Category Theory 7

Limits

This is to accompany the reading of 14–21 November. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Define the term **limit**. In what sense are limits unique? Prove your uniqueness statement.

2. *Limit is a process that takes a diagram of shape $\mathbb{I}$ in a category $\mathcal{A}$, and produces from it a new object of $\mathcal{A}$. Later we'll see that this process is functorial. Here we show this in the special case of binary products.*

   Let $\mathcal{A}$ be a category with binary products. Choose for each pair (X, Y) of objects a product cone
$$X \xleftarrow{p_1^{X,Y}} X \times Y \xrightarrow{p_2^{X,Y}} Y.$$
   Show that once this choice is made, we have a canonical functor $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ defined on objects by $(X, Y) \longmapsto X \times Y$.

3. Take a commutative diagram
$$\begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \downarrow & & \downarrow & & \downarrow \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \end{array}$$
   in some category. Suppose that the right-hand square is a pullback. Show that the left-hand square is a pullback if and only if the outer rectangle is a pullback.

4. Let $E \xrightarrow{i} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ be an equalizer (in some category). Is
$$\begin{array}{ccc} E & \xrightarrow{i} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$
   necessarily a pullback? Give a proof or a counterexample.

5. A map $m : A \longrightarrow B$ in a category is **regular monic** if there exist an object C and maps $B \rightrightarrows C$ of which m is an equalizer. It is **split monic** if there exists a map $e : B \longrightarrow A$ such that $em = 1_A$.
   (a) Show that split monic $\Rightarrow$ regular monic $\Rightarrow$ monic.
   (b) In **Ab**, show that all monics are regular but not all monics are split. *(Hint for the first part: equalizers in* **Ab** *are calculated as in* $\mathbf{Vect}_k$*.)*
   (c) In **Top**, describe the regular monics and find a monic that is not regular.