Category Theory 5

Representables

This is to accompany the reading of 31 October–7 November. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

Some questions on these sheets require knowledge of other areas of mathematics; skip any that you haven't the background for. That aside, I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Let $\mathcal{A}$ be a locally small category. What does it mean for a presheaf X on $\mathcal{A}$ to be **representable**? What is a **representation** of X?

2. Write down five examples of representable functors. *(It's possible to answer this with almost no inventiveness at all: just look at the definition of representability.)*

3. Let $\mathcal{A}$ be a locally small category and $A, B \in \mathcal{A}$. Show that if $H_A \cong H_B$ then $A \cong B$.

4. *One way to understand the Yoneda Lemma is to think about special cases. Here we think about one-object categories.*

   Let M be a monoid. The underlying set of M can be given a right M-action by multiplication: $x \cdot m = xm$ for all $x, m \in M$. This M-set is called the **right regular representation** of M. I will write it as $\underline{M}$.

   (a) When M is regarded as a one-object category, functors $M^{\mathrm{op}} \longrightarrow \mathbf{Set}$ correspond to right M-sets. Show that the M-set corresponding to the unique representable functor $M^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is the right regular representation.

   (b) Now let X be any right M-set. Show that for each $x \in X$, there is a unique map $\alpha : \underline{M} \longrightarrow X$ of right M-sets such that $\alpha(1) = x$. *(See 1.3.3(b) for the definition of a map of M-sets; those are left M-sets but you can dualize.)* Deduce that there is a bijection between {maps $\underline{M} \longrightarrow X$ of right M-sets} and X.

   (c) Deduce the Yoneda Lemma for one-object categories.

5. *Here we consider natural transformations between functors whose domain is a product category $\mathcal{A} \times \mathcal{B}$. Your task is to show that naturality in two variables simultaneously is equivalent to naturality in each variable separately.*

   Take functors $F, G : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$. For each $A \in \mathcal{A}$ there are functors $F^A, G^A : \mathcal{B} \longrightarrow \mathcal{C}$, as in Sheet 1, q.5; similarly, for each $B \in \mathcal{B}$, there are functors $F_B, G_B : \mathcal{A} \longrightarrow \mathcal{C}$.

   Let $(\alpha_{A,B} : F(A, B) \longrightarrow G(A, B))_{A \in \mathcal{A}, B \in \mathcal{B}}$ be any family of maps. Show that this family is a natural transformation $F \longrightarrow G$ if and only if

   - for each $A \in \mathcal{A}$, the family $(\alpha_{A,B} : F^A(B) \longrightarrow G^A(B))_{B \in \mathcal{B}}$ is a natural transformation $F^A \longrightarrow G^A$, and
   - for each $B \in \mathcal{B}$, the family $(\alpha_{A,B} : F_B(A) \longrightarrow G_B(A))_{A \in \mathcal{A}}$ is a natural transformation $F_B \longrightarrow G_B$.