Category Theory 4
Adjoints and sets

This is to accompany the reading of 24–31 October. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

Some questions on these sheets require knowledge of other areas of mathematics; skip any that you haven't the background for. That aside, I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Let $G : \mathcal{B} \longrightarrow \mathcal{A}$ be a functor.
   (a) For $A \in \mathcal{A}$, define the comma category $(A \Rightarrow G)$.
   (b) Suppose that G has a left adjoint F, and let η be the unit of the adjunction. Show that η_A is an initial object of $(A \Rightarrow G)$, for each $A \in \mathcal{A}$.
   (c) Conversely, suppose that for each $A \in \mathcal{A}$, the category $(A \Rightarrow G)$ has an initial object. Show that G has a left adjoint.

2. State the dual of Corollary 2.3.6. What would you do if someone asked you to prove your dual statement? *(Duality is discussed in Remark 2.1.7.)*

3. The **diagonal functor** $\Delta : \mathbf{Set} \longrightarrow \mathbf{Set}^2$ is defined by $\Delta(A) = (A, A)$ for all sets A. Exhibit left and right adjoints to Δ.

4. Let $O : \mathbf{Cat} \longrightarrow \mathbf{Set}$ be the functor sending a small category to its set of objects. Exhibit a chain of adjoints
$$C \dashv D \dashv O \dashv I.$$