \quad m \geq 2.$$

Counterexample 6. S^m knots in $S^1 \times S^{2m}$, $m \geq 2$.

Notice that $d = -1$ and $S^1 \times S^{2m}$ just fails to be $(d+2)$-connected.

Proof of the knotting. We shall give two embeddings $S^m \subset S^1 \times S^{2m}$ such that one bounds a disk and the other doesn't. Therefore they cannot be ambient isotopic, but they must be homotopic because any two maps are homotopic. It is trivial to choose the first one. For the other choose the embedding described in Example 3 after Theorem 19 in Chapter 7. It consists of two little linked m-spheres connected by a pipe round the S^1.



We showed in Chapter 7 that this embedding cannot bound a disk.

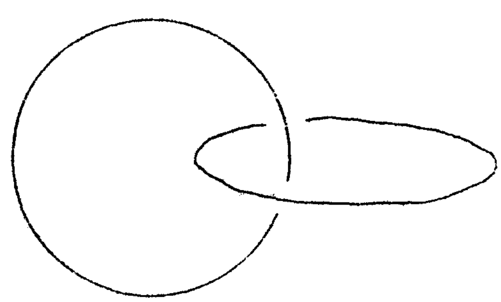
Remark. We shall shortly furnish a large class of alternative counterexamples by giving conditions for S^m to knot in the solid torus $S^r \times E^{q-r}$. The conditions are given in terms of homotopy groups of spheres, in Corollary 2 to Theorem 25 below. The simplest example is that S^4 knots in $S^3 \times E^4$. We shall show there are an infinite number of knots, although $\pi_4(S^3) = Z_2$. Notice that here $d = 1$ and $S^3 \times E^4$ just fails to be $(d+2)$-connected.

This completes the six counterexamples that were designed to show the dimensional restrictions in Theorems 23 and 24 were the best possible.

We want next to classify the links of two disjoint spheres S^m, S^p in a larger sphere S^q, up to ambient isotopy. The classical situation of two curves linking in S^3 is somewhat deceptive because knotting is confused with linking, but it does illustrate the three types of linking that can occur.

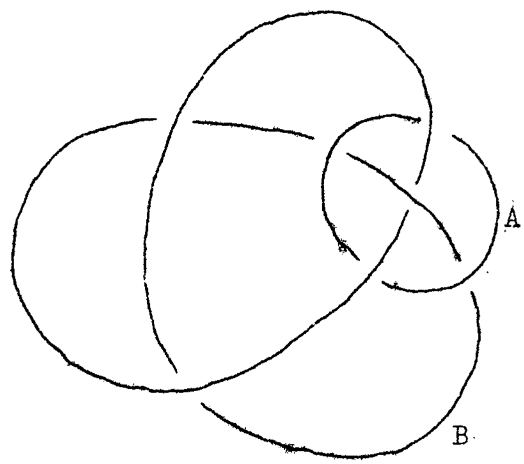
<u>(1) Homological linking</u>.

Each curve is non-homologous to zero in the complement of the other. By duality this is a symmetrical situation.



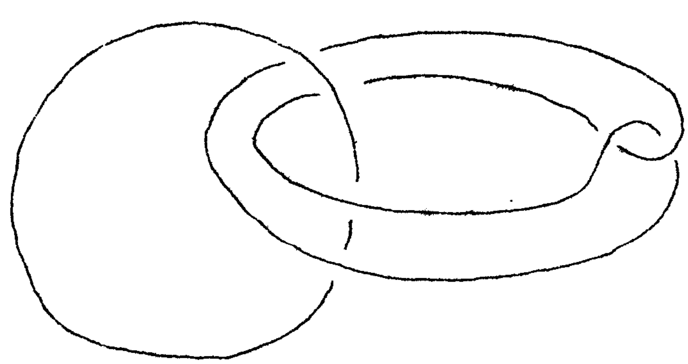
<u>(2) Homotopic linking</u>.

Here A, B are homoligically unlinked, but A is essential in the complement of B. This situation can be unsymmetric, because, as we have drawn them, B is inessential in the complement of A.



(3) Geometric linking.

Two curves are geometrically unlinked if they can be ambient isotoped with opposite hemispheres. We illustrate geometrically linked curves that are homotopically unlinked.



Summarising we have:

$$\begin{pmatrix}\text{homologically}\\ \text{linked}\end{pmatrix} \underset{\not\Longleftarrow}{\Longrightarrow} \begin{pmatrix}\text{homotopically}\\ \text{linked}\end{pmatrix} \underset{\not\Longleftarrow}{\Longrightarrow} \begin{pmatrix}\text{geometrically}\\ \text{linked.}\end{pmatrix}$$

In higher dimensions we shall stick to codimension $\geqslant 3$, so as to separate knotting and linking and be able to concentrate on the latter. Therefore we shall assume

$$m \leqslant p \leqslant q - 3,$$

so that each of S^m, S^p is unknotted in S^q.

There are three cases.

<u>Case (i) $m + p < q - 1$.</u> Then S^m, S^p are geometrically unlinked by Corollary 1 to Theorem 24.

<u>Case (ii) $m + p = q - 1$.</u> In this case homological linking can occur, and this is the only case in which it can occur. We shall show in Corollary 1 to Theorem 25 that the link is classified by the linking number, which is an integer. To be more precise there are two linking numbers which differ only by the sign $(-)^{mp+1}$. However we shall not bother to define the homology linking numbers, because they are special cases of the more general homotopy linking numbers.

<u>Case (iii) $m + p > q - 1$.</u> Homotopy linking can occur in both this and the previous case. We shall define the homotopy linking numbers, and show in Theorem 25 that one of them classifies the link, provided $2m + p \leqslant 2q - 4$.

<u>The homotopy linking numbers of S^m, $S^p \subset S^q$.</u>

Since each sphere is unknotted we have

$$S^m \subset S^q - S^p \cong S^{q-p-1} \times E^{p+1}$$
$$S^p \subset S^q - S^m \cong S^{q-m-1} \times E^{m+1}.$$

We assume that all three spheres S^m, S^p, S^q are oriented, and so orientations are induced on S^{q-p-1}, S^{q-m-1} (we shall examine these induced orientations more carefully in a moment). Therefore the link determines <u>homotopy linking numbers</u>

$$\alpha \in \pi_m(S^{q-p-1})$$
$$\beta \in \pi_p(S^{q-m-1}).$$

Notice that both these are in the (m+p-q+1)-stem, and we shall show in Lemma 62 that they have a common stable suspension (to within sign). We call α, β <u>stable</u> if they lie in stable homotopy groups. Recall that $\pi_i(S^j)$ is stable if $i \leq 2j - 2$. Therefore

$$\alpha \text{ is stable if } m + 2p \leq 2q - 4$$
$$\beta \text{ is stable if } 2m + p \leq 2q - 4.$$

Since $m \leq p$, we can have α unstable while β is stable, and this will be a particularly interesting situation; for example S^3, $S^4 \subset S^7$. Let Σ denote the suspension homomorphism, and Σ^{p-m} the composite suspension

$$\pi_m(S^{q-p-1}) \to \pi_{m+1}(S^{q-p}) \to \dots \to \pi_p(S^{q-m-1}).$$

When there is no confusion we shall abbreviate Σ^{p-m} to Σ.

Theorem 25. *Let S^m, $S^p \subset S^q$ be a link such that $m \leqslant p \leqslant q - 3$. If β is stable then α classifies the link. In other words*

$$\mathrm{Iso}(S^m,\ S^p \subset S^q) \xrightarrow[\alpha]{\cong} \pi_m(S^{q-p-1}).$$

Also $\beta = (-)^{m+p+q+pm}\Sigma\alpha$.

Before we prove the theorem we deduce a corollary and a couple of examples and prove two lemmas.

Corollary 1. *If $m + p = q - 1$, then*

$$\mathrm{Iso}(S^m,\ S^p \subset S^q) \cong \pi_m(S^m) \cong H_m(S^m) \cong Z,$$

and the homological linking number classifies the link.

Example (1). Two 50-spheres can be linked in 101, 100, 99, 98 dimensions, but in 97, 96 they become unlinked, and then can be linked again in 95, 94, ... ??? ..., 52. The explanation is that the words

link/unlink ≅ nonzero/zero

of certain stable homotopy groups, and the unlinking is 97, 96 correspond to the vanishing of the stable 4, 5 stems.

