Category Theory 9

Limits and colimits of presheaves

This is to accompany the reading of 28 November–5 December. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Let $\mathbb{A}$ be a small category.

   (a) What does it mean to say that limits and colimits are computed pointwise in $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$? Prove that this is so.

   (b) Describe explicitly the monics and epics in $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$. (Now see if you can do this without the aid of (a).)

2. Let $\mathbb{A}$ be a small category.

   (a) Show that for each $A \in \mathbb{A}$, the representable functor $H^A : \mathbb{A} \longrightarrow \mathbf{Set}$ preserves limits.

   (b) Show that the Yoneda embedding $H_\bullet : \mathbb{A} \longrightarrow [\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ preserves limits.

3. Let $\mathbb{A}$ be a small category and $A, B \in \mathbb{A}$. Show that the sum $H_A + H_B$ in $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ is never representable.

   (Warning 5.1.13 might give a clue. You might also want to use the description of representability in terms of universal elements, though you don't need to.)

4. Let X be a presheaf on a small category. Show that X is representable if and only if its category of elements $\mathbb{E}(X)$ has a terminal object.

   Since a terminal object is a limit of the empty diagram, this means that the concept of representability can be derived from the concept of limit. Since a terminal object of a category $\mathcal{E}$ is a right adjoint to the unique functor $\mathcal{E} \longrightarrow \mathbf{1}$, representability can also be derived from the concept of adjoint.

5. Let $\mathcal{A}$ be a category and $A \in \mathcal{A}$. A **subobject** of A is an isomorphism class of monics into A. More precisely, let $\mathbf{Monic}(A)$ be the category whose objects are the monics with codomain A and whose maps are commutative triangles; this is a full subcategory of the slice category $\mathcal{A}/A$ (Example 2.3.3(a)). Then a subobject of A is an isomorphism class of objects of $\mathbf{Monic}(A)$.

   (a) Let $X \xrightarrow{m} A$ and $X' \xrightarrow{m'} A$ be monics in $\mathbf{Set}$. Show that m and m' are isomorphic in $\mathbf{Monic}(A)$ if and only if they have the same image. Deduce that subobjects of A correspond one-to-one with subsets of A.

   (b) Part (a) says that in $\mathbf{Set}$, subobjects are subsets. What are subobjects in $\mathbf{Gp}$, $\mathbf{Ring}$ and $\mathbf{Vect}_k$? How about in $\mathbf{Top}$? *(Careful!)*