Category Theory 3

Adjoints

This is to accompany the reading of 17–24 October. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

Some questions on these sheets require knowledge of other areas of mathematics; skip any that you haven't the background for. That aside, I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Write down two examples of (a) adjunctions, (b) initial objects, and (c) terminal objects, that aren't in the notes.

2. What can you say about adjunctions between discrete categories?

3. What is an **adjunction**? Show that left adjoints preserve initial objects, that is, if $\mathcal{A} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{B}$ and I is an initial object of $\mathcal{A}$, then $F(I)$ is an initial object of $\mathcal{B}$. Dually, show that right adjoints preserve terminal objects.

   (Later we'll see this as part of a bigger picture: right adjoints preserve limits and left adjoints preserve colimits.)

4.(a) Let $\mathcal{A} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{B}$ be an adjunction. Define the **unit** η and **counit** ε of the adjunction. Prove the triangle identities, $(\varepsilon F) \circ (F\eta) = 1_F$ and $(G\varepsilon) \circ (\eta G) = 1_G$.

   (b) Prove that given functors $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ and natural transformations $\eta : 1 \longrightarrow GF$, $\varepsilon : FG \longrightarrow 1$ satisfying the triangle identities, there is a unique adjunction between F and G with η as its unit and ε as its counit.

5. Let $\mathcal{A} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{B}$ be an adjunction with unit η and counit ε. Let $\mathbf{Fix}(GF)$ be the full subcategory of $\mathcal{A}$ whose objects are those $A \in \mathcal{A}$ for which η_A is an isomorphism, and dually $\mathbf{Fix}(FG) \subseteq \mathcal{B}$. Prove that the adjunction $(F, G, \eta, \varepsilon)$ restricts to an equivalence $(F', G', \eta', \varepsilon')$ between $\mathbf{Fix}(GF)$ and $\mathbf{Fix}(FG)$.

   In this way, any adjunction restricts to an equivalence between full subcategories. Take some examples of adjunctions and work out what this equivalence is.