Example (2). There are exactly two links of S^9, $S^{10} \subset S^{16}$. One is geometrically unlinked, and the other is half-homotopically linked as in the second diagram above, because

$$\alpha \neq 0,\ \alpha \in \pi_9(S^5) \cong Z_2.$$
$$\beta = 0,\ \beta \in \pi_{10}(S^6) = 0.$$

Lemma 59. If S^m is unknotted in S^q, then an orientation preserving homeomorphism of S^q keeping S^m fixed is isotopic to the identity keeping S^m fixed.

Proof by induction on q, keeping the codimension fixed, the induction beginning trivially with $m = -1$. Let $h: S^q \to S^q$ be the given homeomorphism. Choose triangulation K, L of S^q, S^m and a vertex $x \in L$. Choose subdivisions such that $h: K_1 \to K_2$ is simplicial. Let B^q, B^m be the closed stars of x in K_1, L_1. Then h maps B^q, B^m linearly into $st(x, K)$, $st(x, L)$ and so by pseudo radial projection (see Lemma 8, Chapter 3) we can ambient isotop h to k, keeping S^m fixed, such that $kB^q = B^q$. Now $\dot{B}^m$ is unknotted in $\dot{B}^q$, since S^m is locally unknotted, and $k|\dot{B}^q$ is orientation preserving, and so by induction we can isotop $k|\dot{B}^q$ to the identity keeping $\dot{B}^m$ fixed. By Alexander's trick (c.f. the proof of Lemma 16) we can extend the isotopy to each of B^q, $S^q - \mathring{B}^q$ keeping S^m fixed, and so isotop k to the identity.

For the next lemma we want to compare links in spheres with knots in solid tori. Write $S^q = S^{q-p-1} * S^p$. Let $g: S^p \to S^q$ denote the embedding onto the right-hand end of the join, and let $e: S^{q-p-1} \times E^{p+1} \to S^q$ denote the embedding onto the complement. Then any embedding $f: S^m \to S^{q-p-1} \times E^{p+1}$ determines a link

$$\varphi(f) = (ef, g): S^m, S^p \to S^q.$$

<u>Lemma 60.</u> <u>If $p \leq q - 3$ (with no restriction on m) then φ induces an isomorphism</u>

$$\underline{\mathrm{Iso}(S^m \subset S^{q-p-1} \times E^{p+1}) \cong \mathrm{Iso}(S^m, S^p \subset S^q).}$$

<u>Proof.</u> Let $\approx$ denote ambient isotopic. If $f \approx f'$, then we can choose the ambient isotopy to have compact support, by Theorem 12 in Chapter 5, and it can be extended to S^q. Therefore $\varphi(f) \approx \varphi(f')$. Therefore φ induces a map, $\bar{\varphi}$ say, of isotopy classes $\bar{\varphi}$ is injective: for, given $\varphi(f) \approx \varphi(f')$, then the end of the ambient isotopy gives an orientation preserving homeomorphism keeping S^p fixed, which by Lemma 59 is isotopic to the identity keeping S^p fixed. The restriction of this to the complement of S^p gives $f \approx f'$. Finally $\bar{\varphi}$ is surjective: for given a link k, ambient isotop the embedding of S^p onto g (using $p \leq q - 3$), and hence $k \approx (ef, g)$ for some f.

<u>Proof of Theorem 25.</u>

S^m unknots in $S^{q-p-1} \times E^{p+1}$ because

$$\begin{aligned} d + 2 &= (2m - q) + 2 \\ &\leq q - p - 2 \text{ by the stability of } \beta. \end{aligned}$$

$$\begin{aligned} \text{Therefore } \mathrm{Iso}(S^m, S^p \subset S^q) &\cong \mathrm{Iso}(S^m \subset S^{q-p-1} \times E^{p+1}) \text{ by Lemma 60} \\ &\cong \pi_m(S^{q-p-1}) \text{ by Corollary 2 to Theorem 24.} \end{aligned}$$

There remains to show that $\beta = (-)^{m+p+q+mp}\Sigma\alpha$, but first we must be more explicit about our orientation conventions.

Suppose we are given orientations of S^m, S^p, S^q. We define the induced orientation on S^{q-p-1} in $S^q - S^p \cong S^{q-p-1} \times E^{p+1}$ as follows. At a point of S^p choose a local coordinate system in which S^p appears as a linear subspace. Choose axes $1, \ldots, q$ to give the orientation of S^q so that $1, \ldots, p$ gives the orientation of S^p. Then $p+1, \ldots, q$ determine an orientation in a transverse disk D^{q-p}, and hence induce the require orientation on $\dot{D}^{q-p} = S^{q-p-1} \subset S^{q-p-1} \times E^{p+1}$. To see that this is a topological invariant definition, observe that it can be expressed homologically: if $x \in H_q(S^q)$, $y \in H^p(S^p)$ are the given orientations, then $\delta y \in H^{p+1}(S^q, S^p)$ and the cap product

$$x \cap \delta y \in H^{q-p-1}(S^q - S^p)$$

gives the induced orientation. We use the given orientation on S^m and the induced orientation on S^{q-p-1} to define the linking number $\alpha \in \pi_m(S^{q-p-1})$. Similarly define β.

Suspended links. Suppose that we are given a link S^m, $S^p \subset S^q$, with linking numbers α, β. Let ΣS^m, $S^p \subset \Sigma S^q$ denote the link formed by suspending S^m and S^q, while keeping S^p the same. Orient the suspension ΣS^q by choosing axes $0, 1, \ldots, q$ at a point of S^q so that 0 points towards the north pole and $1, \ldots, q$ gives the orientation of S^q. Let α_*, β_* denote the linking numbers of the suspended link.

Lemma 61. $\alpha_* = (-)^p \Sigma\alpha$ and $\beta_* = \beta$.

Proof. First look at β_*. At a point of S^m we can choose axes so that $0, 1, \ldots, m$ orients ΣS^m and $0, 1, \ldots, q$ orients ΣS^q. Therefore $m+1, \ldots, q$ orients the same transverse disk as in the unsuspended link. Meanwhile S^p is unchanged. Therefore β is unchanged.

Now look at α_*. At a point of S^p we can choose axes so that $1, \ldots, p$ orients S^p and $0, 1, \ldots, q$ orients ΣS^q. By the prescribed rule we must reorder the axes so that $1, \ldots, p$ come first. Therefore this introduces a factor of $(-)^p$ into the orientation induced on the transverse disk by $0, p+1, \ldots, q$. For the transverse disk we can choose ΣD^{q-p}, the suspension of the transverse disk D^{q-p} in the unsuspended link. In the unsuspended link the class $\alpha \in \pi_m(S^{q-p-1})$ is determined by homotoping the embedding $S^m \subset S^q - S^p$ into a map $f: S^m \to \dot{D}^{q-p}$ say. In the suspended link we can homotop $\Sigma S^m \subset \Sigma S^q - S^p$ into the suspension $\Sigma f: \Sigma S^m \to \Sigma \dot{D}^{q-p}$, which determines the class $\Sigma\alpha \in \pi_{m+1}(S^{q-p})$. Adding in the factor $(-)^p$ we have

$$\alpha_* = (-)^p \Sigma\alpha.$$

<u>Lemma 62.</u> <u>If we are given a link such that $m = p \leqslant q - 3$ and α, β are stable then $\beta = (-)^{q-m}\alpha$.</u>

<u>Proof.</u> The link consists of disjoint embeddings $f: S^m \to S^q$ and $g: S^m \to S^q$, where

α is the class of $f: S^m \to S^q - gS^m$

β is the class of $g: S^m \to S^q - fS^m$.

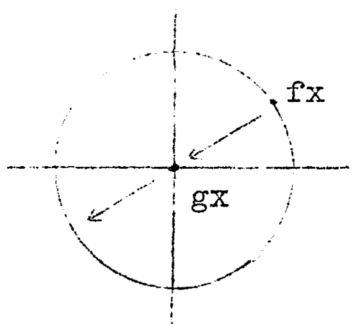
Write $S^q = S^m * S^{q-m-1}$, and let $j: S^m \to S^q$ be the inclusion of the left hand end of the join. Ambient isotop g onto j, and assume from now on that $g = j$. Then fS^m lies in the complement $S^q - S^m \cong E^{m+1} \times S^{q-m-1}$. If $e: E^{m+1} \times S^{q-m-1} \to S^{q-m-1}$ denotes projection, then $ef: S^m \to S^{q-m-1}$ represents α. Let $S^m \times S^{q-m-1}$ denote the torus half-way between the two ends of the join. Let $\Gamma(ef): S^m \to S^m \times S^{q-m-1}$ denote the graph of ef. Then both f and $\Gamma(ef)$ represent α, and since β is stable α classifies the link, by the part of Theorem 25 that we have proved already. Therefore we can ambient isotop f onto $\Gamma(ef)$ keeping gS^m fixed. Consequently assume $f = \Gamma(ef)$ from now on.

We have reached a situation where both spheres are embedded in the complement of the right hand end $S^q - S^{q-m-1} = S^m \times E^{q-m}$. More precisely

$$f: S^m \to S^m \times E^{q-m} \text{ is given by } fx = (x, efx)$$
$$g: S^m \to S^m \times E^{q-m} \text{ is given by } gx = (x, 0).$$

Let T be the antipodal map of S^{q-m-1}. There is a homeomorphism h of $S^m \times E^{q-m}$, isotopic to the identity such that

$$hf = g$$
$$hg = (1 \times T)f.$$



If we were content to have a topological homeomorphism, then h would be easy to describe: for each $x \in S^m$ merely translate $x \times E^{q-m}$ by the vector $-fx$. However such an h is not in general piecewise linear and so the best way to construct h is as follows. First ambient isotop g to $(1 \times T)f$ keeping fS^m fixed, which is possible because they are homotopic in the complement of fS^m, and the stability of α, β ensures unknotting. Then ambient isotop f to g keeping $(1 \times T)fS^m$ fixed, for similar reasons. Since the ambient isotopy can be chosen to have compact support, by Theorem 12, it can be extended to S^q, and so the link is unchanged. In the new position we see that, removing hfS^m,

$$\beta = [ehg] = [e(1 \times T)f] = [Tef] = T[ef] = T\alpha.$$

But the antipodal map T of S^{q-m-1} has degree $(-)^{q-m}$. Therefore $\beta = (-)^{q-m}\alpha$.

<u>Completion of the proof of Theorem 25.</u>

We are given S^m, $S^p \subset S^q$ with linking numbers α, β. Suspend the smaller sphere $p - m$ times, to give a link $\Sigma^{p-m}S^m$, $S^p \subset \Sigma^{p-m}S^q$ with linking numbers α_*, β_* say. Then

$$\alpha_* = (-)^{p(p-m)}\Sigma^{p-m}\alpha$$

$$\beta_* = \beta$$

by Lemma 61, and

$$\beta_* = (-)^{(qp+-m)-p}\alpha_*$$

by Lemma 62, because β is stable. Therefore

$$\beta = (-)^{q-m+p(p-m)}\Sigma^{p-m}\alpha$$

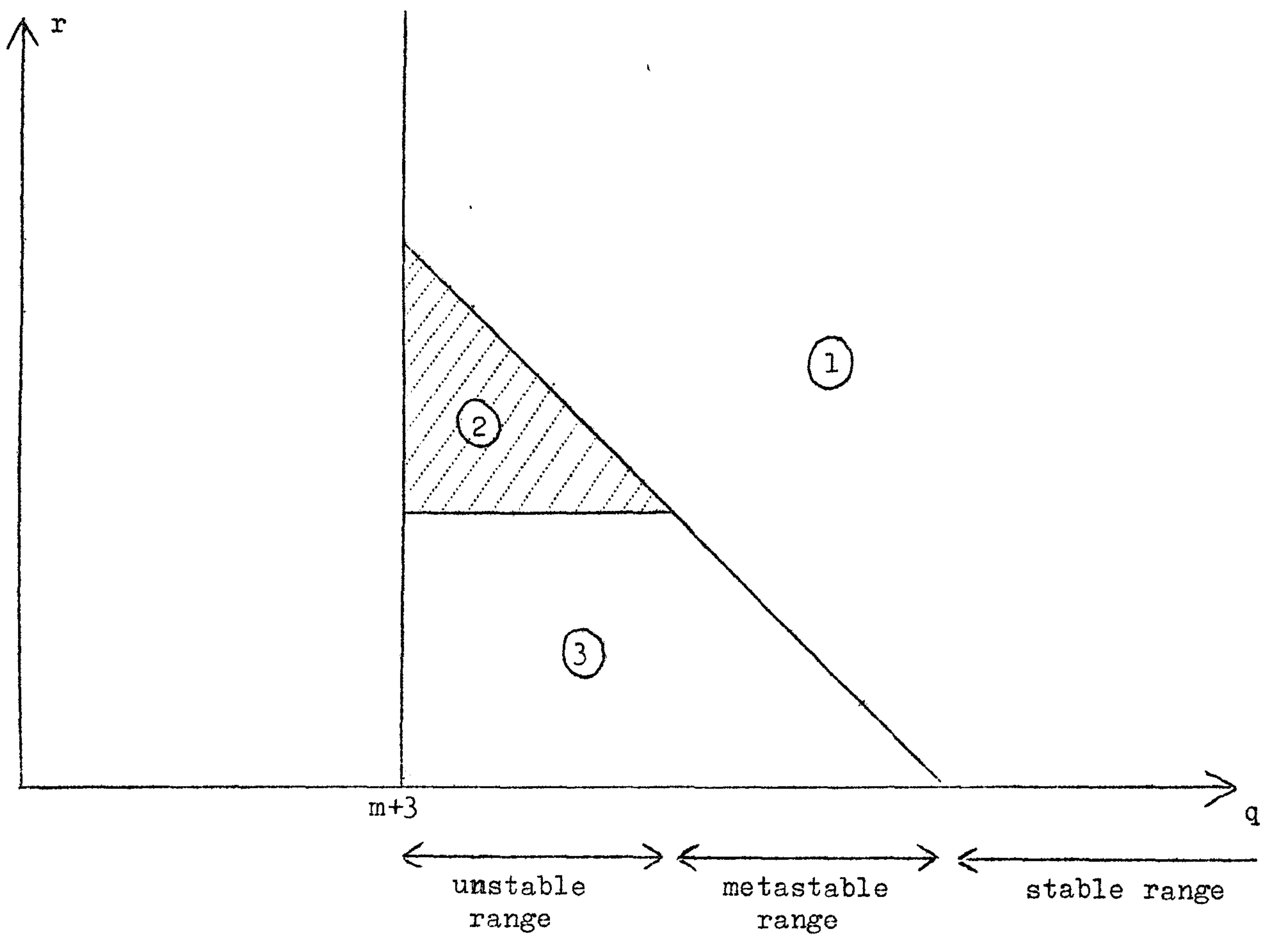
$$= (-)^{p+m+q+pm}\Sigma^{p-m}\alpha.$$

This completes the proof of the theorem.

Remark. If neither of α, β are stable then we can show they have a common stable suspension to within the sign $(-)^{m+p+q+pm}$ by suspending both sides of the link sufficiently and applying Lemmas 61 and 62.

Knots of spheres in solid tori.

Consider embeddings $S^m \subset S^r \times E^{q-r}$. Keeping m fixed, let us plot against coordinates q and r the three regions in which different types of knotting can occur. As usual we restrict attention to $q \geqslant m + 3$.



1. Unknotting. Region (1) is bounded by $q + r \geqslant 2m + 3$, which is the condition by S^m to unknot in $S^r \times E^{q-r}$ by Theorem 24 Corollary 1. Therefore the classes of embeddings are classified by $\pi_m(S^r)$.

2. Homotopy-stable knotting. Region (2) is bounded by $q + r < 2m + 3$ and $r \geqslant m/2 + 1$. We call these homotopy-stable because the condition $r \geqslant m/2 + 1$ is exactly the condition for $\pi_m(S^r)$ to be stable. In the next corollary we classify homotopy-stable knots.

3. Homotopy-unstable knotting. Region (3) is the complement. Here we can tie knots like the knotted $S^m \subset S^{2m} \times E^1$ described in Counter example 6 above.

Remark. It is interesting that region (2) lies entirely in the unstable range. Therefore all the knots that we classify in the following corollary lie in the unstable range. So far there are no analogous results in the smooth category; the smooth situation is complicated by the presence of sphere knots in spheres, and it is difficult to disentangle them from the situation.

Corollary 2 to Theorem 25. Homotopy-stable knots are classified by the diagram

$$\begin{tikzcd} \mathrm{Iso}(S^m \subset S^r \times E^{q-r}) \arrow[d, "\theta"', "\cong"] \arrow[dr, "\beta"] & \\ \pi_{q-r-1}(S^{m-r-1}) \arrow[r, "\Sigma"'] & \pi_m(S^r). \end{tikzcd}$$

Therefore S^m unknots in $S^r \times E^{q-r}$ if and only if the suspension Σ is a monomorphism.

Proof. Notice that conditions for region (2) imply

$$q - r - 1 \leqslant (2m - r + 2) - r - 1, \text{ because } 2m - r \geqslant q - 2$$
$$< m, \quad \text{because } m \leqslant 2r - 2.$$

Also it is easy to show $r \geqslant 2$. Therefore

$$\begin{aligned} \mathrm{Iso}(S^m \subset S^r \times E^{q-r}) &\cong \mathrm{Iso}(S^m, S^{q-r-1} \subset S^q) \text{ by Lemma 60} \\ &\cong \mathrm{Iso}(S^{q-r-1}, S^m \subset S^q) \text{ putting the smaller sphere first.} \\ &\cong \pi_{q-r-1}(S^{q-m-1}), \text{ by Theorem 25 since } \beta \text{ stable.} \end{aligned}$$

Define the isomorphism $\theta = (-)^{(q-r-1)+m+q+(q-r-q)m}\alpha$. Then, by Theorem 25, $\Sigma\theta = \beta$ and so the diagram is commutative.

Finally we can write $\beta = \lambda\mu$ where

$$\mathrm{Iso}(S^m \subset S^r \times E^{q-r}) \xrightarrow[\text{surjective}]{\mu} [S^m \subset S^r \times E^{q-r}] \xrightarrow[\text{injective}]{\lambda} \pi_m(S^r).$$

Therefore S^m unknots in $S^r \times E^{q-r}$ $\Longleftrightarrow$ μ injective

$\Longleftrightarrow$ β injective

$\Longleftrightarrow$ Σ monomorphism.

This completes the proof of the Corollary.

Examples of homotopy-stable knots.

sphere $\subset$ torus	isotopy classes of embeddings $\longrightarrow$ homotopy classes of maps
$S^4 \subset S^3 \times E^4$	$Z \xrightarrow{\text{epi}} Z_2$
$S^{10} \subset S^6 \times E^{10}$	$Z_2 \longrightarrow 0$
$S^{14} \subset S^8 \times E^{10}$	$Z_3 \longrightarrow Z_2$.

In the last example there are three knots all homotopically trivial, and no realisation as an embedding of the other homotopy class.

Problems.

It would be interesting to extend the results to:

(i) the region 3.

(ii) knots of S^m in arbitrary q-dimensional regular neighbourhoods of S^r, rather than just the product neighbourhood.

(iii) knots of S^m in an (r-1)-connected manifold, where r is not big enough for unknotting. This would be the beginning of an obstruction theory.

We now return to the task of proving Theorems 23 and 24, which occupies the rest of this chapter.

Proof of the Embedding Theorem 23 when M is closed.

We are given a continuous map $f:M \to Q$ which we have to homotop into a piecewise linear embedding in the interior, and we are given that

$$m \leqslant q - 3$$

M is d-connected

Q is (d+1)-connected,

where $d = 2m - q$.

The first step is to make f piecewise linear by simplicial approximation. Next homotopy f into the interior of Q as follows. Let Q_1 be a regular neighbourhood of fM in Q. Since

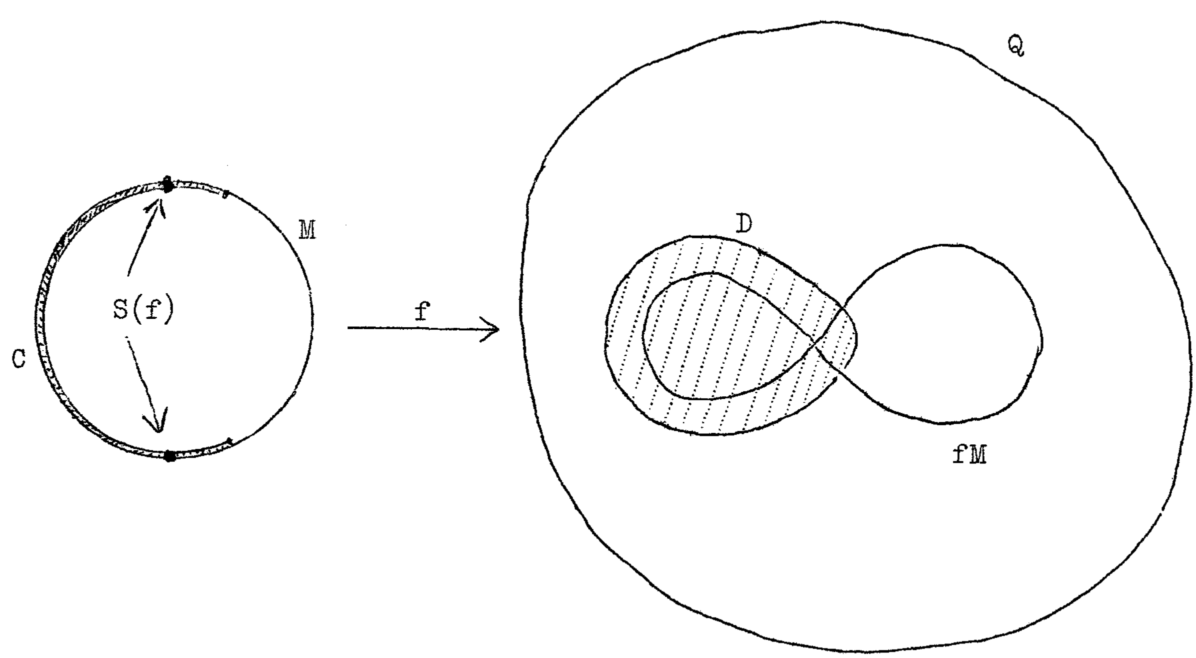
M is compact so is Q_1, and therefore Q_1 has a collar. By shrinking this collar to half its length (the inner half) homotop Q_1 into $\mathring{Q}_1$. This homotopy carries fM into $\mathring{Q}_1, \subset \mathring{Q}$.

Now homotop f into general position in $\mathring{Q}$ by Theorem 18 Corollary 1 of Chapter 6. Therefore singular set $S(f)$ of f will have dimension

$$\dim S(f) \leqslant d,$$

the double point dimension. The next main step of the proof is contained in the following lemma.

<u>Lemma 63</u>. <u>There exist collapsible subspaces C, D of M, $\mathring{Q}$ respectively, such that $S(f) \subset C = f^{-1}D$.</u>



Proof. The main idea of the proof is to use the engulfing Theorem 20 or Chapter 7 several times in an inductive process.

Since M is d-connected, we can start by engulfing $S(f)$ in a collapsible subspace C_1^{d+1},

$$S(f) \subset C_1 \subset M.$$

Of course when C_1 is mapped by f into Q it no longer remains collapsible, because bits of $S(f)$ get glued together to form non-bounding cycles. Nevertheless, since Q is $(d+1)$-connected, we can engulf fC_1 in a collapsible subspace D_1^{d+2},

$$fC_1 \subset D_1 \subset \mathring{Q}.$$

We are not finished yet, because although $f^{-1}D_1$ contains C_1, it may contain other stuff as well. The idea is to move D_1 so as to minimise the dimension of this other stuff and then engulf it. More precisely we shall define an induction on i, where the i^{th} induction statement is as follows:

There exist three collapsible subspaces C_i in M and $D_i \supset E_i$ in $\mathring{Q}$, such that

(1) $S(f) \subset C_i$

(2) $f^{-1}E_i \subset C_i \subset f^{-1}D_i$

(3) $\dim (D_i - E_i) \leqslant d - i + 3$.

The induction begins at $i = 1$, by constructing C_1, D_1 as above, and choosing E_1 to be a point of fC_1. The induction ends at $i = d+4$, because then $D_i = E_i$ and so we have $S(f) \subset C_i = f^{-1}D_i$ as required.

There remains to prove the inductive step, and so assume the i^{th} inductive statement is true, where $i \geq 1$. Then $fC_i \cup E_i \subset D_i$ by (2). Let

$$F = D_i - (fC_i \cup E_i).$$

Then

$$\dim F \leq d - i + 3$$

by (2). Using Theorem 15 of Chapter 6 ambient isotop D_i in Q keeping $fC_i \cup E_i$ fixed until F is in general position with respect to fM. Then

$$\begin{aligned} \dim (F \cap fM) &\leq (d - i + 3) + m - q \\ &\leq d - i, \text{ because } m \leq q - 3. \end{aligned}$$

Therefore $\qquad \dim f^{-1}F \leq d - i,$

because f is non-degenerate, being in general position. Let E_{i+1} denote the new position of D_i after the isotopy. Then E_{i+1} is collapsible because it is homeomorphic to D_i. Since M is d-connected, we can engulf $f^{-1}F$ (or more precisely the closure of $f^{-1}F$) by pushing out a feeler from C_i. That is to say there exists a subspace C_{i+1} of M such that

$$f^{-1}F \subset C_{i+1} \searrow C_i$$

$$\dim (C_{i+1} - C_i) \leq d - i + 1.$$

Then C_{i+1} is collapsible because $C_{i+1} \searrow C_i \searrow 0$;

$S(f) \subset C_{i+1}$, because $S(f) \subset C_i$, by induction $\subset C_{i+1}$.

$f^{-1}E_{i+1} \subset C_{i+1}$, because $f^{-1}E_{i+1} = C_i \cup f^{-1}E_i \cup f^{-1}F$
$= C_i \cup f^{-1}F$, by induction
$\subset C_{i+1}$, by engulfing.

Since Q is $(d+1)$-connected we can now engulf $f(C_{i+1} - C_i)$ by pushing out a feeler from E_{i+1}. That is to say there exists a subspace D_{i+1} of $\mathring{Q}$ such that

$$f(C_{i+1} - C_i) \subset D_{i+1} \searrow E_{i+1}$$
$$\dim (D_{i+1} - E_{i+1}) \leqslant d - i + 2.$$

Then D_{i+1} is collapsible because $D_{i+1} \searrow E_{i+1} \searrow 0$. We have constructed the three spaces, and verified all the conditions of the $(i+1)^{th}$ inductive statement except $C_{i+1} \subset f^{-1}D_{i+1}$. This follows because $fC_i \subset E_{i+1}$, since the isotopy kept fC_i fixed, and so

$$\begin{aligned} fC_{i+1} &= fC_i \cup f(C_{i+1} - C_i) \\ &\subset E_{i+1} \cup D_{i+1}, \text{ by engulfing} \\ &= D_{i+1}. \end{aligned}$$

This completes the proof of the inductive step, and hence the proof of Lemma 63.

We return to the proof of Theorem 23. Choose a compact submanifold Q_* of Q containing $fM \cup D$ in its interior. Triangulate M, Q_* such that f is simplicial and C, D are subcomplexes. If we pass to the barycentric second derived complexes then f remains simplicial because f is non-degenerate (being in general position). Let B^m, B^q denote the second derived neighbourhoods of C, D in M, Q_* respectively; these are balls by Theorem 5, because C, D are collapsible. Then Lemma 63 implies $S(f) \subset B^m = f^{-1}B^q$. In fact the lemma implies more : it implies that

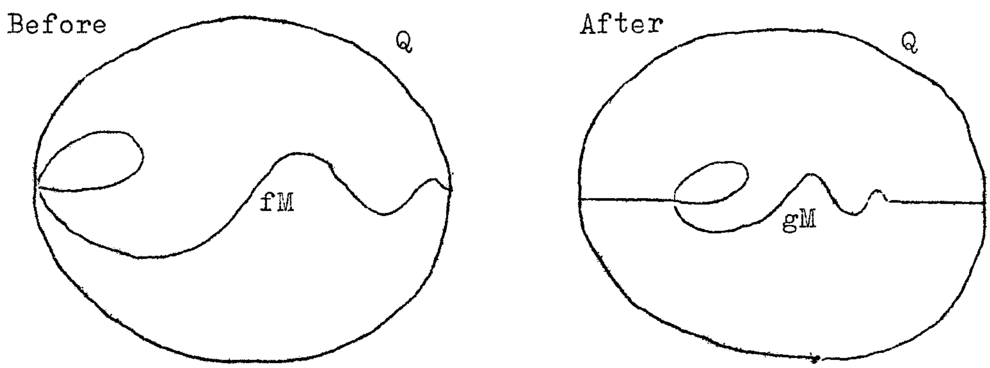
$$f \begin{cases} \text{maps } \mathring{B}^m \to \mathring{B}^q \\ \text{embeds } \dot{B}^m \to \dot{B}^q \\ \text{embeds } M - B^m \to Q - B^q. \end{cases}$$

Now we see our way clear: we have localised all the singularities of f inside balls, where it is easy to straighten them out. More precisely let $g: B^m \to B^q$ be an embedding such that $g|\dot{B}^m = f|\dot{B}^m$, obtained by joining the boundary to an interior point in some linear representation of B^q. Extend g to an embedding $g: M \to Q$ by making g equal to f outside B^m. Then $g \simeq f$. Notice that the homotopy is global, but takes place inside the ball B^q. This completes the proof of Theorem 23 in the case M closed.

Proof of Theorem 23 when M is bounded.

We are given a continuous map $f: M \to Q$ such that $f|\dot{M}$ is a piecewise linear embedding of $\dot{M}$ in $\dot{Q}$. First make f piecewise linear keeping $\dot{M}$ fixed by relative simplicial approximation. The next thing to do is to straighten up the map near the boundary. Call a map $g: M \to Q$ proper (as in the case of embeddings) if $g^{-1}\dot{Q} = \dot{M}$.

Lemma 64. *f is homotopic to a proper map $g:M \to Q$ keeping $\dot{M}$ fixed, such that $S(g) \subset \mathring{Q}$.*



Once the lemma is proved we can apply the same arguments as in the unbounded case to eliminate the singularities of g, working entirely in the interior of Q. The proof of the theorem will therefore be complete.

Proof of Lemma 64. Since M is compact we can choose a collar by the Corollary to Lemma 24, that is to say an embedding

$$c_M : \dot{M} \times I \to M,$$

such that $c_M(x, 0) = x$, for all $x \in \dot{M}$. Let $M_0 = \text{closure}(M - \text{im } c_M)$. Let $h_t : M \to M$ be a homotopy that starts with the identity and finishes with a map h_1 that shrinks the collar onto the boundary and maps M_0 homeomorphically onto M. Such a homotopy can be easily defined by stretching the inner half of a collar twice as long. In particular

$$h_1 c_M(x, u) = x,$$

for all $x \in \dot{M}$, $u \in I$. Then h_t keeps $\dot{M}$ fixed, and therefore $f \simeq fh_1$ keeping $\dot{M}$ fixed.

Now let Q_1 be a regular neighbourhood of fM in Q. Since M is compact so is Q_1, and we can choose a collar

$$c_Q : \dot{Q}_1 \times I \to Q_1$$

such that $c(y, 0) = y$ for all $y \in \dot{Q}_1$. Let $Q_0 = \text{closure}(Q_1 - \text{im}\, c_Q)$. Let $k_t : Q_1 \to Q_1$ be the homotopy shrinking the collar onto its inner boundary and keeping Q_0 fixed. More precisely for $0 \leqslant t \leqslant 1$ define

$$k_t c_Q(y, u) = \left.\begin{cases} c_Q(y, t+u), & 0 \leqslant u \leqslant 1-t \\ c_Q(y, 1), & 1-t \leqslant u \leqslant 1 \end{cases}\right\} y \in \dot{Q}_1$$

$$k_t | Q_0 = \text{identity}.$$

We now use k to construct a homotopy $g_t : M \to Q$ that moves M_0 into Q_0 and sketches the collar c_M out again compatibly with c_Q. More precisely for $0 \leqslant t \leqslant 1$ define

$$g_t c_M(x, u) = \left.\begin{cases} c_Q(fx, u), & 0 \leqslant u \leqslant t \\ c_Q(fx, t), & t \leqslant u \leqslant 1 \end{cases}\right\} x \in \dot{M}$$

$$g_t | M_0 = k_t f h_1 | M_0.$$

Notice that g_t keeps $\dot{M}$ fixed. Define $g = g_1$, and we have

$$f \simeq fh_1 = g_0 \simeq g_1 = g,$$

all keeping $\dot{M}$ fixed. Meanwhile $gM_0 \subset Q_0$, and the collars c_M, c_Q are compatible with g in the sense that the diagram

$$\begin{tikzcd} \dot{M} \times I \arrow[r, "c_M"] \arrow[d, "f|\dot{M}\times 1"'] & M \arrow[d, "g"] \\ \dot{Q}_1 \times I \arrow[r, dashed, "c_Q"] & Q_1 \end{tikzcd}$$

is commutative. Therefore $g\mathring{M} \subset \mathring{Q}_1 \subset \mathring{Q}$ and $g\dot{M} = f\dot{M} \subset \dot{Q}$, and so g is proper. Also the restriction of g to the collar is an embedding, and so

$$S(g) \subset \text{closure } Q_0 = Q_0 \subset \mathring{Q}_1 \subset \mathring{Q}.$$

This completes the proof of Lemma 64 and Theorem 23.

Proof of Theorem 24 when M is closed.

We are given homotopic embeddings $f, g: M \to \mathring{Q}$, we have to show they are ambient isotopic keeping $\dot{Q}$ fixed, provided

$$m \leq q - 3$$

$$M \text{ is } (d+1)\text{-connected}$$

$$Q \text{ is } (d+2)\text{-connected.}$$

Without loss of generality we can assume that M is unbounded, because if we prove the result for the unbounded case then f, g are ambient isotopic in $\mathring{Q}$. Then, by Theorem 12, it is possible to choose an ambient isotopic with compact support, which is therefore extendable to an ambient isotopy of Q keeping $\dot{Q}$ fixed. Therefore assume Q unbounded.

Remark. If we had Hudson's concordance $\Longrightarrow$ isotopy result available, then the theorem could be deduced immediately from Theorem 23, as follows. The homotopy gives a continuous map

$$F : M \times I \to Q \times I$$

such that $F|(M \times I)^{\boldsymbol{\cdot}}$ is an embedding in $(Q \times I)^{\boldsymbol{\cdot}}$. The connectivities of M, Q have to be increased by one each, because the double-point dimension of F is

$$2(m+1) - (q+1) = (2m-q)+1 = d+1.$$

By Theorem 23 homotop F into an embedding

$$G : M \times I \to Q \times I$$

keeping $(M \times I)^{\boldsymbol{\cdot}}$ fixed. In other words G is a concordance between f and g. Therefore they are ambient isotopic. However as we have not proved the concordance $\Longrightarrow$ isotopy in these notes, we give a separate proof of Theorem 24, similar to that of Theorem 23 above.

Begin the proof by ambient isotoping g into general position with respect to fM, by Theorem 15. The given homotopy is a continuous map

$$h : M \times I \to Q$$

which we can make piecewise linear keeping $(M \times I)^{\boldsymbol{\cdot}}$ fixed, by relative simplicial approximation.

<u>Lemma 65.</u> <u>After a suitable homotopy of h keeping $(M \times I)^{\boldsymbol{\cdot}}$ fixed, we can find collapsible subspaces C, D of M, Q such that $S(h) \subset C \times I = h^{-1}D$.</u>

We prove the lemma in two stages in order to make the proof more translucent. In the first stage we prove a weaker result by assuming the stronger hypotheses

$$m \leqslant q - 5$$
$$M \text{ is } (d+2)\text{-connected}$$
$$Q \text{ is } (d+4)\text{-connected.}$$

The proof follows the pattern of the proof of Lemma 63. In the second stage we show how to sharpen the proof as various points in order get by with the correct hypotheses of Theorem 24, namely

$$m \leqslant q - 3$$
$$M \text{ is } (d+1)\text{-connected}$$
$$Q \text{ is } (d+2)\text{-connected.}$$

We achieve the sharpening by using the piping techniques of the last chapter.

<u>Proof of First Stage</u>. Let $\pi : M \times I \to M$ denote projection. Notice that $h|(M \times I)^{\cdot}$ is in general position because we have already isotoped g into general position with respect to fM. By Theorem 18 homotop h into general position keeping $(M \times I)^{\cdot}$ fixed. Therefore

$$\begin{aligned} \dim S(h) &\leqslant 2(m + 1) - q \\ &= d + 2. \end{aligned}$$

There is an induction on i, as follows. There exist collapsible subspaces C_i in M and $D_i \supset E_i$ in Q such that

(1) $S(h) \subset C_i \times I$

(2) $h^{-1}E_i \subset C_i \times I \subset h^{-1}D_i$

(3) $\dim (D_i - E_i) \leqslant d - i + 6$.

The induction begins at $i = 1$. Since under the stronger hypotheses we are assuming M to be $(d+2)$-connected, engulf $\pi S(h)$ in a collapsible subspace C_1^{d+3} of M. Since Q is assumed to be $(d+4)$-connected, engulf $h(C_1 \times I)$ in a collapsible subspace D_1^{d+5} of Q. Choose E_1 to be a point of $h(C_1 \times I)$, and the conditions for $i = 1$ are satisfied.

For the inductive step, assume true for i. Obtain E_{i+1} by ambient isotoping D_i in Q keeping $h(C_i \times I) \cup E_i$ fixed, until the complement F is in general position with respect to $h(M \times I)$. Then

$$\begin{aligned}\dim \pi h^{-1}F &\leqslant \dim h^{-1}F \\ &= \dim [F \cap h(M \times I)] \\ &\leqslant (d - i + 6) + (m + 1) - q \\ &\leqslant d - i + 2,\end{aligned}$$

because we are assuming $m \leqslant q - 5$. Engulf in M

$$\pi h^{-1}F \subset C_{i+1} \searrow C_i$$
$$\dim (C_{i+1} - C_i) \leqslant d - i + 3.$$

Therefore

$$\dim (C_{i+1} \times I - C_i \times I) \leqslant d - i + 4.$$

Engulf in Q

$$h(C_{i+1} \times I - C_i \times I) \subset D_{i+1} \searrow E_{i+1}$$
$$\dim (D_{i+1} - E_{i+1}) \leqslant d - i + 5.$$

Verify the $(i+1)^{th}$ induction statement as in the proof of Lemma 63. The induction ends at $i = d + 7$ with $D_i = E_i$, and consequently $S(h) \subset C_i \times I = h^{-1}D_i$, as required.

Proof of the Second Stage. When homotoping h into general position, the full strength of Theorem 18 was not used. We now use the additional information that h is in general position for the pair $M \times I$, $(M \times I)^{\bullet}$. In particular this implies that the (d+2)-dimensional stuff of $S(h)$ all lies in the interior of $M \times I$, at places where exactly two sheets of $M \times I$ cross one another. The trick now is to punch holes in this top dimensional stuff, by piping one of the sheets over the free-end $M \times 0$. More precisely we use the piping Lemma 48 of Chapter 7. The triple $M \times 0$, $M \times 1 \subset M \times I$ is "cylinderlike" in the sense of Chapter 7, and Q has no boundary by assumption. Therefore by the piping lemma, we can homotop h keeping $(M \times I)^{\bullet}$ fixed, and then find a subspace T of $M \times I$ such that

(1) $S(h) \subset T$

(2) $\dim T \leqslant d + 2$

(3) $\dim [(M \times 1) \cap T] \leqslant d + 1$

(4) $M \times I \searrow (M \times 1) \cup T \searrow M \times 1.$

By being a little more precise in the proof of Lemma 48 at one point, we can factor the first of these collapses

(5) $M \times I \searrow (M \times 1) \cup (\pi T \times I) \searrow (M \times 1) \cup T.$

Since M is (d+1)-connected it is possible, using (3), to engulf $(M \times 1) \cap T$ in a collapsible subspace R^{d+2} of $M \times 1$. Define $C_1^{d+2} = \pi(R \cup T)$. Notice that compared with the dimension of C_1 in the proof of the first stage, we have scored an improvement of 1. Then

$$C_1 \times I = (\pi R \times D) \cup (\pi T \times I)$$
$$\searrow R \cup (\pi T \times I), \text{ cylinderwise}$$
$$\searrow R \cup T \text{ by } (5).$$

Therefore $h(C_1 \times I) \searrow h(R \cup T)$ by Lemma 38 and (1) above. But

$$\dim h(C_1 \times I) \leqslant d + 3$$
$$\dim h(R \cup T) \leqslant d + 2.$$

Therefore $h(C_1 \times I)$ can be "furled" in the sense of Chapter 7. Since Q is $(d+2)$-connected engulf $h(C_1 \times I)$ in a collapsible subspace D_1^{d+2} of Q, of the same dimension, by the furling Corollary to Theorem 20. As before define E_1 to be a point of $h(C_1 \times I)$. Notice that we have scored an improvement of 2 in the dimension of D_1.

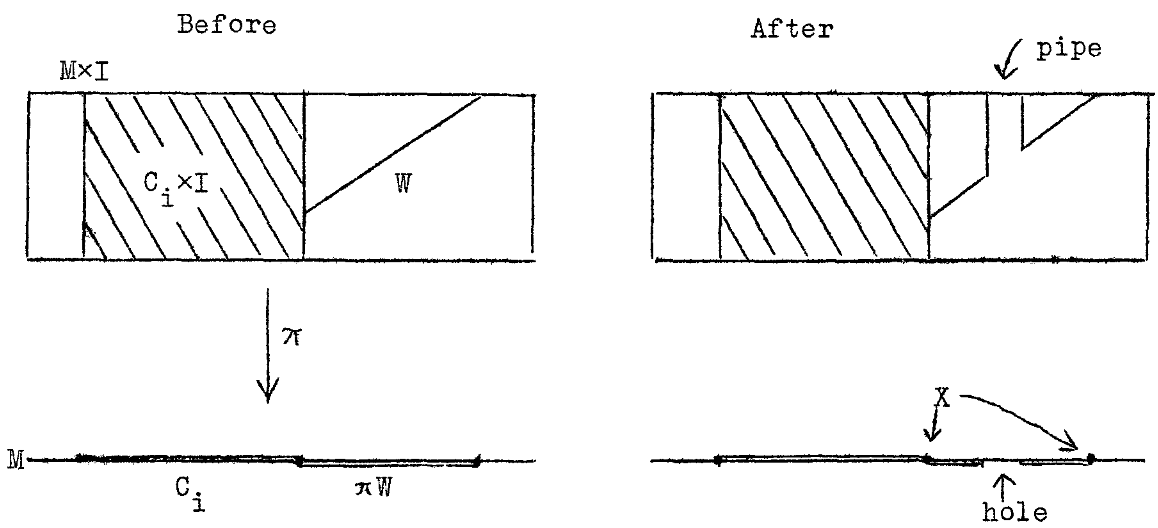
We can write this improvement into the i^{th} inductive statement by replacing condition (3) by

$$(3)^* \quad \dim (D_i - E_i) \leqslant d - i + 4.$$

We now have to do some more piping and furling for the inductive step. As before obtain E_{i+1} by ambient isotoping D_i keeping $h(C_i \times I) \cup E_i$ fixed, until the complement F is in general position with respect to $h(M \times I)$. Let $W = \text{closure}(h^{-1}F)$. Then

$$\begin{aligned} \dim \pi W &\leqslant \dim W \\ &= \dim [F \cap h(M \times I)] \\ &\leqslant (d - i + 4) + (m + 1) - q \\ &\leqslant d - i + 2 \end{aligned}$$

because $m \leqslant q - 3$. We now want to engulf πW from C_1 but with a feeler of the same dimension, and the way to do this is to furl πW. We furl πW by punching holes in the top dimensional simplexes (of some suitable triangulation), by piping the relevant top dimensional piece of F off the end $h(M \times 0)$ of $h(M \times I)$.



Notice that $S(h) \subset C_i \times I$, and F does not meet $h(C_i \times I)$, and so that the self-intersections of $h(M \times I)$ do not get in the way of the pipe.

More precisely, we can adapt Lemma 48 to give the following result. We can find an ambient isotopy of D_i keeping $h(C_i \times I) \cup E_i$ fixed, such that in the new position πW^{d-i+2} can be furled to a subspace X^{d-i+1} relative to C_i, and

$$(C_i \times I) \cup (\pi W \times I) \searrow (C_i \times I) \cup (X \times I) \cup W.$$

By the Corollary to Theorem 20, engulf in M

$$\pi W \subset C_{i+1} \searrow C_i$$
$$\dim (C_{i+1} - C_i) \leqslant d - i + 2.$$

Let Y^{d-i+3} = closure $(C_{i+1} \times I - C_i \times I)$, $Z^{d-i+2} = (X \times I) \cup W \cup (C_{i+1} \times$
Then Y can be furled to Z relative to $C_i \times I$ because

$$\begin{aligned}(C_i \times I) \cup Y = C_{i+1} \times I \\ &\searrow (C_i \times I) \cup (\pi W \times I) \cup (C_{i+1} \times 1) \text{ cylinderwise} \\ &\searrow (C_i \times I) \cup Y \textbf{ by } \text{above.}\end{aligned}$$

Therefore hY can be furled to hZ relative to $h(C_i \times I)$ by Lemma 38, because $S(h) \subset C_i \times I$. Moreover hY can be furled to hZ relative to E_{i+1}, because

$$\begin{aligned}hY \cap E_{i+1} &\subset h(M \times I) \cap E_{i+1} \\ &= h(C_i \times I) \cup hW \\ &\subset hZ \cap E_{i+1},\end{aligned}$$

and therefore $hY \cap E_{i+1} = hZ \cap E_{i+1}$. Therefore engulf in Q

$$hY \subset D_{i+1} \searrow E_{i+1}$$
$$\dim (D_{i+1} - E_{i+1}) \leqslant d - i + 3.$$

Verify the $(i+1)^{th}$ inductive statement as in the proof of the first stage, and the proof of Lemma 65 is complete.

Lemma 66. <u>There exist balls $B^m \subset M$, $B^q \subset Q$ such that</u>

$$h \begin{cases} \underline{\text{maps } \mathring{B}^m \times I \to \mathring{B}^q} \\ \underline{\text{embeds } \dot{B}^m \times I \to \dot{B}^q} \\ \underline{\text{embeds } (M - \mathring{B}^m) \times I \to Q - \mathring{B}^q.} \end{cases}$$

Proof. The obvious way is to take derived neighbourhoods of the collapsible subspaces C, D of Lemma 66. However we run into the technical difficulty of not being able to find triangulations such that both maps

$$\begin{array}{ccc} M \times I & \xrightarrow{h} & Q \\ \big\downarrow \pi & & \\ M & & \end{array}$$

are simplicial (as is illustrated in the Example at the end of Chapter 1). Therefore first choose triagulation K, L of $M \times I$, M such that $\pi : K \to L$ is simplicial, and C is a full subcomplex of L. Let $\lambda : L \to I$ be the unique simplicial map such that $\lambda^{-1}0 = C$. Choose ε, $0 < \varepsilon < 1$ of a smallness to be specified later. Define $B^m = \lambda^{-1}[0, \varepsilon]$ which is ball, because it is a regular neighbourhood of C by Lemma 14.

Now choose a subdivision K_1 of K, and a triangulation Q_1 of a regular neighbourhood of $h(M \times I)$ in Q such that $h : K_1 \to Q_1$ is simplicial. If K_2, Q_2 are barycentric first deriveds then $h : K_2 \to Q_2$ remains simplicial because h is non-degenerate. Now choose ε such that $\varepsilon < \lambda\pi v$ for all vertices $v \in K_2$ not in $C \times I$. Call a simplex of K_2 <u>exceptional</u> if it meets $C \times I$, but is not contained in $C \times I$. Then $(\lambda\pi)^{-1}\varepsilon$ meets only exceptional simplexes, and meets each exceptional simplex in a hyperplane.

Let K_3 be a first derived of K_2 obtained by starring all exceptional simplexes on $(\lambda\pi)^{-1}\varepsilon$, and the rest barycentrically. Since $S(h) \subset C \times I$, no exceptional simplex is identified with any other simplex by h. Therefore we can define a first derived Q_3 of Q_2, such that $h:K_3 \to Q_3$ remain simplicial, by starring images of exceptional simplexes at the image of the star-point, and the rest barycentrically. Define $B^q = N(D, Q_3)$, which is a ball, being a second derived neighbourhood of the collapsible subspace D. Then

$$h^{-1}B^q = N(C \times I, K_3) = (\lambda\pi)^{-1}[0, \varepsilon] = B^m \times I.$$

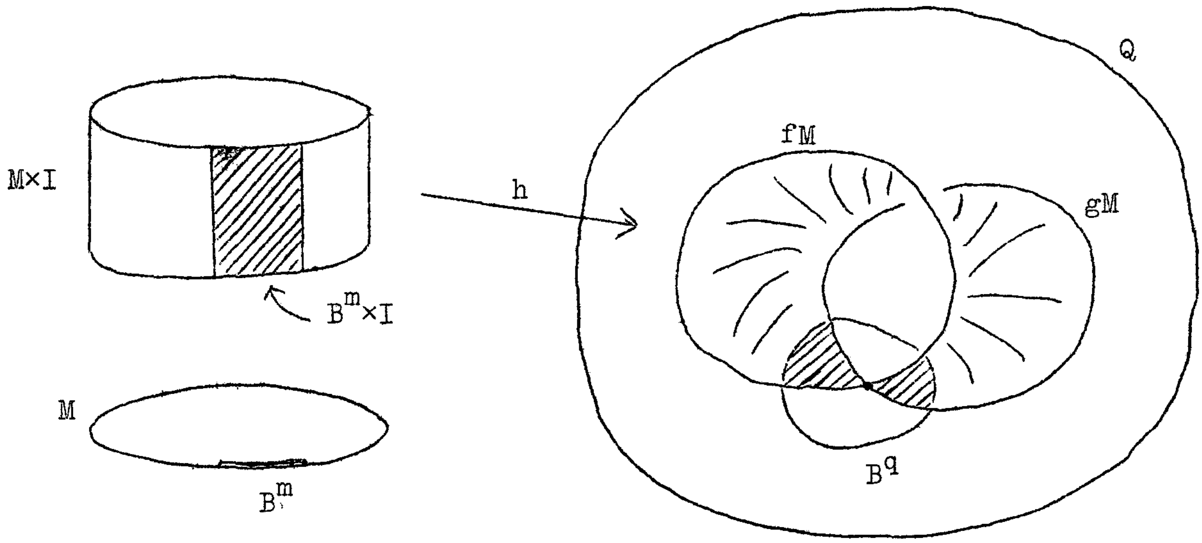
$$h^{-1}\dot{B}^q = \dot{B}^m \times I$$

$$S(h) \subset \mathring{B}^m \times I.$$

The proof of Lemma 66 is complete.

<u>Continuing the proof of Theorem 24.</u>

So far we have the picture:

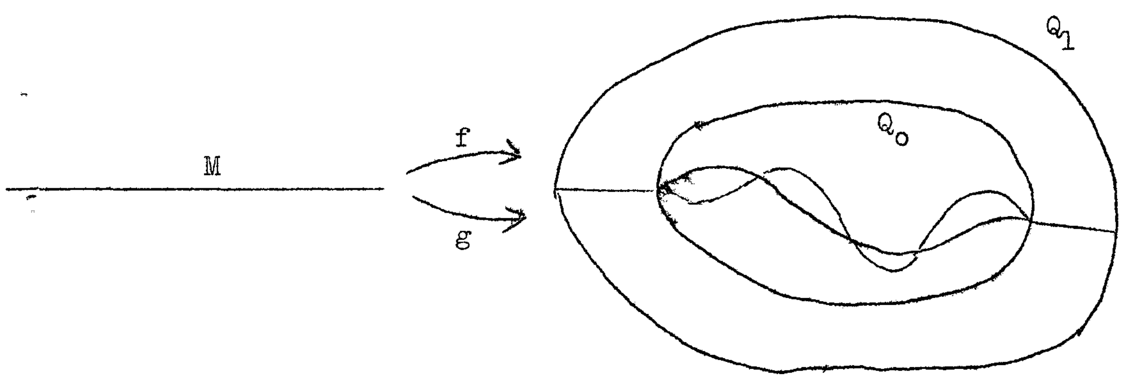


Since $h|M - \mathring{B}^m$ is a proper embedding of $M - \mathring{B}^m$ in $Q - \mathring{B}^q$, this means that $f|M - \mathring{B}^m$ is isotopic to $g|M - \mathring{B}^m$. Therefore by Theorem 12 Corollary 1 of Chapter 5 they are ambient isotopic. Extend the ambient isotopy of $Q - \mathring{B}^q$ arbitrarily over B^q to give an ambient isotopy of Q. The latter moves f to f', say, where f' agrees with g except on $\mathring{B}^m$, and $f'|B^m$, $g|B^m$ are proper embeddings of B^m in B^q that agree on the boundary. Since $m \leq q - 3$, by Theorem 9 Corollary 1 of Chapter 4 we can ambient isotop B^q keeping $\dot{B}^q$ fixed so as to move $f'|B^m$ onto $g|B^m$. This ambient isotopy extends trivially to an ambient isotopy of Q, moving f' onto g. Hence f and g are ambient isotopic. This completes the proof of Theorem 24 when M is closed.

Proof of Theorem 24 when M is bounded.

We are given embeddings $f, g: M \to Q$ that are homotopic keeping $\dot{M}$ fixed, and we have to show they are ambient isotopic keeping $\dot{Q}$ fixed. The first thing to do is to make them agree on a collar.

Let Q_1 be a compact submanifold of Q containing the image $h(M \times I)$ of the given homotopy h. Choose a collar c_M of M. By Theorem 10 of Chapter 5 choose two collars c_Q^f, c_Q^g of Q_1 such that c_M, c_Q^f are compatible with f, and c_M, c_Q^g are compatible with g. By Theorem 13 there is an ambient isotopy k of Q_1 keeping $\dot{Q}_1$ fixed, moving c_Q^f onto c_Q^g. Since $\dot{Q}_1$ is kept fixed, k extends trivially to an ambient isotopy of Q keeping $\dot{Q}$ fixed. Therefore if we replace f by $k_1 f$, and write $c_Q = c_Q^g$, then f, g agrees on the collar im c_M, and c_M, c_Q are compatible with both f and g. Let $M_0 =$ closure $(M - \text{im } c_M)$, $Q_0 =$ closure $(Q_1 - \text{im } c_Q)$ Then $fM_0, gM_0 \subset Q_0$. The picture now looks like:



<u>Lemma 67.</u> <u>The maps $f|M_0, g|M_0 : M_0 \to Q_0$ are homotopic in Q_0 keeping $\dot{M}_0$ fixed.</u>

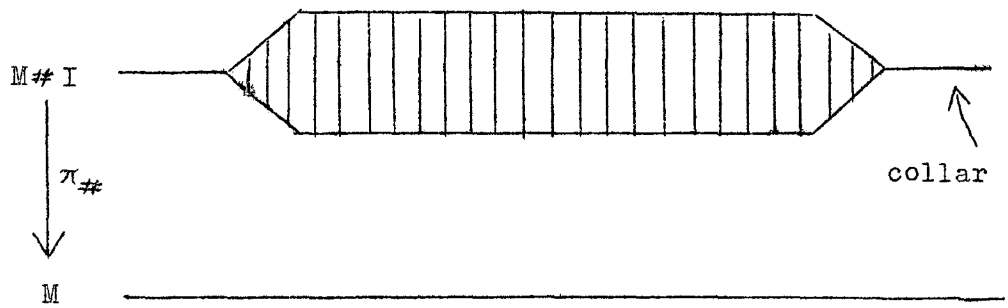
<u>Proof.</u> Let $e_t : M_0 \to M$ be a homotopy starting with the inclusion and ending with a homeomorphism that stretches a collar of M_0 over that of M. In particular $e_t c_M(x, 1) = c_M(x, 1-t)$, all $x \in \dot{M}$, $t \in I$. Let $j : Q_1 \to Q_0$ be the retraction that shrinks the collar c_Q onto its inner boundary, $jc_Q(y, t) = c_Q(y, 1)$ all $y \in \dot{Q}_1$, $t \in I$. Then the required homotopy is obtained by timewise composition of the three homotopies

$$jfe_t, \; jh_te_1, \; jge_{1-t}.$$

This completes the proof of the lemma.

The purpose of what we have done so far is to push the singularities of the homotopy into the interiors of M, Q so that the boundaries do not interfere with the engulfing. However there is the trivial technical difficulty that a constant homotopy of the collar is of course a singular map of (collar) $\times$ I. Nor can we ambient isotop $g|\text{im } c_M$ away from $f|\text{im } c_M$ for two reasons: firstly we have got to keep $\dot{M}$ fixed, and secondly there is an obstruction in $H^{2q-2m}(\dot{Q})$. Therefore we get round this difficulty by defining the homotopy to be a map of a reduced product $M \# I$, obtained from $M \times I$ by shrinking $x \times I$ to a point for each $x \in \text{im } c_M$.

More precisely, identify the collars of $M \times 0$, $M \times 1$ (but not the complements of the collars) and define $M \# I$ to be the relative mapping cylinder of the homeomorphism $M \times 0 \to M \times 1$. Let $\pi_{\#}: M \# I \to M$ denote the projection.



If $X \subset M$, define $X \# I = \pi_{\#}^{-1} X$. Then

$$M \# I = (M_0 \# I) \cup (\text{collar}).$$

Define $h_{\#}: M \# I \to Q$ by $\begin{cases} \text{mapping } M_0 \# I \text{ by Lemma 67} \\ \text{embedding the collar by } f\pi_{\#}. \end{cases}$

Then $\pi_{\#} S(h_{\#}) \subset M_0 \subset \mathring{M}$.

$h_{\#} S(h_{\#}) \subset Q_0 \subset \mathring{Q}_1 \subset \mathring{Q}$.

Therefore we can apply engulfing arguments in the interiors of M, Q, as in the unbounded case, and obtain balls $B^m \subset \mathring{M}$, $B^q \subset \mathring{Q}$, such that

$$h_{\#} \begin{cases} \text{maps } \mathring{B}^m \# I \to \mathring{B}^q \\ \text{embeds } \dot{B}^m \# I \to \dot{B}^q \\ \text{embeds } (M - \mathring{B}^m) \# I \to Q - \mathring{B}^q. \end{cases}$$

Therefore, as before, f and g are isotopic keeping $\dot{M}$ fixed. By Theorem 12 they are ambient isotopic keeping $\dot{Q}$ fixed. The proof of Theorem 24 is complete.